

2.9 The FEL Equations with Betatron Oscillation

In this section we will take betatron oscillation into account in the wiggler. We will also look at the behavior of the electron beam when it is off-axis and find some solutions to get maximum matching in the wiggler.

Natural focusing

From the Maxwell's equations we have the relation between the components of the magnetic field as

$$\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y} \quad \text{Eqn 2.9-1}$$

$$\text{Thus } \Delta B_z = \frac{\partial B_y}{\partial z} \Delta y \quad \text{Eqn 2.9-2}$$

Electron moves in the wiggler as in Figure 2.9-1 and if the electron is not centered vertically, it will experience a Lorentz Magnetic force because of the magnetic field that we obtained in Eqn 2.9-2.

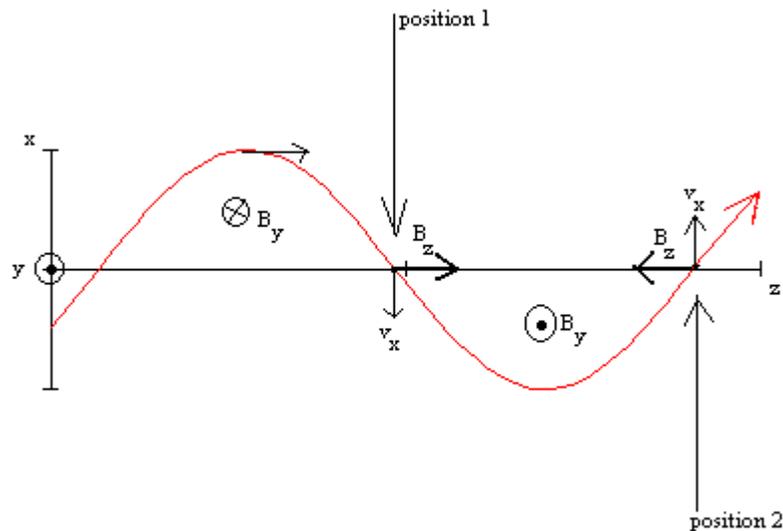


Figure 2.9-1 Off axis electron trajectory in the wiggler, showing fields and forces. The paper plane is the wiggler-plane, the offset is in a direction perpendicular to the paper.

In Figure 2.9-1, if the electron is at position 1 and $\Delta y > 0$, then $\Delta B_z > 0$.

Therefore $F_y < 0$ which means electromagnetic force is towards the axis of the motion in vertical direction.

If the electron is in position 2 and $\Delta y > 0$, then $\Delta B_z < 0$ because $\frac{\partial B_y}{\partial z}$ term in Eqn 2.9-2 is negative. The Lorentz force would still be negative toward the axis.

If we have $\Delta y < 0$ then we would have positive Lorentz force which is also towards axis. Thus the Lorentz force always points to the y-axis and provide vertical focusing in the wiggler.

Parabolic pole face – horizontal focusing

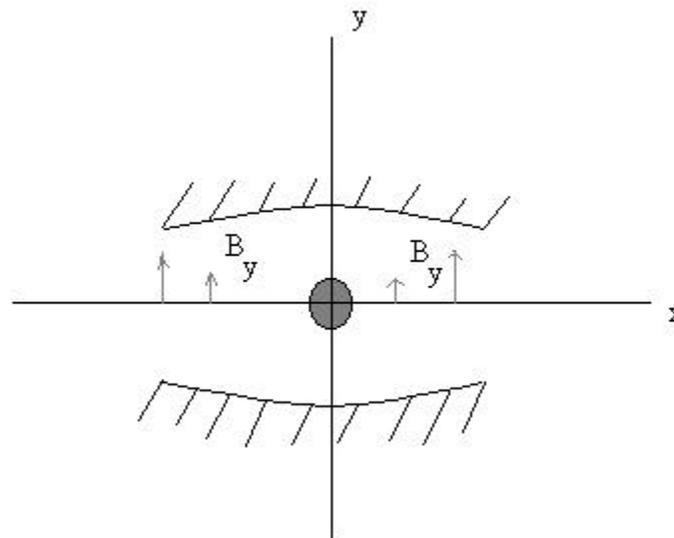


Figure 2.9-2 Parabolic pole face design. The beam direction is perpendicular to the plane of the paper.

Assuming the z-axis is going into the page in Figure 2.9-2 if we had parabolic faced poles in the wiggler we would have a variable strength magnetic field in y direction. The more the electrons are off-axis, the more magnetic field strength they would experience, and therefore they will experience more electromagnetic force toward the center. This causes horizontal focusing.

