2.4 The Equations of Motion in the High Gain Regime

In section 2.3 we analyzed the small signal case where we assumed the field amplitude is slow varying. In high gain regime we can no longer assume the field as constant thus we need to consider the evolution of the radiation along the wiggler using Maxwell's equations.

We have the Maxwell's equation as follows

$$(\nabla_{\perp}{}^{2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})A_x = -m_0 J_x \qquad \text{Eqn 2.4-1}$$

The current density can be represented as delta functions because the electrons are point particles.

$$J_x = ecK\cos(k_w z)\sum_j \frac{1}{g_j} d(x - x_j)\, d(y - y_j)\, d(z - z_j) \qquad \text{Eqn 2.4-2}$$

The vector potential is

$$A_x = \frac{1}{2} A_s e^{i(k_s z - w_s t)} + c.c. \equiv A_x{}^{(+)} + c.c. \qquad \text{Eqn 2.4-3}$$

$$(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})A_x{}^{(+)} = (\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t})(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t})A_x{}^{(+)} \qquad \text{Eqn 2.4-4}$$

A_s is slowly varying therefore only z dependence comes from the exponential term.

$$\frac{\partial A_x{}^{(+)}}{\partial z} \cong ik_s A_x{}^{(+)}$$

$$\frac{1}{c}\frac{\partial}{\partial t} A_x{}^{(+)} \cong -i\frac{w_s}{c} A_x{}^{(+)} = -ik_s A_x{}^{(+)}$$

$$(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t})A_x{}^{(+)} \cong 2ik_s A_x{}^{(+)}$$

Thus we can reduce the Eqn 2.4-1 to

$$[\nabla_{\perp}{}^{2} + 2ik_s(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t})]A_x{}^{(+)} + c.c. = -m_0 J_x \qquad \text{Eqn 2.4-5}$$

The electric field has exponential dependence similar to the vector field.

$$E_x^{(+)} = -\frac{\partial}{\partial t} A_x^{(+)} \cong i w_s A_x^{(+)} \qquad \text{Eqn 2.4-6}$$

If we drop the exponential terms we obtain

$$E = i w_s A_s$$

$$(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{2ik_s}\nabla_\perp^{\ 2})E = -m_o c J_x e^{-i(k_s z - w_s t)}$$

$$= -m_0 c^2 \sum_j (-\frac{K}{g_j}) \cos k_w z e^{-i(k_s z - w_s t)}\, d(\vec{r}_\perp - \vec{r}_{\perp j})\, d(z - z_j(t))$$

$$= -\frac{m_0 c^2 K}{g_0} \sum_j \cos k_w z e^{-i(k_s z - w_s t_j(z))}\, d(\vec{r}_\perp - \vec{r}_{\perp j})\, d(z - z_j(t)) \qquad \text{Eqn 2.4-7}$$

We can replace t by $t_j(z)$ for a fixed z because there is a delta function $d(z - z_j(t))$

If we separate slow varying components

$$\cos k_w z e^{-i(k_s z - w_s t_j)} = \frac{1}{2} e^{-iq_j}[JJ] + \text{high frequency terms}$$

$$(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{2ik_s}\nabla_\perp^{\ 2})E = \frac{m_0 e c^2 K[JJ]}{2\, g_0} \sum_j e^{-iq_j}\, d(\vec{r}_\perp - \vec{r}_{\perp j})\, d(z - z_j) \qquad \text{Eqn 2.4-8}$$

We extract slow components by averaging over an optical period λ

$$\frac{1}{l}\int_l dz \sum_j d(z - z_j) = \frac{1}{l}\sum_{z \le z_j \le z + l} = \frac{1}{l}\sum_{j=1}^{N_l}$$

Let N be the number of electrons per meter, $N_l = N.\, l$ the number of electrons in λ

$$\frac{ec}{l} = \frac{ecN}{lN} = \frac{I_0}{N_l}$$

I_0 is the current

$N.\, l$ is the number of electrons in l

Thus the full set of equations is as follows

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}+\frac{1}{2ik_s}\nabla_\perp^2\right)E=\frac{\mu_0 cK[JJ]}{2\gamma_0}\frac{I_0}{N_l}\sum_{j=1}^{N_l}e^{-i\theta_j}\,\delta(\vec{r}_\perp-\vec{r}_{\perp j})$$ Eqn 2.4-9

$$\frac{d\theta_j}{dz}=2k_w\frac{\gamma_j-\gamma_0}{\gamma_j}$$ Eqn 2.4-10

$$\frac{d\gamma_j}{dz}=-\frac{k_w D_2}{\gamma_0}\left(Ee^{i\theta_j}+c.c.\right)$$ Eqn 2.4-11