2.7 The Initial Value Problem in 1-D Theory

In this section we will consider the initial conditions and solve the electric field. In the previous part we had obtained the expression for Fourier Laplace transform of E (Eqn 2.6-9). Using that expression, we can get the equation below

$$\bar{E} = \frac{\Omega^2}{\Omega^3 - \Omega^2 q - (2r)^3}\bar{S} \qquad \text{Eqn 2.7-1}$$

Remembering Eqn 2.6-10

$$\bar{S} = i\tilde{E}(t=0,q) - \frac{D_1}{g_0}\int \frac{\tilde{F}(t=0,q,g)}{\Omega - 2\frac{g-g_0}{g_0}}dg \qquad \text{Eqn 2.7-2}$$

We assume initially there is no energy modulation so we assume $\tilde{F}$ is proportional to delta function:

$$\tilde{F}(t=0,q,g) \sim d(g-g_0)$$

Then

$$\bar{S} = i\tilde{E}(t=0) - \frac{1}{\Omega}\frac{D_1}{g_0}\int \tilde{F}(t=0,q)dg \qquad \text{Eqn 2.7-3}$$

There are two initial conditions.

1) The first one is the Amplifier case where there is radiation initially but no distribution change in the electron bunch:

$$\tilde{E}(t=0) \neq 0, \tilde{F}(t=0) = 0$$

2) The second one is SASE case where there is no radiation initially so it starts from noise: $\tilde{E}(t=0) = 0, \tilde{F}(t=0) \neq 0$

2.7.1 FEL Amplifier:

Here we solve the first case where we have initial radiation but no density modulation. In this case $\bar{S} = i\tilde{E}(t=0)$ Therefore Eqn 2.7-1becomes

$$\bar{E} = \bar{g}\tilde{E}(t=0) \qquad \text{Eqn 2.7-4}$$

Where

$$\bar{\tilde{g}} = \frac{i\Omega^2}{\Omega^3 - \Omega^2 - (2r)^3}$$ Eqn 2.7-5

If we take the inverse Laplace transform of the Eqn 2.7-4 we get

$$\tilde{E}(t,q) = \tilde{\tilde{g}}\tilde{E}(t=0)$$ Eqn 2.7-6

The Inverse Fourier transform of $\bar{\tilde{g}}$ can be calculated using residue integral.

(Remembering $\Omega = 2rl$ and $\Delta \equiv \frac{q}{2r} = \frac{\Delta w}{2rw_s}$)

When τ>0 the integral becomes equal to summation of residues around the poles.

We obtain the poles as $l^3 - \Delta l^2 - 1 = (l - l_1)(l - l_2)(l - l_3)$

So $\tilde{\tilde{g}}(t,\Delta)$ would be equal to

$$\tilde{\tilde{g}}(t,\Delta) = \frac{l_1^2 e^{-i2rl_1 t}}{(l_1 - l_2)(l_1 - l_3)} + \frac{l_2^2 e^{-i2rl_2 t}}{(l_2 - l_1)(l_2 - l_3)} + \frac{l_3^2 e^{-i2rl_3 t}}{(l_3 - l_1)(l_3 - l_2)}$$ Eqn 2.7-7

The physical meaning of $\tilde{\tilde{g}}$ is the gain of the electric field

Limiting cases:

1) $t = 0$, $\tilde{\tilde{g}} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ which is the beginning of the wiggler.

2) $\Delta = 0$, $rt >> 1$ first term dominates

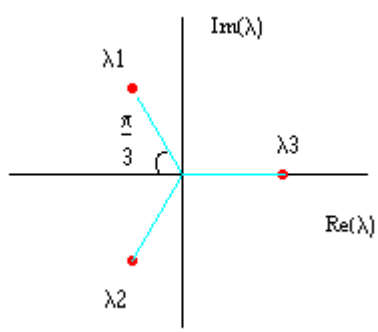

Figure 2.7-1 The roots of the cubic equation in λ, providing the growth rate

$|l_1| = 1$, $\mathrm{Im}(l_1) = \frac{\sqrt{3}}{2}$

$|l_1| = 1$ Therefore $|\tilde{\tilde{g}}| = \frac{1}{3} e^{\sqrt{3}rt}$ Eqn 2.7-8

$$P_{out} = \frac{1}{9} e^{4p\sqrt{3}rN_w} P_{in} = \frac{1}{9} e^{\frac{L_w}{L_G}} P_{in}$$ Eqn 2.7-9