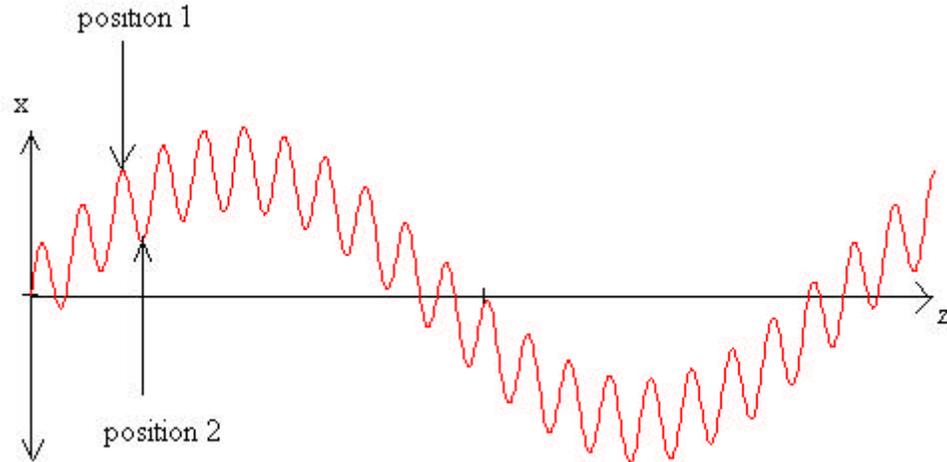


Figure 2.9-3 An electron trajectory with betatron oscillation (long period) and wiggle motion (short period).

Thus the trajectory would be similar to what we see in Figure 2.9-3. When the electrons are in position 1 they will experience more electromagnetic field compared to position 2, therefore the curvature is larger in position 1.

More quantitative description of the betatron-motion has been done by E.T Scharlemann [3].

In section 2.3 we had described the vector potential as

$$\begin{aligned}\bar{A}_w &\cong A_w \hat{x} \cos k_w z \\ \bar{B}_w &= \nabla \times \bar{A}_w \cong -k_w A_w \hat{y} \sin k_w z\end{aligned}\tag{Eqn 2.9-3}$$

However this is only an approximation near the axis, not a solution of Maxwell's equations. A solution of Maxwell's equations from Scharlemann's paper is as follows:

$$\bar{B}_w = -\frac{B_{w0}}{k_y} \left\{ \hat{x} k_x \text{sh}(k_x x) \text{sh}(k_y y) \sin k_w z + \hat{y} k_y \text{ch}(k_x x) \text{ch}(k_y y) \sin k_w z + \hat{z} k_w \text{ch}(k_x x) \text{sh}(k_y y) \cos k_w z \right\}\tag{Eqn 2.9-4}$$

$$k_x^2 + k_y^2 = k_w^2\tag{Eqn 2.9-5}$$

So near axis y-component of the magnetic field can be expanded as

$$B_y = -B_{w0} \left(1 + \frac{1}{2} k_x^2 x^2 + \frac{1}{2} k_y^2 y^2\right) \sin k_w z \quad \text{Eqn 2.9-6}$$

What we are interested in is having equal focusing in horizontal and vertical directions.

Therefore we would like to have

$$k_x + k_y = \frac{1}{\sqrt{2}} k_w \quad \text{Eqn 2.9-7}$$

Results of Scharlemann's analysis is as follows:

We have **b**-oscillation plus the wiggling motion as shown in Figure 2.9-4

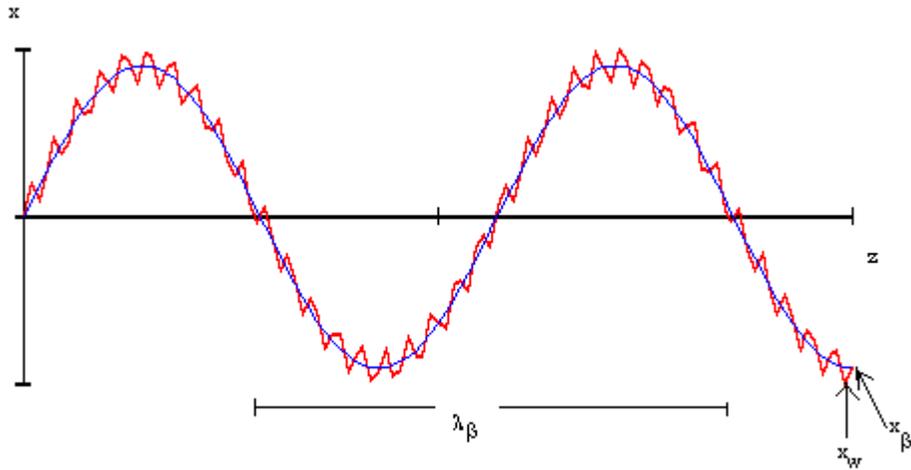


Figure 2.9-4 Decomposition of the electron's trajectory into wiggling motion and betatron motion.

We define displacement from the axis of the wiggler x as the summation of betatron displacement and wiggling displacement.

$$\begin{aligned} x &= x_b + x_w \\ y &= y_b \end{aligned} \quad \text{Eqn 2.9-8}$$

We have the equation for betatron oscillation as from Scharlemann's paper as:

$$\begin{aligned} x_b'' &= -k_{bx}^2 x_b \\ y_b'' &= -k_{by}^2 y_b \end{aligned} \quad \text{Eqn 2.9-9}$$

Where

$$k_{bx} = \frac{K}{\sqrt{2g}} k_x \text{ and } k_{by} = \frac{K}{\sqrt{2g}} k_y \quad \text{Eqn 2.9-10}$$

For equal focusing we have $k_x = k_y = \frac{1}{\sqrt{2}} k_w$ so

$$k_{bx} = k_{by} \equiv k_{bn} = \frac{K}{2g} k_w \quad \text{Eqn 2.9-11}$$

where k_{bn} is called natural focusing wavenumber.

