2 FEL THEORY

2.1 Introduction to the Theory of the FEL

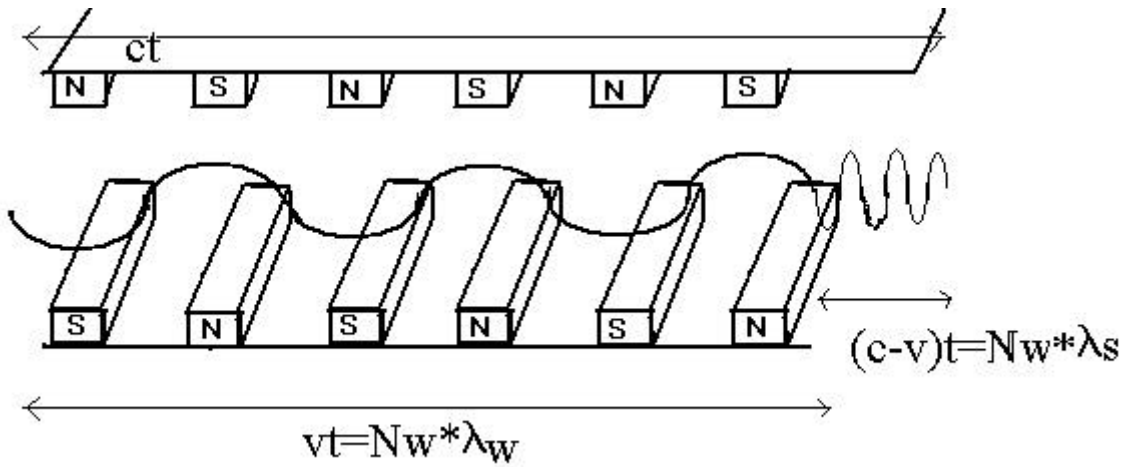

Figure 2.1-1 Simplified diagram showing the electron beam motion in the wiggler and the radiation process

Electron beam undergoes axial bunching in the wiggler due to energy spread combined with the radiation field. Slightly higher and lower energetic electrons covers slightly shorter and longer distance in the wiggler so head and tail catch up. Because of this bunching the coherent radiation is produced.

By the time the first photon covers *ct* distance electron can cover *vt* distance. As shown in Figure 2.1-1 *vt* is the length of the wiggler.

$$vt = N_w \, l_w \qquad \text{Eqn 2.1-1}$$

where N_w is the number of wiggle in the wiggler and l_w is wiggler period. The path difference between electrons and first photon emitted must be equal to $N_w \, l_s$ where l_s is the radiation wavelength.

Thus

$$(c - v)t = N_w \, l_s \qquad \text{Eqn 2.1-2}$$

If we substitute N_w with $vt/\ l_w$ we get

$$l_s = l_w \frac{(1-b)}{b} \qquad \text{Eqn 2.1-3}$$

which is approximately

$$l_s = \frac{l_w}{2\,g^2} \qquad \text{Eqn 2.1-4}$$

Since the actual velocity in z direction is less than the electron velocity this expression changes as follows

$$l_s = \frac{l_w}{2\,g^2}[1+\frac{1}{2}(\frac{eB_w\ l_w}{2pmc})^2] \qquad \text{Eqn 2.1-5}$$

We will come to this a bit later.

There are basically two kinds of wigglers

1) Helical
2) Planar

2.2 The FEL Equations with a Helical Wiggler in a Small Signal Approximation

In this section we will introduce the FEL equations of motion with a helical wiggler, which is a simpler case. We will solve the equations using a small signal approximation.

A helical wiggler is characterized by a vector potential of a helical form:

$$\vec{A}_w = A_w(\hat{e}_- e^{ik_w z} + c.c)\frac{1}{\sqrt{2}} \qquad \text{Eqn 2.2-1}$$

where

$$\hat{e}_\pm = \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}, \hat{e}_+\hat{e}_- = 1, \hat{e}_\pm\hat{e}_\pm = 0 \qquad \text{Eqn 2.2-2}$$

As will be shown below, the electrons will follow helical path as depicted in Figure 2.2-1.