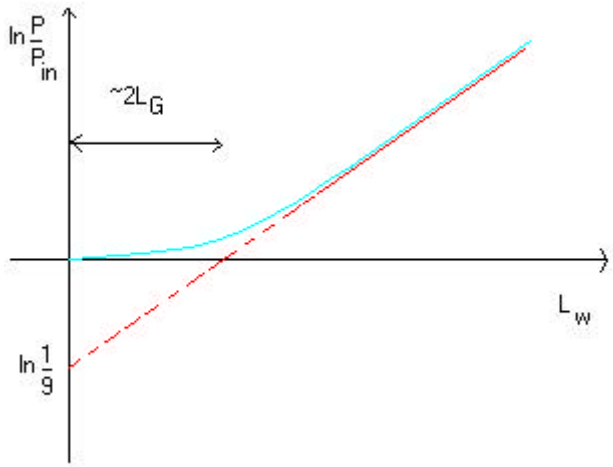

Figure 2.7-2 Gain vs. wiggler length (log scale)

Synchrotron rotation Frequency and Saturation

As described in section 2.2 the synchrotron frequency is the oscillation frequency between the energy modulation and the phase. As the field grows the synchrotron frequency also grows. This means, as the bucked height shown in Figure 2.7-3 increases, the energy spread of the beam increases due to the interaction with the radiation. Thus the whole picture just expands. However when we reach the saturation the field can no longer grow. The energy spread causes over bunching as shown in Figure 2.7-4 and electrons cannot lose energy any more.

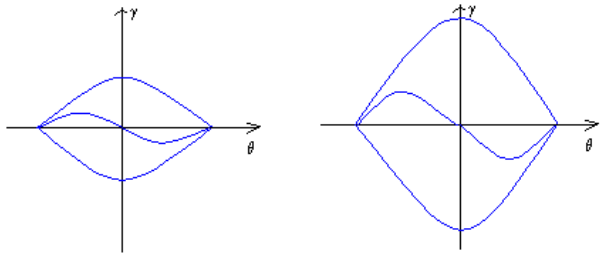

Figure 2.7-3 Electrons phase space plot in the exponential growth regime. The figure on the right shows growth in both electron modulation and bucket size.

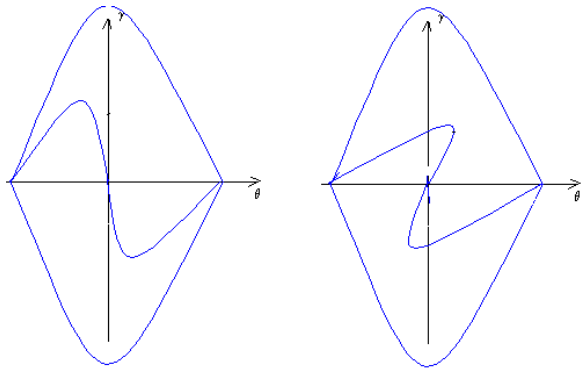

Figure 2.7-4 Electrons phase space plot at the saturation regime. The modulation and the separatrix are shown. The separatrix does not grow and the electron distribution starts executing a synchrotron oscillation.