The solution to Eqn 2.9-9 is

$$\begin{aligned} x_b &= x_{b0} \cos(k_{bn} z + \mathbf{f}_x) \\ y_b &= y_{b0} \cos(k_{bn} z + \mathbf{f}_y) \end{aligned} \quad \text{Eqn 2.9-12}$$

Thus

$$\begin{aligned} x_b' &= -k_{bn} x_{b0} \sin(k_{bn} z + \mathbf{f}_x) \\ y_b' &= -k_{bn} y_{b0} \cos(k_{bn} z + \mathbf{f}_y) \end{aligned} \quad \text{Eqn 2.9-13}$$

x_w' which was calculated in Eqn 2.3-4 becomes

$$x_w' \cong -\frac{K}{g} \left(1 + \frac{1}{2} k_x^2 x_b^2 + \frac{1}{2} k_y^2 y_b^2\right) \cos k_w z \quad \text{Eqn 2.9-14}$$

using Eqn 2.9-6.

Longitudinal velocity:

The longitudinal velocity would be the summation of the betatron and the wiggling velocities.

$$b_{\parallel} = \sqrt{b^2 - b_{\perp}^2} = \sqrt{1 - \frac{1}{g^2} - b_{\perp}^2} \quad \text{Eqn 2.9-15}$$

$$b_{\perp}^2 = x'^2 + y'^2 \quad \text{Eqn 2.9-16}$$

If we average over the wiggler period we get

$$b_{\perp}^2 = x_b'^2 + y_b'^2 + \overline{x_w'^2} = \frac{1}{2} \frac{K^2}{g^2} + k_{bn}^2 (x_{b0}^2 + y_{b0}^2) \quad \text{Eqn 2.9-17}$$

Therefore

$$b_{\parallel} \cong 1 - \frac{1}{2g^2} - \frac{2}{2g^2} - \frac{1}{2} k_{bn}^2 (x_{b0}^2 + y_{b0}^2) = 1 - \frac{1 + \frac{K^2}{2}}{2g^2} - \frac{1}{2} k_{bn}^2 (x_{b0}^2 + y_{b0}^2) \quad \text{Eqn 2.9-18}$$

The underlined term in the longitudinal velocity is reduction and spread. b_{\parallel} is constant during the betatron oscillation.

Emittance

One of the most important parameters of an electron beam is its emittance. Before we give the definition of emittance it would be beneficial to introduce the concept and the notation.

In a bunch each electron has position and momentum (\bar{r}, \bar{p}) coordinates, where $\bar{r} \equiv (x, y, z)$ and $\bar{p} \equiv (p_x, p_y, p_z)$. We use the convenient notation (x, x') and (y, y') for the transverse vectors, where $p_x = mc\mathbf{b}\mathbf{g}'$ and $p_y = mc\mathbf{b}\mathbf{g}'$, \mathbf{b} and \mathbf{g} are the Lorentz factors. Each electron is described by six dimensional phase space coordinates but for convenience we use two-dimensional pairs as mentioned above. The collection of electrons forms an ellipse in each phase space as shown in Figure 2.9-5.

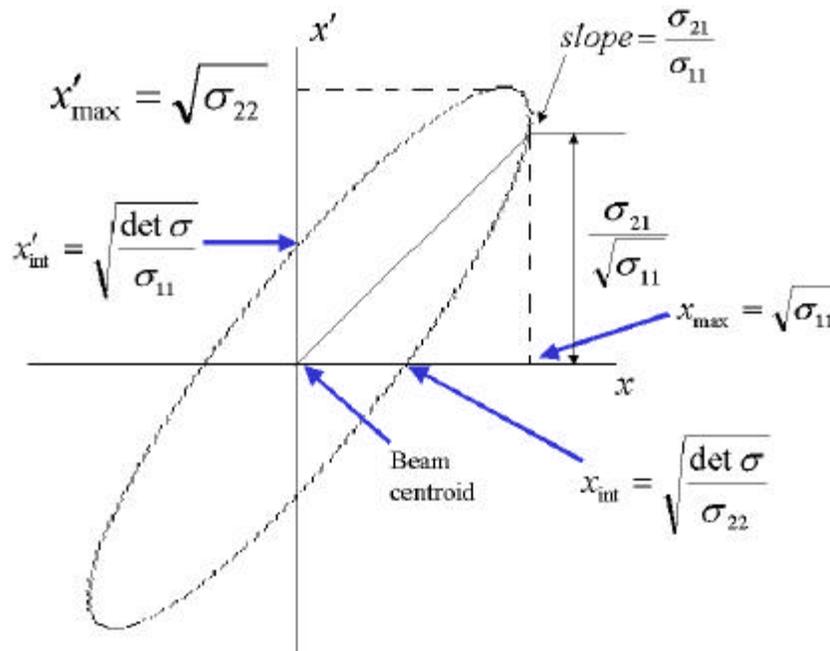


Figure 2.9-5 $x - x'$ phase space ellipse of the beam, relating the shape to various functions of the beam matrix elements.