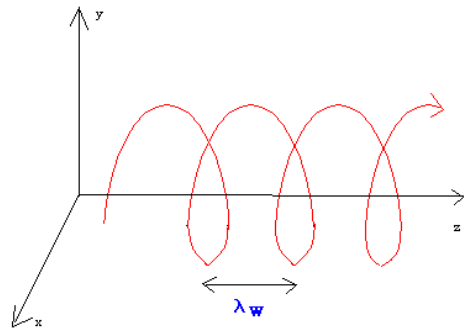

Figure 2.2-1 The electron trajectory in a helical wiggler

Eqn 2.2-1 can be written as:

$$\vec{A}_w = A_w[\hat{x}\cos(k_w z) + \hat{y}\sin(k_w z)] \qquad \text{Eqn 2.2-3}$$

The magnetic field is derived from the vector potential:

$$\vec{B}_w = \nabla \times \vec{A}_w = -k_w A_w[\hat{x}\cos(k_w z) + \hat{y}\sin(k_w z)] \qquad \text{Eqn 2.2-4}$$

The Mechanical momentum Π can be written in terms of the canonical momentum P.

$$\vec{\Pi} = \vec{P} - e\vec{A} \qquad \text{Eqn 2.2-5}$$

The canonical momentum is conserved before and after the field. So let's consider only the perpendicular component of the equation. Since initially there is no momentum in the perpendicular direction $\vec{P}_\perp$ will be zero.

$$\vec{\Pi}_\perp = \vec{P}_\perp - e\vec{A}_\perp \qquad \text{Eqn 2.2-6}$$

Therefore

$$\vec{\Pi}_\perp = -e\vec{A}_\perp \qquad \text{Eqn 2.2-7}$$

We can write the momentum in terms of the velocity.

$$\gamma m\vec{v}_\perp = -e\vec{A}_\perp \qquad \text{Eqn 2.2-8}$$

$$\vec{v}_\perp = -e\frac{\vec{A}_w}{m\,\gamma} = -\frac{eA_w}{m\,\gamma}[\hat{x}\cos(k_w z) + \hat{y}\sin(k_w z)] \qquad \text{Eqn 2.2-9}$$

$$\vec{b}_\perp = -\frac{K}{\gamma}[\hat{x}\cos(k_w z) + \hat{y}\sin(k_w z)] \qquad \text{Eqn 2.2-10}$$

where K, which is called the wiggler magnetic strength parameter, is defined by

$$K = \frac{eA_w}{mc} = \frac{eB_w}{mck_w} = \frac{e\, l_w B_w c}{2pmc^2} \qquad \text{Eqn 2.2-11}$$

In practical units $K \approx \frac{3\times10^8}{2p.0.511\times10^6}\, l_w B_w$

$K \approx 0.94\, l_w(cm) B_w(Tesla)$ Which is usually of the order of 1

Energy Exchange

We want to use the wiggler and electrons to impart energy to an electromagnetic field. So we have to introduce a radiation field and study the energy exchange between the electrons and this field. As we understand from Eqn 2.2-10, the electrons move in a helical orbit. This is not surprising because the vector potential is helical. We expect that the electromagnetic radiation emitted by the electrons will also have a helical form. Thus let us define:

$$\vec{E} = \frac{1}{\sqrt{2}} E(z,t) e^{i(k_s z - w_s t)} \hat{e}_+ + c.c. \qquad \text{Eqn 2.2-12}$$

where k_s is the wave number and w_s is the frequency of the wave according to the plane wave solution. The relation between the k_s and w_s is as follows:

$$k_s = \frac{w_s}{c} \qquad \text{Eqn 2.2-13}$$

In anticipation of the evolution of the electric field intensity along the wiggler, we introduced a time-dependence in the amplitude. Normally there is no energy exchange between the EM field and electron beam propagating in the same direction because their scalar product would be zero (they are perpendicular to each other). Since there is a transverse component of the electron's velocity in the wiggler there will be interaction between the EM field and the electron beam.

Energy exchange is

$$dW = mc^2 d\,g = e\vec{E}.d\vec{r}_\perp \qquad \text{Eqn 2.2-14}$$

If we take the time derivative of the equation Eqn 2.2-14 we get

$$mc^2 \frac{d\,g}{dt} = e\vec{v}_\perp \vec{E} \qquad \text{Eqn 2.2-15}$$

We can change the independent variable from time to position:

$$mc^2 \frac{d\,g}{dz} = e\vec{b}_\perp \vec{E} \qquad \text{Eqn 2.2-16}$$

Let us substitute the form of E from Eqn 2.2-12

$$mc^2 \frac{d\,g}{dz} = -\frac{K}{2\,g}\left(E(z,t)e^{i(k_s z - w_s t + k_w z)} + c.c\right) \qquad \text{Eqn 2.2-17}$$