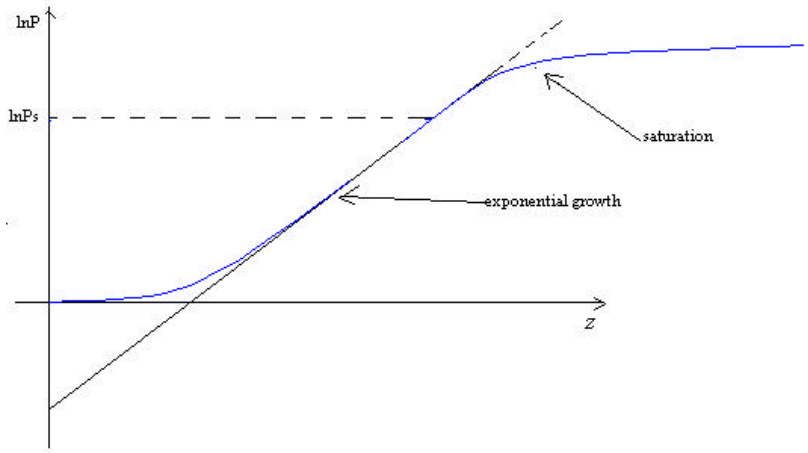

Figure 2.7-5 The FEL gain along the wiggler, showing start-up, the exponential regime and saturation

Saturation condition

Given the growth rate $\mathrm{Im}(\Omega) = \sqrt{3}\,r$ FEL reaches the saturation condition when $\mathrm{Im}(\Omega) = \Omega_s$. The solution of the pendulum equations for saturation yields that

$$\Omega_s^{\,2} = \frac{2D_2E}{g_0^{\,2}} \text{ therefore } 3r^2 = \frac{2D_2E}{g_0^{\,2}} \qquad \text{Eqn 2.7-10}$$

We can calculate Electric field as

$$E = \frac{3r^2 g_0^{\,2}}{2D_2} \qquad \text{Eqn 2.7-11}$$

Thus the power density

$$\frac{|E|^2}{2Z_0} = \frac{9}{4}(r\,g_0)^4 \frac{1}{2Z_0D_2^{\,2}} \qquad \text{Eqn 2.7-12}$$

Using the definition of r

$(2r)^3 = \dfrac{2D_1D_2}{g_0^{\,3}}$ the power density becomes

$$\frac{|E|^2}{2Z_0} = \frac{9}{64}\,2D_1D_2\,r\,g_0\,\frac{1}{Z_0D_2^{\,2}} = \frac{9}{16}\,rn_0cmc^2\,g_0 \qquad \text{Eqn 2.7-13}$$

Saturation power

$$P_s = \frac{|E|^2}{2Z_0}A = \frac{9}{16}\,r.n_0cA.mc^2\,g_0 \qquad \text{Eqn 2.7-14}$$

where A is the cross section area.

n_0cA is the number of electron/sec, thus $n_0cA.mc^2\,g_0$ is the electron beam power P_e. As a result we achieve the simple expression for the saturation power as

$$P_s \cong rP_e \qquad \text{Eqn 2.7-15}$$

2.7.2 1-D SASE Theory

In this section we are going to explain the other initial condition which is SASE case where

$$\tilde{E}(t=0)=0, \tilde{F}(t=0)\neq 0 \qquad \text{Eqn 2.7-16}$$

We had defined $\bar{\ddot{g}}=\dfrac{i\Omega^2}{\Omega^3-\Omega^2-(2r)^3}$ in Eqn 2.7-5. We obtain $\tilde{\ddot{g}}$ by inverse Fourier-Laplace transform of this expression.

$$\tilde{\ddot{g}}=\frac{1}{2p}\int_{-\infty+is}^{\infty+is}\bar{\ddot{g}}e^{-i\Omega t}\,d\Omega \qquad \text{Eqn 2.7-17}$$

If we integrate Eqn 2.7-17 with respect to t we get

$$\int\tilde{\ddot{g}}dt=\tilde{\dot{g}}=\frac{1}{2p}\int_{-\infty+is}^{\infty+is}\bar{\ddot{g}}\frac{e^{-i\Omega t}}{-i\Omega}d\Omega$$

which yields

$$\bar{\dot{g}}=-\frac{\Omega}{\Omega^3-\Omega^2 q-(2r)^3} \qquad \text{Eqn 2.7-18}$$

$\tilde{F}(t=0)\sim d(g-g_0)$ Thus if we substitute these Eqn 2.7-2 we get

$$\bar{E}=\bar{\dot{g}}\tilde{J}_0 \qquad \text{Eqn 2.7-19}$$

where $\tilde{J}_0$ is defined as

$$\tilde{J}_0\equiv\frac{D_1}{g_0}\int\tilde{F}(t=0,q)d\,g \qquad \text{Eqn 2.7-20}$$