$\sqrt{\mathbf{s}_{11}}$ is the radius of the beam size, $\sqrt{\mathbf{s}'_{22}}$ is the divergence of the beam and $\mathbf{s}_{12} = \mathbf{s}_{21}$ is the correlation between the two axes defined as in Figure 2.9-5.

Using the definition of an ellipse we obtain

$$\mathbf{s}_{22}x^2 - 2\mathbf{s}_{12}xx' + \mathbf{s}_{11}x'^2 = \det(\mathbf{s}) \quad \text{Eqn 2.9-19}$$

where $\mathbf{s} \equiv \begin{pmatrix} \mathbf{s}_{11} & \mathbf{s}_{12} \\ \mathbf{s}_{21} & \mathbf{s}_{22} \end{pmatrix}$ is called the beam matrix and, and the ellipse's area is given

$$\text{by } A = p\sqrt{\det(\mathbf{s})} \quad \text{Eqn 2.9-20}$$

When we represent a particle in the beam with a column vector $X \equiv \begin{pmatrix} x \\ x' \end{pmatrix}$, we can write

Eqn 2.9-19 as

$$X' \mathbf{s}^{-1} X = 1 \quad \text{Eqn 2.9-21}$$

The emittance is defined as the volume occupied by the particles in the electron beam in six-dimensional phase-space. The emittance is thus a conserved quantity by Liouville's theory. Under certain conditions the two-dimensional phase spaces (x, P_x) , (y, P_y) are conserved. At a constant energy (x, x') and (y, y') would be conserved. Thus the emittance is related to the area of the ellipse. Conventionally this emittance (the geometrical emittance) is defined as

$$\mathbf{e} \equiv \sqrt{\det(\mathbf{s})} \quad \text{Eqn 2.9-22}$$

Beam Matching

Assuming

$$x = x_b \cos(k_b z + \mathbf{f}_x) \quad \text{Eqn 2.9-23}$$

$$\text{then } x' = -k_b x_b \sin(k_b z + \mathbf{f}_x)$$

Therefore

$$x'^2 + k_b^2 x^2 = x_b^2 = \text{constant} \quad \text{Eqn 2.9-24}$$

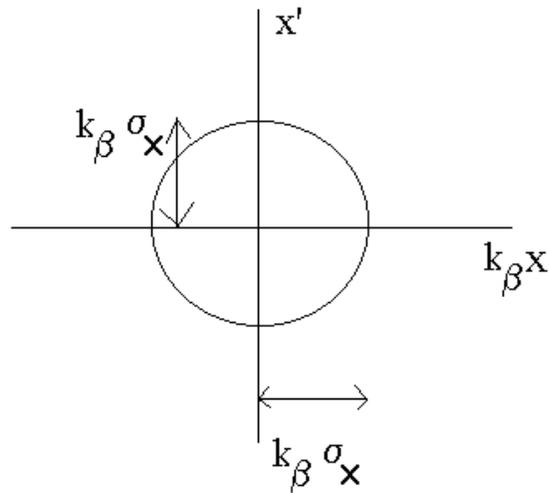


Figure 2.9-6 Phase space plot for a matched beam: It is a circle in the $k_b x - x'$ phase space.

If this matching is not perfect then we would have a tilt ellipse instead a circle above.

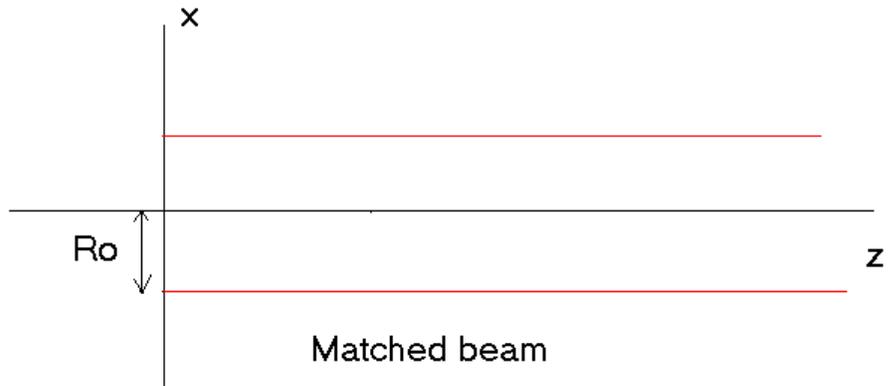


Figure 2.9-7 Beam envelope of a matched beam.

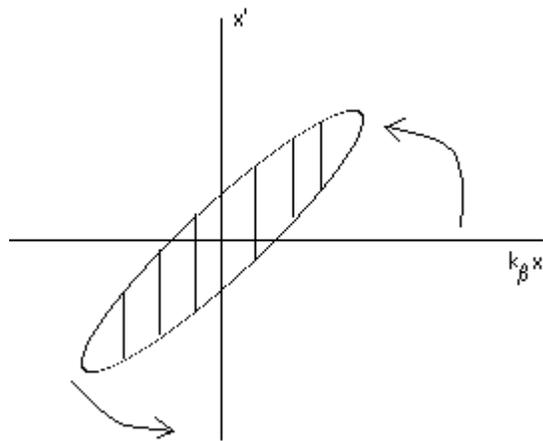


Figure 2.9-8 Phase space plot for an unmatched beam. The arrows show the direction of rotation of the beam ellipse.

