

2.3 The Case of a Planar Wiggler

In this section we will obtain the FEL equations with a planar wiggler, which is more practical one.

In the planar wiggler the vector potential is in the horizontal direction so electrons wiggle in the horizontal direction. The physics is exactly the same as helical wigglers but the potential has the form:

$$\vec{A} = A_w \hat{x} \cos(k_w z) \quad \text{Eqn 2.3-1}$$

Similar to the helical case the canonical momentum is conserved, therefore in the wiggler the mechanical momentum has the form

$$\Pi_x = \mathbf{g} v_x = p_x - eA_x = -eA_w \cos k_w z \quad \text{Eqn 2.3-2}$$

We convert the variable from time to longitudinal position, and get

$$\frac{dx}{dz} = -\frac{eA_w}{mc\mathbf{g}} \cos(k_w z) \quad \text{Eqn 2.3-3}$$

The transverse velocity is

$$\mathbf{b}_\perp = \frac{v_x}{c} = \frac{dx}{cdt} \cong \frac{dx}{dz} = -\frac{K}{\mathbf{g}} \cos k_w z \quad \text{Eqn 2.3-4}$$

where $K \equiv \frac{eA_w}{mc}$ as defined before.

Energy Exchange:

As explained previously in section 2.2, we expect the interaction between the electrons and a radiation field (in the presence of a wiggler field) to produce an energy exchange between the electrons and an electromagnetic field. Therefore we introduce a signal electromagnetic field similar to helical wiggler case.

The equation of motion

$$mc^2 \frac{d\xi_j}{dz} = e \frac{dx}{dz} E_x \quad \text{Eqn 2.3-5}$$

We define the electric field similarly to the helical wiggler case (except for the polarization). In the planer wiggler the electric field would be linearly polarized, with the electric field in the wiggling plane (here assumed to be the x-z, or horizontal plane.)

$$E_x(z,t) = \frac{1}{2} E(z,t) e^{i(k_s z - w_s t)} + c.c \quad (w_s = k_s c) \quad \text{Eqn 2.3-6}$$

The magnitude of the electric field amplitude $E(z,t)$ varies slowly on the scale of $1/W$. Therefore we will neglect its derivative, and write Eqn 2.3-5 as

$$\frac{d\xi_j}{dz} = -\frac{K}{g_j} \frac{E}{2mc^2} \cos k_w z e^{i(k_s z - w_s t)} + c.c \quad \text{Eqn 2.3-7}$$

Phase Equation:

From the expansion of $\cos(k_w z)$ we have two phase factors

$$e^{i(k_w z + k_s z - w_s t_j)} \quad \text{and} \quad e^{i(-k_w z + k_s z - w_s t_j)} \quad \text{Eqn 2.3-8}$$

Let's define the phase y_j , in similarity to the helical wiggler case, as

$$y_j = k_w z + k_s z - w_s t_j \quad \text{Eqn 2.3-9}$$

We neglect the second term in the phase equation for now. If we differentiate the equation with respect to z we get

$$\frac{dy_j}{dz} = k_w + k_s - \frac{w_s}{\left(\frac{dz}{dt}\right)_j} = k_w + k_s - k_s \frac{1}{b_{\parallel}} \quad \text{Eqn 2.3-10}$$

Where the parallel velocity is

$$b_{\parallel} = \sqrt{b^2 - b_{\perp}^2} = \left[1 - \frac{1}{g^2} - \frac{K^2}{g^2} \cos^2(k_w z)\right]^{\frac{1}{2}} \quad \text{Eqn 2.3-11}$$

After performing a Taylor expansion we get

$$b_{\parallel}^{-1} \cong 1 + \frac{1}{2g^2} + \frac{K^2}{2g^2} \cos^2(k_w z) = 1 + \frac{1 + K^2 \cos^2(k_w z)}{2g^2} \quad \text{Eqn 2.3-12}$$

If we take the average over one wiggler period we obtain

$$b_{\parallel}^{-1}{}_{avr} \cong 1 + \frac{1 + \frac{K^2}{2}}{2g^2} \quad \text{Eqn 2.3-13}$$

The resonance condition is obtained when $k_w + k_s - ks\mathbf{b}_{\parallel}^{-1}{}_{avr} = 0$, and the average \mathbf{y}_j does not change after a period. If we convert parallel velocity into \mathbf{g} we get the resonant condition for $\mathbf{g} = \mathbf{g}$

$$k_w + k_s - k_s \left(1 + \frac{K^2}{2\mathbf{g}^2}\right) = 0 \Rightarrow k_w = k_s \frac{1 + \frac{K^2}{2}}{2\mathbf{g}^2} \quad \text{Eqn 2.3-14}$$

Thus the resonant wavelength is $\lambda_s = \lambda_w \frac{1 + \frac{K^2}{2}}{2\mathbf{g}^2}$

Next we consider energies near the resonance, that is \mathbf{g} is close to, but not equal \mathbf{g} .