The meaning J_0 is transverse current and

$$J_0(q)=\frac{D_1}{g_0}\int F(t=0,q)d\,g \qquad \text{Eqn 2.7-21}$$

Therefore inverse Fourier-Laplace transform of Eqn 2.7-19 gives us

$$\tilde{E}(t,q)=\tilde{\dot{g}}(t,q)\frac{D_1}{g_0}\int\tilde{F}(t=0,q)d\,g \qquad \text{Eqn 2.7-22}$$

and $E(t,q)=\int J_0(q')\dot{g}(t,q-q')dq'$ Eqn 2.7-23

is the convolution integral.

The distribution function was given as $f\cong f_0+(Fe^{iq}+c.c.)$

$F(t=0,q)=fe^{-iq}+\text{other freqauncies}$

As we mentioned earlier in section 2.5, during the FEL process, the components that are proportional to phase factors other than e^{iq} can be neglected because they will be off resonance. Therefore we can write Eqn 2.7-21 as

$$J_0(q) = \frac{D_1}{g_0}\int F(t=0,q)\,d\,g \cong \frac{D_1}{g_0}\int e^{-iq} f(q,\ g,\ t=0)\,d\,g \qquad \text{Eqn 2.7-24}$$

Then the convolution integral (Eqn 2.7-23) becomes a summation because of the $d(z - z_j)$

$$E(t,q) \cong \frac{D_1 k_s}{g_0 n_l}\sum_j e^{-iqj}\dot{g}(t,q-q_j) \qquad \text{Eqn 2.7-25}$$

The contribution from every electron evolves from a delta function to a greens function $\dot{g}$. The sum is over these green functions with a phase factor e^{-iqj} for each electron.

SASE power

Now we would like to calculate the SASE power. The expression for power is as follows:

$$P = \frac{|E|^2}{2Z_0}A \qquad \text{Eqn 2.7-26}$$

(A=beam cross section area)

Since E is a summation over all electrons SASE power would be ensemble average of all electrons.

$$|E|^2 = \langle E(t,q)E^*(t,q)\rangle = \left(\frac{D_1 k_s}{g_0 n_l}\right)^2 < \sum_{ij} e^{i(qj-qi)}\dot{g}(t,q-q_j)\dot{g}^*(t,q-q_j) > \qquad \text{Eqn 2.7-27}$$

Thus the power is

$$<P> = \frac{A}{2Z_0}\frac{D_1^2 k_s}{g_0^2 n_l}\int dq'|\dot{g}(t,q-q')|^2 = \frac{1}{2Z_0}\frac{D_1^2 k_s}{g_0^2 n_0}\int dq'|\dot{g}(t,q-q')|^2 \qquad \text{Eqn 2.7-28}$$

where $(n_l = n_0 A)$

The power spectrum

Now we would like to calculate q' integral in Eqn 2.7-28 to get power spectrum. $\dot{g}$ is the inverse Fourier-Laplace transform of $\tilde{\dot{g}}$

$$\dot{g} = \frac{1}{2p}\int \tilde{\dot{g}} e^{iqq} dq \qquad \text{Eqn 2.7-29}$$

Plugging this in Eqn 2.7-28 and taking the proper integrals gives us

$$< P >= \frac{D_1^{\,2} k_s}{4pZ_0\, g_0^{\,2} n_0} \int dq \left| \tilde{g}(t,q) \right|^2 \qquad \text{Eqn 2.7-30}$$

We can get rid of the integral by differentiating both sides with respect to q ($dq = \frac{dw}{w}$ and $k_s = \frac{w}{c}$) and obtain the expression for power spectrum.

$$< \frac{dP}{dw} >= \frac{D_1^{\,2}}{4pZ_0\, g_0^{\,2} n_0 c} \left| \tilde{g}(t,q) \right|^2 = \frac{mc^2 g_0}{4p} (2r)^3 \left| \tilde{g}(t,q) \right|^2 \qquad \text{Eqn 2.7-31}$$