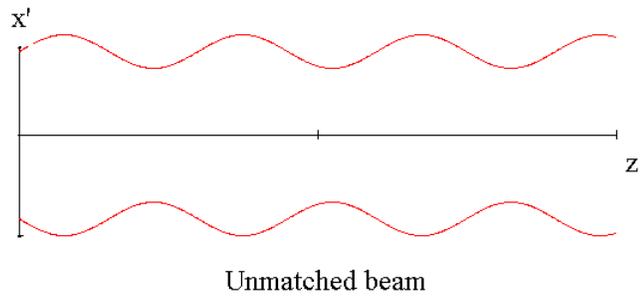


Figure 2.9-9 Beam envelope of an unmatched beam.

if $\mathbf{s}_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ is the rms of x then the for a matched beam the emittance would be

$$\mathbf{e} = k_b \mathbf{s}_x \cdot \mathbf{s}_x = k_b \mathbf{s}_x^2 \quad \text{Eqn 2.9-25}$$

The rms transverse momentum is

$$\mathbf{s}_p = mc \mathbf{g} \mathbf{s}_x = mc \mathbf{g}_b \mathbf{s}_x \quad \text{Eqn 2.9-26}$$

The phase space conservation requires that

$$\mathbf{s}_p \mathbf{s}_x = mc \mathbf{g}_b \mathbf{s}_x^2 = \text{Invariant} \quad \text{Eqn 2.9-27}$$

So the normalized emittance, which is a conserved quantity, is

$$\mathbf{e}_n = \mathbf{g}_b \mathbf{s}_x^2 \quad \text{Eqn 2.9-28}$$

In general we need extra focusing in the wiggler to get better beam matching. In this case the equations modifies as follows.

Phase Equations:

$$\mathbf{y} = k_w z + k_s z - w_s t \quad \text{Eqn 2.9-29}$$

$$\frac{d\mathbf{y}}{dz} = k_w + k_s - k_s \mathbf{b}_{\parallel}^{-1} \quad \text{Eqn 2.9-30}$$

Where

$$\mathbf{b}_{\parallel}^{-1} = \left(1 - \frac{1}{\mathbf{g}^2} - \mathbf{b}_{\perp}^2\right)^{-\frac{1}{2}} \cong 1 + \frac{1}{2\mathbf{g}^2} + \frac{1}{2}\mathbf{b}_{\perp}^2 \quad \text{Eqn 2.9-31}$$

Thus we obtain the phase equation as

$$\frac{dy}{dz} = k_w - k_s \frac{1 + \frac{K^2}{2}}{2g^2} - \frac{k_s}{2} (x_b'^2 + y_b'^2) - \frac{k_s}{2} k_{bn}^2 (x^2 + y^2) \quad \text{Eqn 2.9-32}$$

$$\frac{dy}{dz} \cong 2k_w \frac{g - g_0}{g_0} - \frac{k_s}{2} \left[\left(\frac{d\bar{r}}{dz} \right)^2 + k_{bn}^2 \bar{r}^2 \right]$$

When $k_b = k_{bn}$ (natural focusing) there is no longitudinal velocity modulation in betatron oscillation because \mathbf{b}_{\parallel} is constant as we mentioned above. However when $k_b > k_{bn}$ then

$$\left(\frac{d\bar{r}}{dz} \right)^2 + k_{bn}^2 \bar{r}^2 \neq \text{constant} \quad \text{Eqn 2.9-33}$$

which causes longitudinal velocity modulation

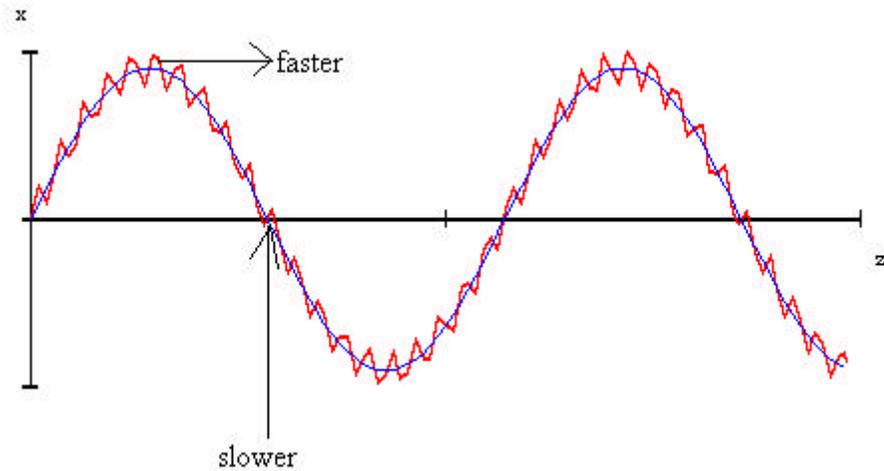


Figure 2.9-10 The trajectory of an electron in the wiggler plane under wiggling and betatron motion, leading to a resultant modulation of the longitudinal velocity.