Let us introduce a phase variable q,

$$\frac{d\,g}{dz} = -\frac{eK}{2mc^2\,g}(E(z,t)e^{iq} + c.c.) \qquad \text{Eqn 2.2-18}$$

where

$$q = (k_s - k_w)z - w_s t \qquad \text{Eqn 2.2-19}$$

Now let's assume that the amplitude of the electromagnetic field has the form:

$$E(z,t) \equiv w_s A_{s0} e^{i(fs - \frac{p}{2})} \qquad \text{Eqn 2.2-20}$$

where w_s is the frequency and A_{s0} is a coefficient and f_s is another phase factor. The meaning of this definition will become clearer when we do the calculations for a bunch of electrons. The phase f_s will provide an individual initial phase for each electron. To work with a dimensionless variable, we define the coefficient a_s as:

$$\frac{eA_{s0}}{mc} \equiv a_s \qquad \text{Eqn 2.2-21}$$

If we substitute these in the Eqn 2.2-18, after a simple algebra we get the important energy equation:

$$\frac{d\,g}{dz} = -\frac{k_s a_s K}{g}\sin(q + f_s) \qquad \text{Eqn 2.2-22}$$

Phase Equation

Using the definition of the phase variable (Eqn 2.2-19) we can take the derivative with respect to z and we get

$$\frac{dq}{dz} = k_s + k_w - w_s \frac{1}{dz/dt} \qquad \text{Eqn 2.2-23}$$

which can be written as

$$\frac{dq}{dz} = k_s + k_w - k_s \frac{1}{b_\parallel} \qquad \text{Eqn 2.2-24}$$

The parallel velocity can be derived using Eqn 2.2-10.

$$b_\parallel = \sqrt{b^2 - b_\perp{}^2} = (1 - \frac{1}{g^2} - \frac{K^2}{g^2})^{\frac{1}{2}} \qquad \text{Eqn 2.2-25}$$

Since $\frac{1+K^2}{g^2}$ term is small compared to 1, we can make a Taylor expansion and get

$$\frac{1}{b_\parallel} \cong 1 + \frac{1+K^2}{2\,g^2} \qquad \text{Eqn 2.2-26}$$

When we plug this in Eqn 2.2-24 we get the other important equation, the phase equation:

$$\frac{dq}{dz} \cong k_w - k_s \frac{1+K^2}{2\,g^2} \qquad \text{Eqn 2.2-27}$$

Resonant Condition

For a certain value of the electron's energy, the phase does not change along the wiggler. This condition is called the resonant condition, and we denote the resonant energy as $g = g_0$. For this value of energy, the phase evolution of Eqn 2.2-27 becomes zero.

$$\frac{dq}{dz} = 0$$

Therefore, the condition for resonance becomes $k_w = k_s \frac{1+K^2}{2\gamma_0^2}$ Eqn 2.2-28

So we get the central wavelength of the spontaneous radiation.

$$l_s = \frac{l_w}{2\gamma_0^2}(1+K^2)$$ Eqn 2.2-29

This is equivalent to the statement that electrons slip one l_s behind the EM wave in one wiggler period l_w.

In the next stage we consider electrons whose energy is near the resonant energy, but not quite.

When $\gamma \neq \gamma_0$, but $|\gamma - \gamma_0| << \gamma_0$

$$\frac{dq}{dz} = k_w - k_s \frac{1+K^2}{2\gamma_0^2}\frac{\gamma_0^2}{\gamma^2} = k_w(1-\frac{\gamma_0^2}{\gamma^2}) \cong 2k_w \frac{\gamma-\gamma_0}{\gamma_0}$$ Eqn 2.2-30

Therefore we can write the energy and phase equations of motion for $\gamma \cong \gamma_0$

$$\frac{d\gamma}{dz} = -\frac{k_s a_s K}{\gamma_0}\sin(q + f_s)$$ Eqn 2.2-31

$$\frac{dq}{dz} = 2k_w \frac{\gamma-\gamma_0}{\gamma_0}$$ Eqn 2.2-32

Small Gain Approximation

Our next task is to solve this set of two equations. Let's consider the case where a_s is essentially constant and f_s is independent of z.

