2.6 The High Gain FEL: 1-D Theory

In this section we will solve the Coupled Vlasov-Maxwell equations (Eqn 2.5-9 and Eqn 2.5-10) in 1-D for $f_0 = \delta(\gamma - \gamma_0)$.

We drop $\nabla_\perp^{\,2}$ because of the 1D solution. Therefore the equations become

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \theta}\right)E = \frac{D_1}{\gamma_0}\int F d\gamma \qquad \text{Eqn 2.6-1}$$

$$\left[\frac{\partial}{\partial \tau} + 2\frac{\gamma - \gamma_0}{\gamma_0} i\right]F = \frac{D_2}{\gamma_0}\frac{\partial f_0}{\partial \gamma}E \qquad \text{Eqn 2.6-2}$$

We define the Fourier Laplace Transforms of E and F.

$$\tilde{E}(\tau, q) = \int_{-\infty}^{\infty} d\theta e^{-iq\theta} E(\tau, \theta) \qquad \text{Eqn 2.6-3}$$

$$\tilde{F}(\tau, q, \gamma) = \int_{-\infty}^{\infty} d\theta e^{-iq\theta} F(\tau, \theta, \gamma) \qquad \text{Eqn 2.6-4}$$

$$\bar{E}(\Omega, q) = \int_{0}^{\infty} d\tau e^{i\Omega\tau} \tilde{E}(\tau, q) \qquad \text{Eqn 2.6-5}$$

$$\bar{F}(\Omega, q, \gamma) = \int_{0}^{\infty} d\tau e^{i\Omega\tau} \tilde{F}(\tau, q, \gamma) \qquad \text{Eqn 2.6-6}$$

The Fourier Laplace transforms of the Eqn 2.6-1 and Eqn 2.6-2 are

$$(-i\Omega + iq)\bar{E} = \frac{D_1}{\gamma_0}\int d\gamma \bar{F} + \tilde{E}(\tau = 0) \qquad \text{Eqn 2.6-7}$$

$$\left(-i\Omega + i2\frac{\gamma - \gamma_0}{\gamma_0}\right)\bar{F} = \frac{D_2}{\gamma_0}\frac{\partial f_0}{\partial \gamma}\bar{E} + \tilde{F}(\tau = 0) \qquad \text{Eqn 2.6-8}$$

If we separate $\bar{F}$ in Eqn 2.6-8 and plug into Eqn 2.6-7 we can get the expression for $\bar{E}$ as follows:

$$\left[\Omega - q - (2\rho)^3 \frac{1}{\Omega^2}\right]\bar{E} = \bar{S} \qquad \text{Eqn 2.6-9}$$

where $\bar{S}(\Omega, q) = i\tilde{\bar{E}}(t=0) - \frac{D_1}{g_0}\int \frac{\tilde{F}(t=0, q, g)}{\Omega - 2\frac{g - g_0}{g_0}} dg$ Eqn 2.6-10

Eqn 2.6-9 can also be written as:

$$[\Omega^3 - \Omega^2 q - (2r)^3]\bar{E} = \Omega^2 \bar{S}$$ Eqn 2.6-11

The inverse Fourier Laplace transforms of this equation is 3rd order partial differential equation: <u>envelope equation</u>

$\frac{\partial}{\partial t} \leftrightarrow i\Omega, \frac{\partial}{\partial q} \leftrightarrow -iq$ Thus

$$[\frac{\partial^3}{\partial t^3} + \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial q} - (2r)^3]E = 0$$ Eqn 2.6-12

Initial conditions given by $\Omega^2 \bar{S}$

The physical meaning of W and *q*

$E \sim e^{-i\Omega t + iqq}$ Therefore the electric field is

$$E.e^{ik_s z - iw_s t} \sim e^{-i\Omega t + iqq} e^{ik_s z - iw_s t} = e^{-i\Omega k_w z + iq(k_w z + k_s z - w_s t + b\sin 2k_w z) + ik_s z - iw_s t}$$

$$= e^{i(k_s + qk_s + qk_w - \Omega k_w)z + iqb\sin 2k_w z} e^{-iw_s(1+q)t}$$ Eqn 2.6-13

As we understand from Eqn 2.6-13 q represents the frequency change

$\Delta w = w_s q$ Therefore $q = \frac{\Delta w}{w_s}$ Eqn 2.6-14

is <u>detuning</u>

In Eqn 2.6-13 the term $e^{-i\Omega k_w z}$ represents the growth rate.

$$e^{-i\Omega k_w z} = e^{-i\,\mathrm{Re}(\Omega) k_w z} e^{\mathrm{Im}(\Omega) k_w z}$$ Eqn 2.6-15

So $\mathrm{Re}(\Omega)$ is just a phase shift rate and $\mathrm{Im}(\Omega)$ is growth rate.

