2.8 3-D FEL Theory and Optical Guiding

As we remember from the Eqn 2.5-9 and Eqn 2.5-10 we had Coupled Maxwell-Vlasov equations. For 1-D solution we had dropped $\nabla_T{}^2$ term. Now for 3-D solution we will include $\nabla_T{}^2$ term.

Now for simplicity let's define a vector

$$\vec{x} \equiv \sqrt{2k_s k_w}\,\vec{r}_\perp \text{ and } \nabla_\perp{}^2 \equiv \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} \qquad \text{Eqn 2.8-1}$$

Therefore the Fourier Laplace transform of the Maxwell-Vlasov equations would become

$$\left(\Omega - q + \nabla_\perp{}^2 + U(\vec{x};\Omega,q)\right)\overline{E}(\vec{x};\Omega,q) = \overline{S}(\vec{x};\Omega,q) \qquad \text{Eqn 2.8-2}$$

Where

$$U(\vec{x};\Omega,q) = \frac{D_1 D_2}{g_0{}^2}\int \frac{\frac{\partial f_0}{\partial g}\,d\,g}{\Omega - 2\frac{g - g_0}{g_0}} \qquad \text{Eqn 2.8-3}$$

Similar to what we have done in 1-D case we will assume f_0 as delta function but with $\vec{x}$ dependence as follows:

$$f_0 = u(\vec{x})\,d(g - g_0)$$

So we can calculate U as

$$U(\vec{x};\Omega,r) = -\frac{(2r)^3}{\Omega^2}u(\vec{x}) \qquad \text{Eqn 2.8-4}$$

Thus Eqn 2.8-2 becomes

$$\left[\Omega - q + \nabla_\perp{}^2 - \frac{(2r)^3}{\Omega^2}u(\vec{x})\right]\overline{E}(\vec{x};\Omega,q) = \overline{S}(\vec{x};\Omega,q) \qquad \text{Eqn 2.8-5}$$

To solve this equation we need find a Green's function, which obeys the equation below:

$$\left[\Omega - q + \nabla_\perp{}^2 - \frac{(2r)^3}{\Omega^2}u(\vec{x})\right]G(\vec{x},\vec{x}';\Omega,q) = d(\vec{x} - \vec{x}') \qquad \text{Eqn 2.8-6}$$

Then the $\overline{E}$ would be

$$\overline{E}(\vec{x};\Omega,q) = \int d^2x' G(\vec{x},\vec{x}';\Omega,q)\overline{S}(\vec{x}';\Omega,q) \qquad \text{Eqn 2.8-7}$$

Similar to Schrodinger equation with potential: $U(\vec{x}) = -\frac{(2r)^3}{\Omega^2}u(\vec{x})$ and energy: $\Lambda = \Omega - q$

Difference from the Schrodinger equation:

1) Λ and U are complex

2) Ω occurs in L and U

Thus we can expand the Green's function by the eigenfunctions. Now let's assume Λ_n are the eigenvalues of Λ, thus we can write the eigenfunction equation as

$$[\Lambda_n + \nabla_\perp{}^2 + U(\vec{x};\Omega,q)]f_n(\vec{x}) = 0 \qquad \text{Eqn 2.8-8}$$

We use a trial Green's function as

$$G(\vec{x},\vec{x}';\Omega,q) = \sum_n \frac{f_n(\vec{x};\Omega,q)\,f_n(\vec{x}';\Omega,q)}{\Omega - q - \Lambda_n(\Omega,q)} \qquad \text{Eqn 2.8-9}$$

When we plug this in Eqn 2.8-6 we see that it is satisfies the equation. Therefore we calculate the

$$\tilde{E}(\vec{x};t,q) = -i\sum_n \frac{e^{-i\Omega_n(q)t}}{1-(\frac{\partial\Lambda_n}{\partial\Omega})_{\Omega=\Omega_n(q)}} y_{\ n}(\vec{x},q)\int d^2x' y_{\ n}(\vec{x}',q)S(\vec{x}';\Omega_n(q),q) \qquad \text{Eqn 2.8-10}$$

Where $y_{\ n}(\vec{x},q) \equiv f_n(\vec{x};\Omega_n(q),q)$ and $\Omega_n(q)$ is the pole in Ω plane.

The physical meaning of this result is as follows: for a frequency $q = \frac{\Delta w}{w}$, the field $\tilde{E}(\vec{x};t,q)$ is a sum over modes $y_{\ n}(\vec{x},q)$, each mode exponentially grows with a rate of $I_m(\Omega_n(q))$, with the mode profile unchanged.