Let's define

$$f \equiv q + f_s$$ Eqn 2.2-33

Also, let us use a dimensionless position variable,

$$t \equiv k_w z$$ Eqn 2.2-34

and a variable h, which we call the detuning variable:

$$h \equiv 2\frac{g - g_0}{g_0}: \qquad \text{Eqn 2.2-35}$$

Next, we define an angular frequency Ω_s, which we will identify later as the synchrotron rotation frequency:

$$\Omega_s^{\ 2} \equiv 2\frac{k_s}{k_w}\frac{a_s K}{g_0^{\ 2}} \qquad \text{Eqn 2.2-36}$$

If we plug these definitions into the equations (Eqn 2.2-31, Eqn 2.2-32) we get the celebrated 'FEL pendulum equations':

$$\frac{df}{dt} = h \qquad \text{Eqn 2.2-37}$$

$$\frac{dh}{dt} = -\Omega_s^{\ 2}\sin f \qquad \text{Eqn 2.2-38}$$

f is analogous to the angle variable of a mechanical pendulum. If the input signal a_s is very small then Ω_s is also very small. In this case Ω_s becomes oscillation frequency in the "ponderomotive potential". This oscillation is called the synchrotron oscillation for historical reasons (it describes the longitudinal motion of particles in a synchrotron). Thus Ω_s is the synchrotron angular frequency.

The initial energy detuning is h_0

Let's define

$$x \equiv h_0 t \quad , \; e \equiv \frac{h}{h_0}, \; m \equiv \frac{\Omega_s^{\ 2}}{h_0^{\ 2}} \qquad \text{Eqn 2.2-39}$$

e is a dimensionless relative energy variable and x is the dimensionless position along the wiggler scaled by the initial detuning.

By plugging these definitions into pendulum equations (Eqn 2.2-37 and Eqn 2.2-38) we can obtain an even simpler form of the equations:

$$\frac{df}{dx} = e \qquad \text{Eqn 2.2-40}$$

$$\frac{de}{dx} = m\sin f \qquad \text{Eqn 2.2-41}$$

We define the small signal regime by $\mu \ll 1$

In this regime we may apply a perturbation method and introduce iterations to solve the equations. Having the solution for one electron (defined by the set of initial conditions) we average over the phase f of the individual electrons, and obtain the change in energy of the ensemble.

We will skip the details of the derivation here and present the result.

$$<\varepsilon>=<\varepsilon_0>+\mu^2 x^3 f(x) \qquad \text{Eqn 2.2-42}$$

where $$f(x)=\frac{\cos x-1+\frac{1}{2}x\sin x}{x^3} \qquad \text{Eqn 2.2-43}$$

By using the definition of the energy variable e from Eqn 2.2-39 we can write

$$<\frac{\Delta g}{g}>=\frac{1}{2}<\Delta h>=\frac{h_0}{2}<\frac{\Delta h}{h_0}> \qquad \text{Eqn 2.2-44}$$

and also by using Eqn 2.2-42 we can obtain the relation for energy loss.

$$<\frac{\Delta g}{g}>=\frac{h_0}{2}<e-e_0>=\frac{h_0}{2}m^2x^3 f(x) \qquad \text{Eqn 2.2-45}$$

We can use the definition of m and x (Eqn 2.2-39) and get the following simplified equation.

$$<\frac{\Delta g}{g}>=\frac{1}{2}\Omega_s^{\,4}t^3 f(x) \qquad \text{Eqn 2.2-46}$$

where

$$\Omega_s^{\,2}=2\frac{k_s}{k_w}\frac{a_s K}{g_0^{\,2}},\ t=k_w z=\frac{2p}{l_w}z$$

which are independent of the initial detuning $h_0=2\frac{g-g_0}{g_0}$

Naturally, when we use x it is at the end of the wiggler, that is at $\tau=k_W L_W=2pN_W$. Thus x is proportional only to the initial detuning. It happens to be the only variable related to the initial detuning,

$$x = h_0 t = 4pN_w \frac{g - g_0}{g_0} \approx \text{detuning} \qquad \text{Eqn 2.2-47}$$

In Figure 2.2-2 below we plot the gain function *f(x)* as a function of *x*. This plot has a significant importance in FEL process, because depending on the value of the detuning we can have positive or negative, minimum or maximum gain values. If we choose a detuning which yields a positive gain value then we will have enhancement in the radiation emitted by the FEL process. That means that electrons transfer their energy to the radiation field. However if we choose a detuning which gives a negative gain value then the electrons will gain energy from the radiation in the wiggler, which causes acceleration of electrons. This process is laser acceleration, in this case an Inverse Free Electron Laser (IFEL) process.