The parallel velocity oscillates as shown in Figure 2.9-10.

We used the scaled \bar{r}_\perp for 3-D theory

$$\bar{x} \equiv \sqrt{2k_s k_w} \bar{r}_\perp \text{ and } \bar{p} = \frac{d\bar{x}}{dt}$$

and we define $k_n = \frac{k_{bn}}{k_w}$

if we get the phase equation

$$\frac{d\mathbf{q}}{dt} = 2 \frac{\mathbf{g} - \mathbf{g}_0}{\mathbf{g}} - \frac{1}{4} (\bar{p}^2 + k_n^2 \bar{x}^2) \quad \text{Eqn 2.9-34}$$

Physically $\frac{1}{8} (\bar{p}^2 + k_n^2 \bar{x}^2)$ is equivalent to energy spread in gain reduction.

(\mathbf{q} is defined as slow varying part of \mathbf{y})

Energy equation:

$$\frac{d\mathbf{g}}{dz} \sim \frac{dx}{dz} E_x$$

$$\frac{dx}{dz} = x' = x'_b + x'_w \text{ and } E_x \sim e^{i(k_s z - w_s t)}$$

x'_b is off-resonance (slow)

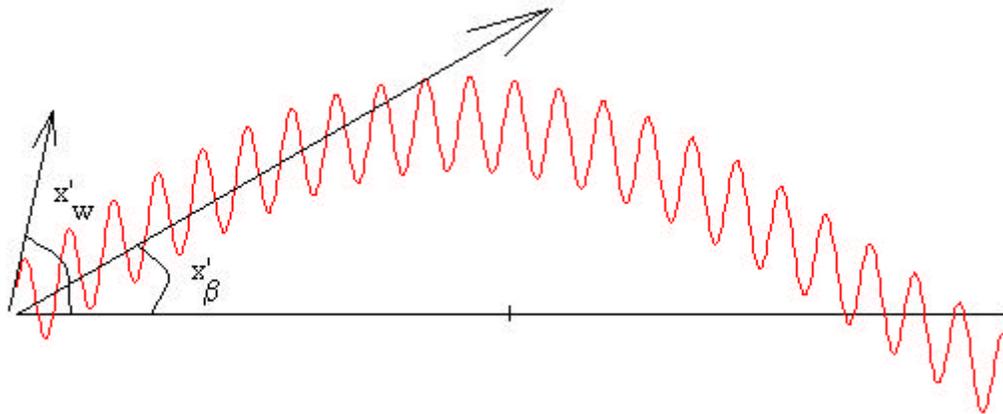


Figure 2.9-11 Comparison of the divergence angles of the wiggler and betatron motions.

$$(x'_b)_{\max} = k_b x_{b0} \cong k_b \mathbf{s}_x \quad \text{Eqn 2.9-35}$$

where \mathbf{s}_x is the beam size.

$$(x'_w)_{\max} = \frac{K}{\mathbf{g}} \quad \text{Eqn 2.9-36}$$

$$\text{For natural focusing } k_{bn} = \frac{K}{2\mathbf{g}} k_w \quad \text{Eqn 2.9-37}$$

$$\frac{(x'_b)_{\max}}{(x'_w)_{\max}} = \frac{1}{2} k_w \mathbf{s}_x = \mathbf{p} \frac{\mathbf{s}_x}{I_w} \quad \text{Eqn 2.9-38}$$

Beam size \mathbf{s}_x is of order of 0.3 mm or less however I_w is of order of 10mm or more.

Therefore $|x'_b| \ll |x'_w|$ usually holds and we get the same 1-D equation for energy exchange

$$\frac{d\xi}{dz} \sim \frac{1}{2} e^{iq} [J_0(b) - J_1(b)] + c.c. \quad \text{and} \quad \frac{d\mathbf{g}}{dt} = -\frac{D_2}{\mathbf{g}} (Ee^{iq} + c.c.)$$

Maxwell Equation

In section 2.5 we derived the results of the Maxwell equations. In this case the derivation exactly same and we use $|x'_b| \ll |x'_w|$ condition to calculate $\cos k_w z e^{i(k_s z - w_s t)}$. Following exact same derivation in Fluid Model section we obtain Eqn 2.5-9

$$\left(\frac{\mathcal{I}}{\mathcal{I}t} + \frac{\mathcal{I}}{\mathcal{I}q} + \frac{1}{2ik_s k_w} \nabla_{\perp}^2 \right) E = \frac{D_1}{\mathbf{g}} \int F d\mathbf{g}$$

Transverse Equation of Motion

$$\frac{d^2 x}{dz^2} + k_b^2 x = 0, \quad \frac{d^2 y}{dz^2} + k_b^2 y = 0$$

$$\frac{d^2 x}{dt^2} + k^2 x = 0, \quad \frac{d^2 y}{dt^2} + k^2 y = 0 \quad \text{where } k \equiv \frac{k_b}{k_w} \quad \text{Eqn 2.9-39}$$