$y_n(\bar{x})$ **: Moore's guided modes**

As we in Figure 2.8-1 the modes starting from noise grow exponentially

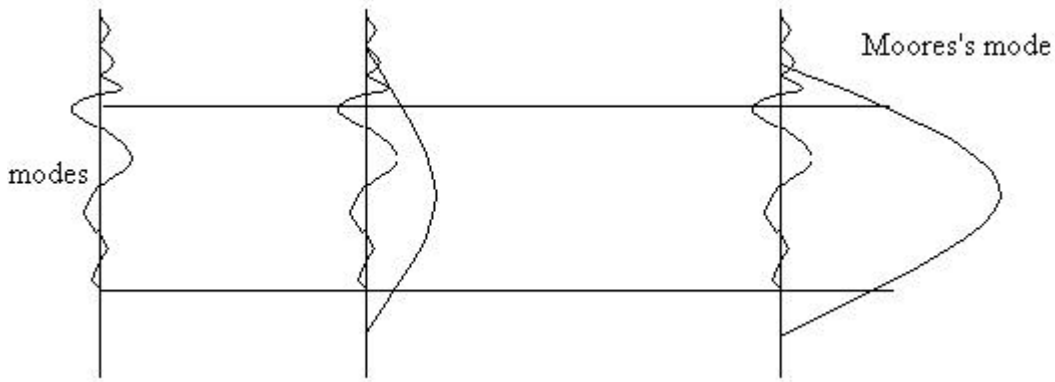

Figure 2.8-1 A schematic diagram of optical guiding. The vertical axis shows spontaneous noise on the left. As the radiation progresses along the wiggler, one mode (Moore's mode) is growing, with gain compensating for diffraction losses.

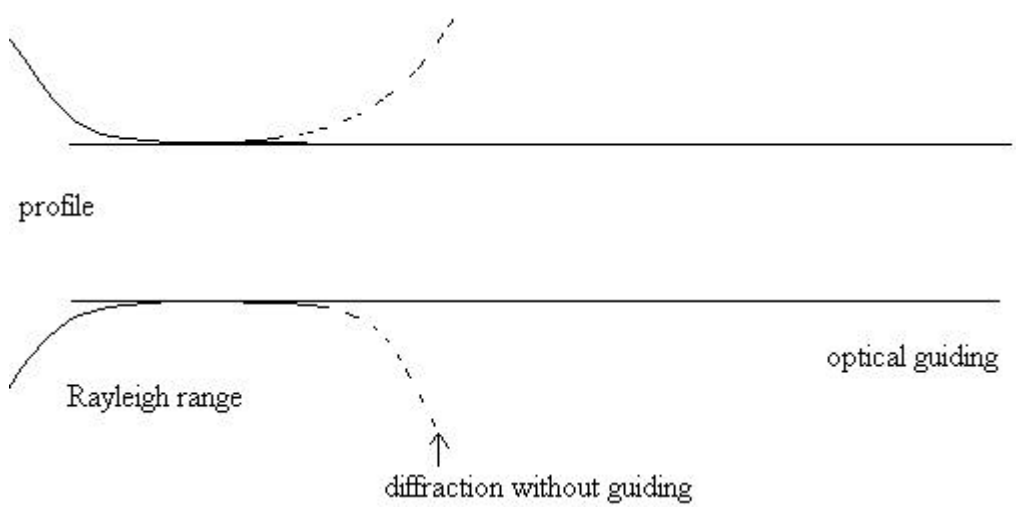

Figure 2.8-2 Increase in the effective Rayleigh range due to optical guiding

As shown in Figure 2.8-2 because of the optical guiding, the effective Rayleigh range becomes very long.

$y_n(\bar{x})$ is found by solving:

$$[\Lambda_n + \nabla_\perp^2 - \frac{(2r)^3}{\Omega^2} u(x)]\psi_n(\bar{x};\Omega,q) = 0 \text{ and } \Omega - q - \Lambda_n = 0$$ Eqn 2.8-11

Step function profile:

Now let's assume $u(x)$ is a step function which is defined as

$$u(x) = 1 \quad |x| < a$$

$$u(x) = 0 \quad |x| > a$$ Eqn 2.8-12

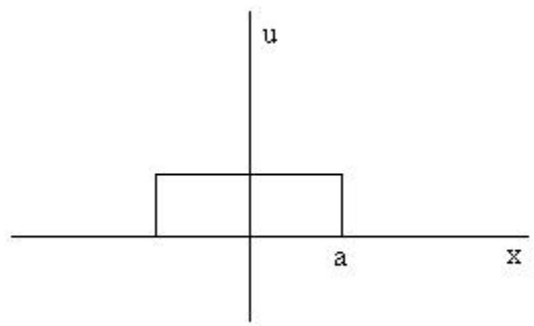

Figure 2.8-3 Step function profile

Therefore we can write the Eqn 2.8-11 as

$$(\Lambda + \nabla_\perp^2 + V(x))\psi(\bar{x}) = 0$$ Eqn 2.8-13

where

$$\Lambda = \Omega - q$$ Eqn 2.8-14

and $V(x) = -\frac{(2r)^3}{\Omega^2} u(x) \equiv Vu(x)$ Eqn 2.8-15

Let's assume that $\Psi(x)$ is in the form of $\Psi(x) = e^{imq} R(x)$, if we plug this in Eqn 2.8-13 we obtain

$$R''' + \frac{1}{x} R' + [\Lambda + Vu(x) - \frac{m^2}{x^2}]R = 0$$ Eqn 2.8-16

which is called radial equation.