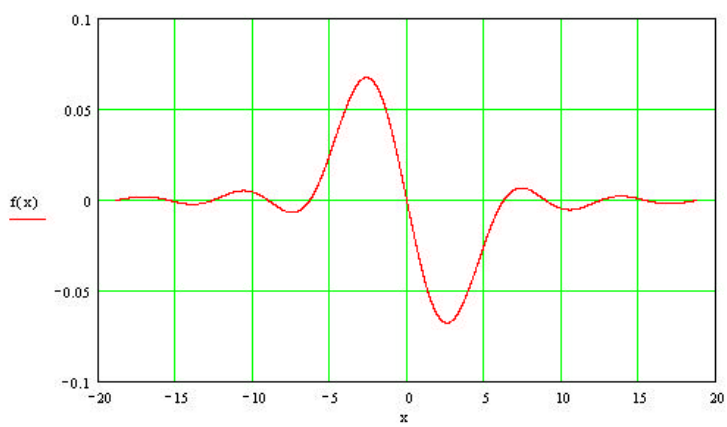

Figure 2.2-2 The gain function as a function of the normalized x variable, which is proportional to the detuning (see text)

Small Gain Formula

Now that we have the energy transfer solved, we will estimate the small gain formula. Let's assume that the optical beam size is equal to electron beam size with the area *A*. We define the electron number density as n_o so that the current would be given by $I_0 = ecn_0A$ and let's take the electron bunch pulse length as *t*. The radiation field is amplified only where it overlaps with the electron bunch, thus *t* is also the length of the radiation pulse.

The radiation field energy is given by

$$Energy = \frac{|E|^2}{Z_0} At \qquad \text{Eqn 2.2-48}$$

The gain is defined as the ratio of energy lost in the electron bunch to the energy of the electric field in the wiggler. So the gain is

$$G = \frac{\left\langle \frac{\Delta g}{g_0} \right\rangle mc^2 g_0 \frac{I_0 t}{e}}{\frac{|E|^2}{Z_0} At} \qquad \text{Eqn 2.2-49}$$

where $\frac{I_0 t}{e}$ is just the number of the electrons in the bunch.

By using Eqn 2.2-46 we get

$$G = -\frac{\Omega_s{}^4 mc^2 g_0 Z_0}{2e|E|^2} \frac{I_0}{A} t^3 f(x) \qquad \text{Eqn 2.2-50}$$

and also using the definitions of Ω_s and τ (Eqn 2.2-36 and Eqn 2.2-20) we can obtain the expression for gain as

$$G = -\frac{(2\frac{k_s}{k_w}\frac{a_s K}{g_0{}^2})^2 mc^2 g_0 Z_0}{2e(w_s A_{s0})^2} en_0 c t^3 f(x) \qquad \text{Eqn 2.2-51}$$

Remembering $a_s = \frac{eA_{s0}}{mc}$ we can also write the gain as

$$G = -\frac{2Z_0 e^2 n_0 K^2}{mck_w{}^2 g_0{}^3} t^3 f(x) \qquad \text{Eqn 2.2-52}$$

In order to simplify the expression we can express the gain in terms of the Pierce parameter r, which is defined as

$$(2r)^3 = \frac{Z_0 e^2 n_0 K^2}{2mck_w{}^2 g_0{}^3} \qquad \text{Eqn 2.2-53}$$

Thus the gain in terms of the Pierce parameter is

$$G = -4(4prN_w)^3 f(x) \qquad \text{Eqn 2.2-54}$$

Hamiltonian and Phase space

We had obtained the pendulum equations before in Eqn 2.2-37 and Eqn 2.2-38. Now we will express the same equation in terms of the Hamilton's Canonical Equations. We will start with a trial Hamiltonian as:

$$H = \frac{1}{2}h^2 - \Omega_s^{\ 2}\cos f \qquad \text{Eqn 2.2-55}$$

By applying the Hamilton's Canonical Equations to this expression we get

$$\frac{df}{dt} = f' = \frac{\P H}{\P h} = h \qquad \text{Eqn 2.2-56}$$

and $$\frac{dh}{dt} = h' = -\frac{\P H}{\P f} = -\Omega_s^{\ 2}\sin f \qquad \text{Eqn 2.2-57}$$

Which reproduces Eqn 2.2-37 and Eqn 2.2-38 (the pendulum equations). Therefore h and f are canonical conjugate variables in Hamilton equations. (Similar to position and momentum in classical mechanics).