Let $e^{\mathrm{Im}(\Omega) k_w z} = e^{\frac{z}{2L_G}}$ Eqn 2.6-16

Then power is $\sim |E|^2 \sim e^{\frac{z}{L_G}}$ Therefore we can define L_G as **<u>gain length</u>**

$$L_G = \frac{1}{2\,\mathrm{Im}(\Omega)k_w}$$ Eqn 2.6-17

Pierce Parameter ρ

$$(2r)^3 = -\frac{2D_1D_2}{g_0^3}$$ Eqn 2.6-18

$$D_1 = \frac{m_0 n_0 e c^2 K[JJ]}{2k_w}$$

$$D_2 = \frac{eK[JJ]}{4k_w mc^2}$$ Eqn 2.6-19

$$(2r)^3 = \frac{m_0 n_0 e^2 K^2 [JJ]^2}{4k_w^2 m\, g_0^3}$$ Eqn 2.6-20

To evaluate we use

$m_0 = \frac{Z_0}{c}$, $Z_0 = 377\Omega$, mc^2=0.511x10^{6}eV

$$(2r)^3 = Z_0 \frac{en_0 c}{k_w^2}\frac{e}{mc^2}\frac{K^2[JJ]^2}{4\,g_0^3}$$

The pierce parameter is dimensionless

$$\Omega \frac{Amp/m^2}{1/m^2}\frac{e}{eV} = \frac{\Omega Amp}{V} = 1$$

Characteristic equation and scaled growth rate

For a fixed detuning *q*, the solution of the envelope equation is the sum of three terms.

$e^{-i\Omega_1 t}, e^{-i\Omega_2 t}, e^{-i\Omega_3 t}$

Ω_1, Ω_2, Ω_3 are solutions of $\Omega^3 - q\Omega^2 - (2r)^3 = 0$

We define $\Omega \equiv 2r\,l$ and $\Delta \equiv \frac{q}{2r} = \frac{\Delta w}{2rw_s}$ Eqn 2.6-21

To simplify the equation.

Then we get $l^3 - \Delta\, l^2 - 1 = 0$

When $\Delta = 0$ (resonance)

$l^3 = 1$

$\lambda = e^{\pm\frac{2\pi}{3}i}, 1$ Therefore $\mathrm{Im}(\lambda) = \pm\frac{\sqrt{3}}{2}, 0$

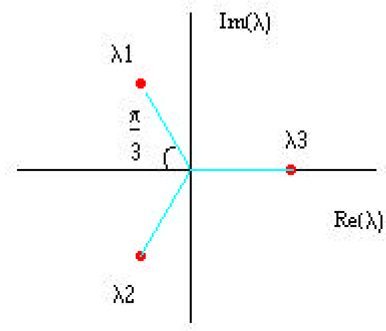

Figure 2.6-1 The roots of the cubic equation in λ providing the growth rate

One solution grows with rate $\frac{\sqrt{3}}{2}$ one decays and the other one oscillates.

$$\mathrm{Im}(\Omega) = 2\rho\,\mathrm{Im}(\lambda) = 2\rho\frac{\sqrt{3}}{2} = \sqrt{3}\rho \qquad \text{Eqn 2.6-22}$$

Therefore we obtain simple expression for the gain length

$$L_G^{1D} = \frac{1}{2\,\mathrm{Im}(\Omega)k_w} = \frac{\lambda_w}{4\sqrt{3}\pi\rho} \qquad \text{Eqn 2.6-23}$$

or $\frac{\lambda_w}{L_G^{1D}} = 4\sqrt{3}\pi\rho$ Eqn 2.6-24