Thus we need to calculate the Green's function $\tilde{g}(t,q)$

From Eqn 2.7-18 we can see that

$\bar{\ddot{g}} = -i\Omega \bar{g}$ where $\Omega = 2rl$

In the previous section we had driven the expression for $\tilde{\ddot{g}}$ by calculating the inverse Fourier-Laplace transform. Similarly we can do the same thing for $\tilde{g}$.

$$\tilde{g} = \frac{1}{2p} \int_{-\infty}^{\infty} \bar{g} e^{-i\Omega t} d\Omega \qquad \text{Eqn 2.7-32}$$

We plug Eqn 2.7-18 in Eqn 2.7-32 we obtain

$$\tilde{g} = \frac{1}{2p} \int_{-\infty}^{\infty} -\frac{\Omega}{\Omega^3 - \Omega^2 q - (2r)^3} e^{-i\Omega t} d\Omega \qquad \text{Eqn 2.7-33}$$

When we do the same $\Omega = 2rl$ and $\Delta \equiv \frac{q}{2r} = \frac{\Delta w}{2rw_s}$ change of variables and we get same contour integral except the nominator will be proportional to λ. We will have same poles so that summation of residues will be

$$\tilde{g}(t,q) = \frac{i}{2r} \left[\frac{l_1 e^{-i2rl_1 t}}{(l_1 - l_2)(l_1 - l_3)} + \frac{l_2 e^{-i2rl_2 t}}{(l_2 - l_1)(l_2 - l_3)} + \frac{l_3 e^{-i2rl_3 t}}{(l_3 - l_1)(l_3 - l_2)} \right] \qquad \text{Eqn 2.7-34}$$

From this expression we can also see that

$$\frac{d}{dt} \tilde{g} = \tilde{\ddot{g}} \qquad \text{Eqn 2.7-35}$$

For $2rt << 1$ we get the spontaneous radiation solution which yields

$$\left|\tilde{g}(t,q)\right|^2 = \left(\frac{\sin\frac{tq}{2}}{\frac{tq}{2}}\right)^2 t^2 \qquad \text{Eqn 2.7-36}$$

For the end of the wiggler ($t = k_w L_w$)

$$\frac{tq}{2} = \frac{k_w L_w}{2}\frac{\Delta w}{w_s} = pN_w(\frac{w}{w_s} - 1)$$

Thus we can express power spectrum as

$$< \frac{dP}{dw} > = \frac{mc^2 g_0}{4p}(2r)^3 t^2 \left[\frac{\sin pN_w(\frac{w}{w_s} - 1)}{pN_w(\frac{w}{w_s} - 1)}\right]^2 \qquad \text{Eqn 2.7-37}$$

Note that $(2r)^3 \propto \frac{I_0}{A} = en_0 c$ Eqn 2.7-38

therefore the spontaneous emission is <u>linearly dependent on the charge</u>.

Qualitative description of spontaneous radiation bandwidth and opening angle

We can plot the power spectrum using Eqn 2.7-37 as follows:

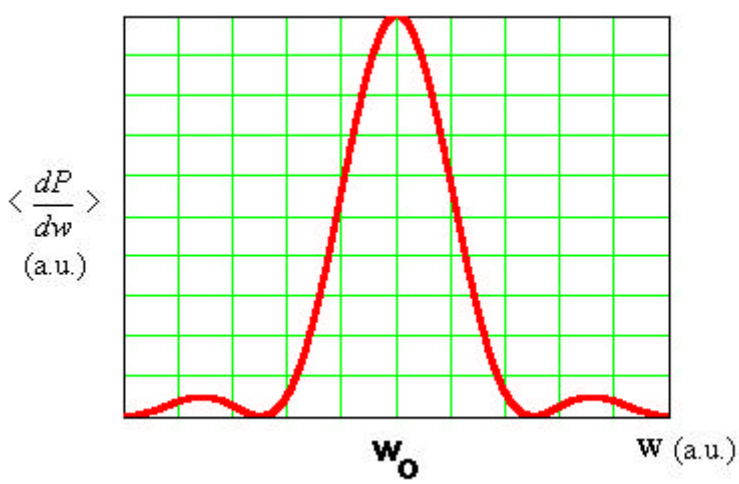

Figure 2.7-6 The power spectrum of the spontaneous radiation.