Since our trial Hamiltonian fits these pendulum equations perfectly, it must be a conserved quantity. Using this conserved Hamiltonian we can plot the phase space of the electron trajectories.

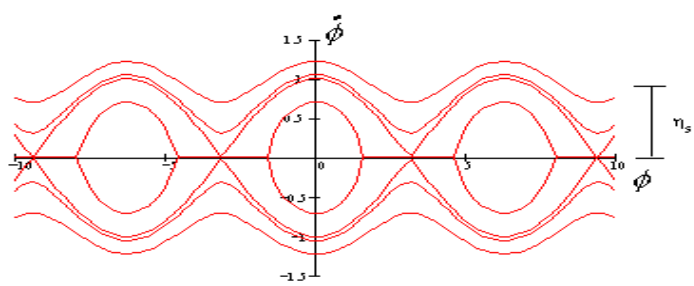

Figure 2.2-3 Electron trajectories in f- f' phase space, showing bounded and unbounded trajectories and the separatrix.

In Figure 2.2-3 above we plot the derivative of the phase versus the phase itself for different values of Hamiltonian H. As shown in Figure 2.2-3 some of the trajectories are unbounded but some are bounded as closed loops.

For a certain value of Hamiltonian the trajectory intersects the phase axis at $\pm p, \pm 3p, \pm 5p.....$ This special trajectory is called the 'separatrix', and it separates the bounded trajectories (loops) from the unbounded ones.

The separatrix height is $\pm 2\Omega_s$

The width of the separatrix is 2π, which can easily be seen on the plot.

For small gain and small signal regime $h_0 >> \Omega_s$. The energy modulation h varies with $x = h_0 t = h_0 k_w z$.

Assume we have electrons with a given energy spread as shown in Figure 2.2-4.

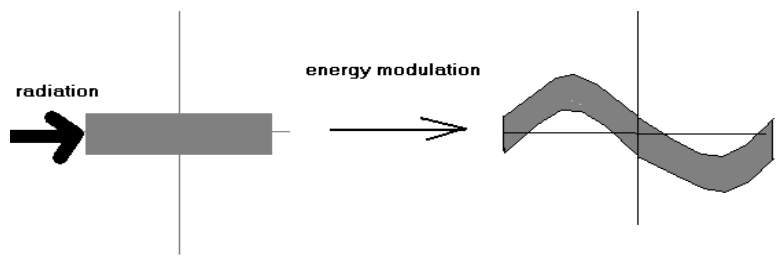

Figure 2.2-4 Electron distribution in phase-energy space, before (left) and after (right energy modulation by the radiation

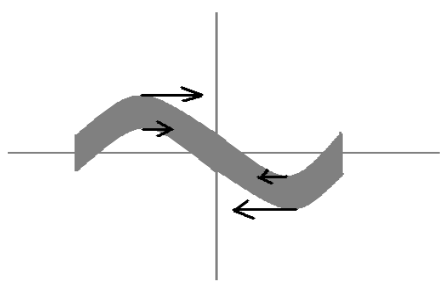

Figure 2.2-5 The process of microbunching in a dispersive medium. The arrows show direction of electron motion due to the modulation.

The interaction with the radiation modulates the energy of the electron beam as shown in Figure 2.2-4. In the wiggler, the more energetic electrons bend less; therefore cover a shorter distance than the less energetic electrons. This phenomenon causes the more energetic electrons to catch up the less energetic electrons and that is called bunch compression (specifically, magnetic bunch compression), leading to micro bunching. The process is shown in Figure 2.2-6. Since the length of these micro-bunches is shorter than

the wavelength, these bunched electrons radiate coherently (in phase with the modulating radiation). The strength of the radiation grows and we have gain. As radiation grows Ω_s also grows, therefore the separatrix height increases and radiation goes into the high gain regime which will be discussed later.

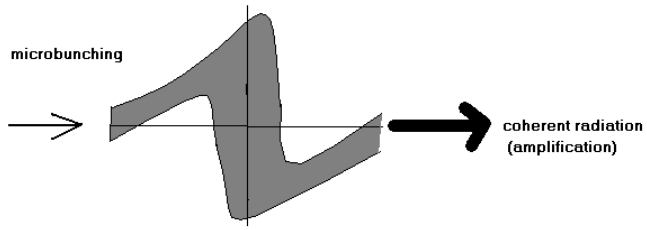

Figure 2.2-6 The microbunched beam