Solution to this equation is Bessel functions:

$$R(x) = CJ_m(c\frac{x}{a}) \text{ for } x \le a$$

$$R(x) = DH^{(1)}{}_m(f\frac{x}{a}) \text{ for } x \ge a \qquad \text{Eqn 2.8-17}$$

where

$$c = a\sqrt{\Lambda + V} \qquad \mathrm{Re}(c) > 0$$
$$f = a\sqrt{\Lambda} \qquad \mathrm{Im}(f) > 0 \qquad \text{Eqn 2.8-18}$$

The condition $\mathrm{Im}(f) > 0$ is the boundary condition so as x goes to infinity y should go to zero.

Due is symmetry of J_m, Re(x) can take either positive or negative value. Here we choose Re(x)>0 arbitrarily

The continuity of $\frac{R'(x)}{R(x)}$ at x=a requires

$$c\frac{J_m'(x)}{J_m(x)} = f\frac{H_m^{(i)}{}'(f)}{H_m(f)} \qquad \text{Eqn 2.8-19}$$

From the definitions of f and c (Eqn 2.8-18) we obtain the following relations:

$$\Lambda = \frac{f^2}{a^2} = \Omega - q \qquad \text{Eqn 2.8-20}$$

$$\Lambda + V = \frac{c^2}{a^2} \qquad \text{Eqn 2.8-21}$$

Therefore $V = \frac{c^2}{a^2} - \frac{f^2}{a^2}$ Eqn 2.8-22

but $V = -\frac{(2r)^3}{\Omega^2} = -\frac{(2r)^3}{(\Lambda + q)^2} = -\frac{(2r)^3}{(\frac{f^2}{a^2} + q)^2}$ Eqn 2.8-23

From Eqn 2.8-23 and Eqn 2.8-22 we can get

$$\frac{f^2}{a^2} - \frac{c^2}{a^2} = \frac{(2r)^3}{(\frac{f^2}{a^2} + q)^2}$$ Eqn 2.8-24

Here we are introducing the scaled beam size $\tilde{a}$, which is defined as

$$\tilde{a}^2 \equiv 2ra^2$$ Eqn 2.8-25

and also we define scaled tune as

$$\Delta \equiv \frac{q}{2r} = \frac{\Delta w}{2rw_s}$$ Eqn 2.8-26

Using these definitions, Eqn 2.8-24 becomes

$$\frac{f^2}{\tilde{a}^2} - \frac{c^2}{\tilde{a}^2} = \frac{1}{(\frac{f^2}{\tilde{a}^2} + \Delta)^2}$$ Eqn 2.8-27

Eqn 2.8-19and Eqn 2.8-27 can be solved numerically for a given $\tilde{a}$ and Δ values.

1-D Limit:

The growth rate is

$$\Omega = \frac{f^2}{a^2} - q = 2r(\frac{f^2}{\tilde{a}^2} + \Delta)$$

Eqn 2.8-28

Similar to what we have done in 1-D case we define

$$l \equiv \frac{\Omega}{2r} = \frac{f^2}{\tilde{a}^2} + \Delta$$ Eqn 2.8-29

If we plug this in Eqn 2.8-27 we get the equation for λ as:

$$l^3 - (\Delta + \frac{c^2}{\tilde{a}^2})\, l^2 - 1 = 0$$ Eqn 2.8-30

For $\Delta = 0$ and as $\tilde{a}$ goes to infinity we obtain the same values for λ as we got in 1-D case:

$$l = 1, e^{\pm\frac{2p}{3}i} \qquad \text{Eqn 2.8-31}$$

Thus f goes to infinity from the definition and $J_m(c)$ goes to zero to satisfy the boundary conditions at the infinity (Eqn 2.8-19).

Therefore c should be a parameter such that

$$J_m(m_{mn}) = 0 \qquad \text{Eqn 2.8-32}$$

And for $l = e^{\frac{2p}{3}i}$

$$f = \tilde{a}^2 e^{\frac{2p}{3}i} \qquad \text{Eqn 2.8-33}$$

The Physical Meaning of $\tilde{a}$

From the definition of $\tilde{a}^2$ we can write

$$\tilde{a}^2 = 2\,ra^2 = 2\,r\,2k_s k_w r_0^{\ 2} \qquad \text{Eqn 2.8-34}$$

This can be written as

$$\tilde{a}^2 = \frac{4}{\sqrt{3}} \frac{4\sqrt{3}pr}{l_w} \frac{pr_0^{\ 2}}{l_s} \sim \frac{L_R}{L_G} \qquad \text{Eqn 2.8-35}$$

Thus we can conclude that

$$\tilde{a}^2 \sim \frac{Rayleigh_Range_with_e-beam_size_waist}{power_gain_length}$$

If $\tilde{a} >> 1$ then $L_R >> L_G$ which means diffraction loss is compensated by gain, system is dominated by gain and we have 1-D limit: diffraction negligible.

But many modes are degenerate so there are many modes.

If $\tilde{a} << 1$ then $L_R << L_G$ which means diffraction is significant, we have 3-D effects therefore gain is less than 1-D case. However higher modes grow much slower, a single mode dominates and size become constant (optical guiding)

If $2 < \tilde{a} < 6$ then growth rate is similar to 1-D, single mode.