This plot is only for one frequency detuning value. Since the process starts from the noise in reality there is no specific frequency detune so we need to integrate over all detune values.

$$\int \left[\frac{\sin pN_w q}{pN_w q} \right]^2 dq = \frac{1}{N_w} \qquad \text{Eqn 2.7-39}$$

Therefore total power spectrum will be

$$P = \left(\frac{dP}{dw} \right)_{peak} \frac{1}{N_w} \text{ where } \left(\frac{dP}{dw} \right)_{peak} = \frac{mc^2 g_0}{4p} (2r)^3 t^2 \qquad \text{Eqn 2.7-40}$$

Also $\frac{0.8895}{N_w} \cong \frac{1}{N_w}$ is the full width at half maximum (FWHM) bandwidth of the Spontaneous radiation spectrum.

The wavelength as function of opening angle q is as follows:

$$l = \frac{l_w}{2g^2} (1 + \frac{K^2}{2} + g^2 q^2) \qquad \text{Eqn 2.7-41}$$

Thus

$$\frac{\Delta l}{l} \cong \frac{g^2 q^2}{1 + \frac{K^2}{2}} \cong \frac{l_w}{2l} q^2 \qquad \text{Eqn 2.7-42}$$

Since the width is $\frac{1}{N_w}$,

$$\frac{\Delta l}{l_s} \cong \frac{1}{N_w} = \frac{l_w}{2l} q^2 \qquad \text{Eqn 2.7-43}$$

Thus we have the opening angle as:

$$q = \sqrt{\frac{2 l_s}{N_w l_w}} = \sqrt{\frac{2 l_s}{L_w}} = \frac{\sqrt{1 + \frac{K^2}{2}}}{g\sqrt{N_w}} \qquad \text{Eqn 2.7-44}$$

This narrow bandwidth-opening angle is $<< \frac{1}{g}$

High gain limit

In the High gain limit $e^{-i2r\lambda_1 t}$ term dominates in Eqn 2.7-34 thus

$$\tilde{g}(t,r) \cong \frac{i}{2r}\frac{\lambda_1 e^{-i2r\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \qquad \text{Eqn 2.7-45}$$

where

$$\lambda_1 \cong e^{\frac{2p}{3}i},\ \ \lambda_2 \cong e^{-\frac{2p}{3}i},\ \lambda_3 \cong 1$$

$$|\ \lambda_1-\lambda_2\ | \cong \sqrt{3},\ |\ \lambda_1\ | \cong 1$$

The λ_1 in the exponent needs to be calculated more accurately

$$\lambda^3 - \Delta\lambda^2 - 1 = 0$$

for small Δ, λ_1 is estimated as

$$\lambda_1 \cong e^{\frac{2p}{3}i} + \frac{1}{3}\Delta + \frac{1}{9}e^{-\frac{2p}{3}i}\Delta^2$$

$$\mathrm{Im}(\lambda_1) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{18}\Delta^2$$

SASE power (1-D)

The power spectrum expression for this case is

$$<\frac{dP}{dw}> = \frac{mc^2 g_0}{4p}(2r)^3\left|\tilde{g}(t,q)\right|^2 = \frac{mc^2 g_0}{4p}(2r)^3\left|\frac{i}{2r}\frac{\lambda_1 e^{-i2r\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right|^2 \qquad \text{Eqn 2.7-46}$$

By integrating over all frequencies we obtain

$$P^{(ID)} = \frac{1}{9}rmc^2 g_0 \frac{s_w}{\sqrt{2p}}e^{2\sqrt{3}rt} \qquad \text{Eqn 2.7-47}$$

where

$$s_w = 2rw_s\left(\frac{3\sqrt{3}}{4rt}\right)^{\frac{1}{2}} \qquad \text{Eqn 2.7-48}$$

Relation to the noise power:

The power can be expressed as

$$P = \frac{1}{9}e^{\frac{L_w}{L_G}}\sqrt{2p}\,s_w\left(\frac{dP}{dw}\right)_{noise} \qquad \text{Eqn 2.7-49}$$

where noise power is

$$\left(\frac{dP}{dw}\right)_{noise} = \frac{r}{2p} mc^2 g_0 \qquad \text{Eqn 2.7-50}$$