2.10 Universal Scaling

The coupled Maxwell-Vlasov equations in 3-D can be reduced to the following:

$$\begin{aligned}
 (-i\Omega + iq - i\nabla_{\perp}^2)\bar{E} &= \frac{D_1}{\mathbf{g}} \int d\mathbf{g} \int d^2 p \bar{F} + \tilde{E}(\mathbf{t} = 0) \\
 (-i\Omega + iq)\bar{F} + \bar{p} \frac{\partial \bar{F}}{\partial \bar{x}} - k^2 \bar{x} \frac{\partial \bar{F}}{\partial \bar{p}} &= \frac{D_2}{\mathbf{g}} \frac{\partial f_0}{\partial \mathbf{g}} \bar{E} + \tilde{F}(\mathbf{t} = 0)
 \end{aligned}
 \tag{Eqn 2.10-1}$$

After some calculation we obtain the dispersion relation for the electric field.

$$\begin{aligned}
 (\nabla_{\perp}^2 + \mathbf{m})E(\bar{x}) &= (2r)^3 \int d\mathbf{g} \int d^2 p u(\bar{p}^2 + k^2 \bar{x}^2) \int_{-\infty}^0 s ds e^{-ias} e^{i\frac{k}{16}(1-\frac{k_n^2}{k^2})(\bar{x}^2 - \frac{\bar{p}^2}{k^2})\sin(2ks) + \frac{2\bar{x}\bar{p}}{k}(1-\cos(2ks))} \\
 &\times E(\bar{x} \cos ks + \frac{\bar{p}}{k} \sin ks)
 \end{aligned}
 \tag{Eqn 2.10-2}$$

where $k = \frac{k_b}{k_w}$, $k_n = \frac{k_{bn}}{k_w}$, $\bar{x} = \sqrt{2k_s k_w} \bar{r}$, $\mathbf{m} = \Omega - q = \Omega - \frac{w - w_s}{w_s}$

$$\mathbf{a} = \mathbf{m} + \frac{w - w_s}{w_s} - 2 \frac{\mathbf{g} - \mathbf{g}}{\mathbf{g}} + \frac{1}{8} \left(1 + \frac{k_n^2}{k^2}\right) (\bar{p}^2 + k^2 \bar{x}^2),$$

$$h(\mathbf{g}) = \frac{1}{\sqrt{2ps_g}} e^{-\frac{(\mathbf{g} - \mathbf{g})^2}{2s_g^2}}$$

$$u(\bar{p}^2 + k^2 \bar{x}^2) = \frac{1}{\mathbf{k}^2 a^2} \Theta(k^2 a^2 - k^2 \bar{x}^2 - \bar{p}^2) \text{ is uniform distribution (step fn),}$$

$a = \sqrt{2k_s k_w} R_0$ is the scaled beam size and R_0 is beam size.

The rms value of the beam is $\mathbf{s}_x = \frac{1}{\sqrt{6}} R_0$. For a matched beam $\mathbf{s}'_x = k_b \mathbf{s}_x$, thus the

$$\text{emittance is } \mathbf{e}_x = k_b \mathbf{s}_x^2 = \frac{1}{6} k_b R_0^2$$

Eqn 2.10-2 can be solved using a trial function

$$E(\bar{x}) = \begin{cases} e^{-\frac{c}{a^2}r^2} & r \leq a \\ CH_0^{(1)}(r\sqrt{\mathbf{m}}) & r \geq a \quad \text{Im}(\mu) > 0 \end{cases} \quad \text{Eqn 2.10-3}$$

The continuity of logarithmic derivative at $r = a$ leads to

$$a\sqrt{\mathbf{m}} \frac{H_0^{(1)'}(a\sqrt{\mathbf{m}})}{H_0^{(1)}(a\sqrt{\mathbf{m}})} = -\mathbf{c} \quad \text{Eqn 2.10-4}$$

Substituting the trial function into Eqn 2.10-2 gives us

$$\mathbf{m}^2(1 - e^{-\mathbf{c}}) - [1 - (1 - \mathbf{c})e^{-\mathbf{c}}] = (2\mathbf{r})^3 a^2 \int_{-\infty}^0 ds \frac{s}{\cos ks} e^{-is[\mathbf{m} + \frac{w-w_s}{w_s}] - 2\frac{\mathbf{s}g^2}{g_0^2}s^2} \left[\frac{1 - e^{-h_+}}{h_+} - \frac{1 - e^{-h_-}}{h_-} \right] \quad \text{Eqn 2.10-5}$$

$$\text{where } h_{\pm} = \frac{\mathbf{c}}{2}(1 \mp \cos ks) + \frac{i}{4}k^2 a^2 s \quad \text{Eqn 2.10-6}$$

Thus we have two equations (Eqn 2.10-4 and Eqn 2.10-5) and two unknown (\mathbf{m} and \mathbf{c}) which we can use simulations to solve.