For $\mathbf{g}_j \neq \mathbf{g}$, Eqn 2.3-10 becomes

$$\frac{d\mathbf{y}_j}{dz} = k_w \left(1 - \frac{\mathbf{g}_j^2}{\mathbf{g}^2}\right) - k_w \frac{\frac{K^2}{2}}{1 + \frac{K^2}{2}} \cos 2k_w z \quad \text{Eqn 2.3-15}$$

The electron motion in the resonant frame (the frame which is moving with the center of the electron bunch), that is \mathbf{y}_j as a function of x , can be obtained by integrating Eqn 2.3-3 and the second term of Eqn 2.3-15. This plot, exhibiting the ‘figure of 8’ phase oscillations is shown in Figure 2.3-1:

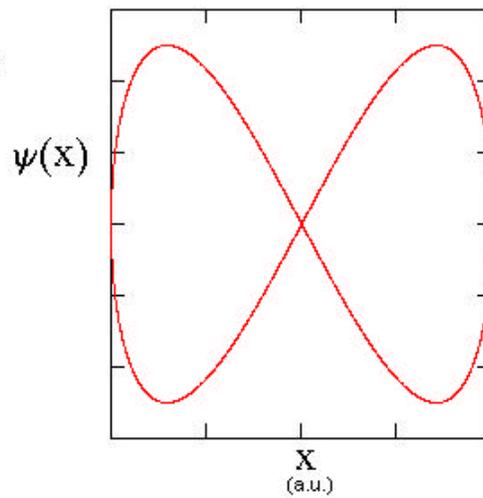


Figure 2.3-1 The electron’s “Figure of 8” motion in the resonant frame

We define phase \mathbf{q} ,

$$\mathbf{q}_j \equiv \mathbf{y}_j + \frac{\frac{K^2}{4}}{1 + \frac{K^2}{2}} \sin(2k_w z)$$

Eqn 2.3-16

This variable is convenient, since we remove the oscillatory term, which is characteristic of a planar wiggler, and obtain a ‘smooth’ phase variable, very similar to the helical wiggler case. Then derivative of \mathbf{q}_j becomes

$$\frac{d\mathbf{q}_j}{dz} = k_w \left(1 - \frac{\mathbf{g}^2}{\mathbf{g}_j^2}\right) = k_w \frac{(\boldsymbol{\varepsilon}_j + \boldsymbol{\varepsilon}_0)(\boldsymbol{\varepsilon}_j - \boldsymbol{\varepsilon}_0)}{\mathbf{g}_j^2} \cong \frac{2k_w(\boldsymbol{\varepsilon}_j - \boldsymbol{\varepsilon}_0)}{\mathbf{g}_0}$$

Eqn 2.3-17

We also define the parameter $b \equiv \frac{\frac{K^2}{4}}{1 + \frac{K^2}{2}}$ for simplicity.

In the plane perpendicular to the magnetic field (the ‘wiggle plane’), the electrons move in a wiggling motion as shown Figure 2.3-2.

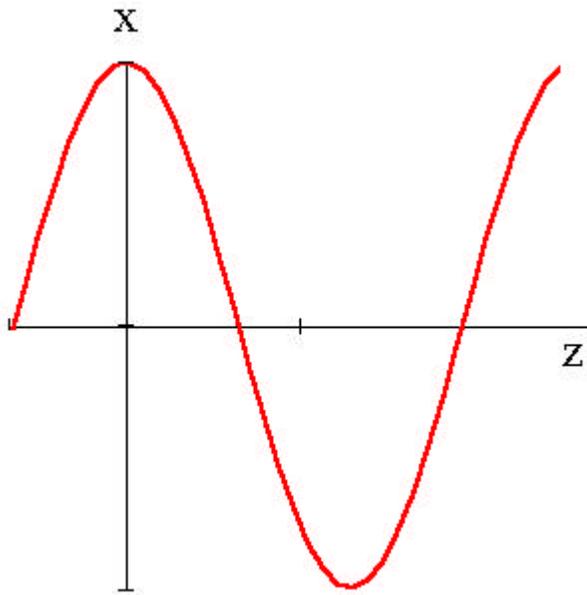


Figure 2.3-2 The electron’s motion in the laboratory frame, in the wiggle plane (x-z)

If we take the second term into account in the phase equation then $K \Rightarrow K[JJ]$ as only difference where $[JJ] = J_0(b) - J_1(b)$ and J_0 and J_1 are the Bessel functions
Thus the equation of motion becomes

$$\frac{d\mathbf{q}_j}{dz} = 2k_w \frac{\boldsymbol{\xi}_j - \boldsymbol{\xi}_0}{\boldsymbol{g}_0} \quad \text{Eqn 2.3-18}$$

$$\frac{d\boldsymbol{\xi}_j}{dz} = -\frac{k_w D_2}{\boldsymbol{g}_0} (Ee^{iq_j} + c.c.) \quad \text{Eqn 2.3-19}$$

where

$$D_2 = \frac{eK[JJ]}{4k_w mc^2} \quad \text{Eqn 2.3-20}$$

The solution of these equations is similar to the helical case.