SASE Bandwidth

Full width of SASE is

$$\frac{\sqrt{2p}\, s_w}{w_s} = \sqrt{2p}.2\, r\left(\frac{3\sqrt{3}}{4rt}\right)^{\frac{1}{2}} = \left(\frac{6\sqrt{3}pr}{t}\right)^{\frac{1}{2}} = \left(\frac{3\sqrt{3}\, r\, l_w}{L_w}\right)^{\frac{1}{2}} = \left(\frac{3\sqrt{3}\, r}{N_w}\right)^{\frac{1}{2}} \qquad \text{Eqn 2.7-51}$$

using the definition of s_w from Eqn 2.7-48.

Full width of spontaneous radiation is $\frac{1}{N_w}$. Thus the ratio would be

$$\frac{\Delta w_{SASE}}{\Delta w_{spon}} = (3\sqrt{3}\, r N_w)^{\frac{1}{2}} = \left(\frac{4p\sqrt{3}\, r}{l_w}\frac{3N_w\, l_w}{4p}\right)^{\frac{1}{2}} = \left(\frac{3}{4p}\frac{L_w}{L_G}\right)^{\frac{1}{2}} \cong \left(\frac{L_w}{4L_G}\right)^{\frac{1}{2}} \qquad \text{Eqn 2.7-52}$$

Therefore when

$L_w > 4L_G$ SASE bandwidth is larger than spontaneous radiation when

$L_w < 4L_G$ exponential growth term is not dominating yet.

Saturation length for SASE

We had obtained the expression $P_{sat} \cong rP_e$ where $P_e = n_0 cA.mc^2 g_0$ is the electron beam power.

From Eqn 2.7-47 we have

$$P^{(ID)} = \frac{1}{9} rmc^2 g_0 \frac{s_w}{\sqrt{2p}} e^{\frac{L_w}{L_G}} \qquad \text{Eqn 2.7-53}$$

$P_e = mc^2 g_0 n_l c$ where $n_l = n_0 A$ Eqn 2.7-54

Thus

$$P^{(ID)} = \frac{1}{9} e^{\frac{L_w}{L_G}} rP_e \frac{1}{l_c}\frac{1}{n_l} = \frac{1}{9} e^{\frac{L_w}{L_G}} rP_e \frac{1}{N_c} \qquad \text{Eqn 2.7-55}$$

where $N_c = n_l l_c$ is the number of electron in the coherence length.

The coherence length is given as $l_c = \frac{\sqrt{2p}}{s_w} c = N_w\, l_s \left(\frac{4p}{3}\frac{L_G}{L_w}\right)^{\frac{1}{2}}$

At saturation

$P^{ID} = P_{sat} = rP_e$ Therefore we need to have $\frac{1}{9N_c} e^{\frac{L_w}{L_G}} = 1$ Eqn 2.7-56

$$N_c = \frac{I_0}{ec} l_c = \frac{I_0}{ec} N_w l_s (\frac{4p}{3} \frac{L_G}{L_w})^{\frac{1}{2}}$$ Eqn 2.7-57

Thus

$$N_c = \frac{I_0}{ec} l_s \frac{L_G}{l_w} (\frac{4p}{3})^{\frac{1}{2}} (\frac{L_w}{L_G})^{\frac{1}{2}}$$ Eqn 2.7-58

Let's take $\frac{L_w}{L_G}$ =x and get Eqn 2.7-56 as

$$\frac{e^x}{\sqrt{x}} = 9 \frac{I_0}{ec} l_s \frac{L_G}{l_w} (\frac{4p}{3})^{\frac{1}{2}}$$ Eqn 2.7-59

Example:

5.3 μm SASE at ATF at BNL

L_G=37 cm λ_w=3.3 cm I_0=110 Amp λ_s=5.3*10^{-6} m

$$\frac{e^x}{\sqrt{x}} = 2.5*10^9 \quad x \cong 23$$

In general saturation length is roughly 20L_G.