The gain length can then be found using

$$\text{Im}(\mathbf{m}) = \frac{1}{2k_w L_G} \quad \text{Eqn 2.10-7}$$

In order to increase the speed of the simulations a scaling method is used by changing the variables as follows:

We define $\tilde{a} = \sqrt{2\mathbf{r}}a = \sqrt{2\mathbf{r}}\sqrt{2k_s k_w} R_0$ and change the integration variable $2\mathbf{r}s \Rightarrow s$

Then the Eqn 2.10-4 becomes

$$\tilde{a} \sqrt{\frac{\mathbf{m}}{2\mathbf{r}}} \frac{H_0^{(1)'}(\tilde{a} \sqrt{\frac{\mathbf{m}}{2\mathbf{r}}})}{H_0^{(1)}(\tilde{a} \sqrt{\frac{\mathbf{m}}{2\mathbf{r}}})} = -\mathbf{c} \quad \text{Eqn 2.10-8}$$

and Eqn 2.10-5 becomes

$$\frac{\mathbf{m}}{2\mathbf{r}} \tilde{a}^2 (1 - e^{-c}) - [1 - (1 - \mathbf{c})e^{-c}] = \tilde{a}^2 \int_{-\infty}^0 ds \frac{s}{\cos \frac{k}{2\mathbf{r}} s} e^{-is[\frac{\mathbf{m}}{2\mathbf{r}} + \frac{w-w_s}{2\mathbf{r}w_s}] - 2(\frac{\mathbf{S}_g}{2\mathbf{r}\mathbf{g}})^2 s^2} \left[\frac{1 - e^{-h_+}}{\mathbf{h}_+} - \frac{1 - e^{-h_-}}{\mathbf{h}_-} \right]$$

where $\mathbf{h}_{\pm} = \frac{\mathbf{c}}{2} (1 \mp \cos \frac{k}{2\mathbf{r}} s) + \frac{i}{4} (\frac{k}{2\mathbf{r}})^2 \tilde{a}^2 s$ Eqn 2.10-9

Thus $\frac{\text{Im}(\mathbf{m})}{\mathbf{r}} = \frac{1}{2k_w L_G \mathbf{r}} = F(\tilde{a}, \frac{\mathbf{S}_g}{\mathbf{g}\mathbf{r}}, \frac{k_b}{k_w \mathbf{r}}, \frac{w-w_s}{w_s \mathbf{r}})$ Eqn 2.10-10

Thus $\frac{\text{Im}(\mathbf{m})}{\mathbf{r}}$ is a function of 4 scaled parameters.

The physical meaning of $\tilde{a}^2 = 2\mathbf{r}a^2 = 2\mathbf{r}2k_s k_w R_0^2 = 2\sqrt{3} \frac{L_R}{L_G}$ Eqn 2.10-11

where $L_R = \frac{4\mathbf{p}\mathbf{s}_x^2}{\mathbf{l}_s}$ (Rayleigh range with waist of electron beam size)

More practical form of scaling function was introduced by Yu, Krinsky and Gluckstern [4] as follows:

We change the variables as

$D = 2\mathbf{r}\tilde{a}$ and $s\tilde{a} \Rightarrow s$ therefore the Eqn 2.10-9 becomes scaled with D instead of $2\mathbf{r}$ as:

$$\frac{\mathbf{m}}{D} \frac{12k_s \mathbf{e}}{k} (1 - e^{-c}) - [1 - (1 - \mathbf{c})e^{-c}] = \tilde{a}^2 \int_{-\infty}^0 ds \frac{s}{\cos \frac{k}{D} s} e^{-is[\frac{\mathbf{m}}{D} + \frac{w-w_s}{Dw_s}] - 2(\frac{\mathbf{S}_g}{D\mathbf{g}})^2 s^2} \left[\frac{1 - e^{-h_+}}{\mathbf{h}_+} - \frac{1 - e^{-h_-}}{\mathbf{h}_-} \right]$$

where $\mathbf{h}_{\pm} = \frac{\mathbf{c}}{2} (1 \mp \cos \frac{k}{D} s) + i3(\frac{k}{D})k_s \mathbf{e}.s$ Eqn 2.10-12

The continuity equation (Eqn 2.10-8) becomes

$$\frac{H_0^{(1)}((12 \frac{k_s \mathbf{e} \mathbf{m}}{k D})^{\frac{1}{2}})}{(12 \frac{k_s \mathbf{e} \mathbf{m}}{k D})^{\frac{1}{2}}} \frac{D}{H_0^{(1)}((12 \frac{k_s \mathbf{e} \mathbf{m}}{k D})^{\frac{1}{2}})} = -\mathbf{c}$$
 Eqn 2.10-13

Therefore the gain function is

$$\frac{\text{Im}(\mathbf{m})}{D} = \frac{1}{2k_w L_G D} = G(k_s \mathbf{e}, \frac{\mathbf{s}_g}{g_0 D}, \frac{k_b}{k_w D}, \frac{w - w_s}{w_s D}) \quad \text{Eqn 2.10-14}$$

$$D \text{ can be calculated from } D^2 = (2\mathbf{r})^3 a^2 = Z_0 \frac{e}{\rho m c^2} \frac{k_s}{k_w} \frac{I_0}{g_0^3} K^2 [JJ]^2 \quad \text{Eqn 2.10-15}$$

$$\text{Thus } D = \left(\frac{2Z_0 e}{\rho m c^2} \frac{I_0}{g_0} \frac{K^2}{1 + \frac{K^2}{2}} \right)^{\frac{1}{2}} [JJ]$$