A PARAMETER STUDY OF DIELECTRIC WAKE FIELD ACCELERATOR PERFORMANCE

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1. INTRODUCTION

We investigate the behavior of the efficiency, accelerating gradient, energy spread, and other measures of performance of a cylindrical multimode monolayer dielectric wake field accelerator (*MM- DWA)* as a function of the bunch dimensions (rms-length, longitudinal shape, etc), driving train profile, geometrical dimensions of a structure (inner, and outer radii), and dielectric constant. The energy balance equation is derived, and on its basis a time efficient, semi- analytical numerical computational algorithm is developed to predict the accelerating gradient, efficiency, and energy spread of the *MM- DWA.* Having analyzed over 2,000 cases we reach conclusions about the quantitative behavior of the *MM- DWA* performance, affected by changes in the structure and/or bunch dimensions, as well as the dielectric material.

The studies take the example of a cylindrically symmetric structure in which a vacuum channel is made for the electrons to pass along the axis through an annular layer of dielectric material. The dielectric is enclosed by a metal casing (Fig.1) A train of colinearly- moving bunches of charge (referred to commonly as a drive train, or drive bunches) is used to excite the structure. A single test bunch that is to be accelerated follows this train. The boundary conditions at the structure ends are assumed to have negligible effect upon the longitudinally- infinite solution, in which the wake field is described as the superposition of many TM_{0m} modes [1][2][3] each having phase velocity equal to that of the electron bunches.

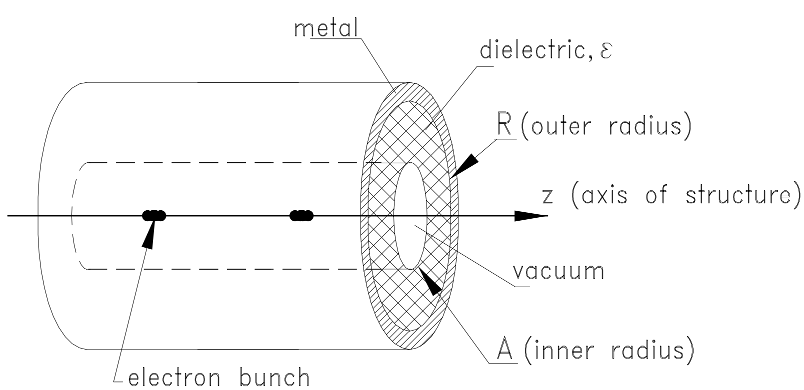

FIG. 1 Example of a DWA structure

Every drive bunch has sufficient initial energy for it to remain synchronous with the wake field for an appreciable amount of time, and, should this bunch fall out of synchronism, it would be replaced by a new highly energetic bunch. We neglect any transverse features of a bunch since it will be demonstrated that for tightly focused beams the transverse distribution has little effect upon the excited wake field. We assume that focusing is provided by external elements of whatever design is found to be appropriate. We also do not address any issues concerning the electric breakdown threshold on the surface of the dielectric.

2. THEORETICAL FORMULATION

For a train of axially symmetric bunches with the transverse half-width $w_{trans} \le A/5$, centered at $r = 0$ the charge distribution is represented as:

$$\rho(r', z'_0) = Q_0 \frac{\delta(r')}{2\pi \cdot r'} \sum_i q_i F_i(z'_0)$$

with $\int F_i(\xi) d\xi = 1$, and $F_i(\xi) \to 0$, when $|\xi| \to \infty$

where the function F_i describes the longitudinal shape of the i^{th} bunch, whose full charge is:

$$Q_i = q_i \cdot Q_0 \qquad (2.1)$$

with Q_0 being some reference charge; $\delta(r')$ is the usual Dirac delta-function; $z'_0 = z' - vt$, r', z' are the radial and longitudinal coordinates, t is time, and $v = c\beta$ is the bunch velocity. Note that $q_i \ge 0$ always.

Searching the fields in the form of TM_{0m} eigenmode decomposition [1][2][3]:

$$E_z = \sum_{m=1}^{\infty} E_0^m \cdot e_z^m , \qquad E_r = \sum_{m=1}^{\infty} E_0^m \cdot e_r^m , \qquad H_\theta = \sum_{m=1}^{\infty} \frac{E_0^m}{\sqrt{\mu_0/\varepsilon_0}} \cdot h_\theta^m \qquad (2.2)$$

where $E_0^m = \dfrac{Q_0}{2\pi\varepsilon_0 A^2} f_m(0)$, and m is mode number, one obtains that:

$$e_z^m = -\sum_i q_i \frac{f_m(r)}{\alpha_m} \int_{-\infty}^{\infty} dz'_0 \cdot F_i(z'_0)\theta_{hv}(z'_0 - z_0) \cos\left(\frac{\omega_m}{v} \cdot (z'_0 - z_0)\right)$$

$$e_r^m = \sum_i q_i \frac{c\beta}{\omega_m \left(1 - \varepsilon(r)\beta^2\right)} \cdot \frac{\partial f_m(r)/\partial r}{\alpha_m} \int_{-\infty}^{\infty} dz'_0 \cdot F_i(z'_0)\theta_{hv}(z'_0 - z_0) \sin\left(\frac{\omega_m}{v} \cdot (z'_0 - z_0)\right)$$

$$h_\theta^m = \sum_i q_i \frac{\varepsilon(r) c\beta^2}{\omega_m \left(1 - \varepsilon(r)\beta^2\right)} \cdot \frac{\partial f_m(r)/\partial r}{\alpha_m} \int_{-\infty}^{\infty} dz'_0 \cdot F_i(z'_0) \cdot \theta_{hv}(z'_0 - z_0) \cdot \sin\left(\frac{\omega_m}{v} \cdot (z'_0 - z_0)\right)$$

where $z'_0 = z' - vt$, $z_0 = z - vt$, α_m is the normalization constant, θ_{hv} is the Heaviside function, ω_m is the m^{th} eigen- frequency [see ref.1], and $f_m(r)$ is the m^{th} eigen- function [1]. In the vacuum channel, one obtains eigen- functions:

$$f_m(r) = I_0(k_{1m} r)$$

Then, the normalization constant is:

$$\alpha_m = \frac{1}{2} \cdot \left\{ \frac{1}{\varepsilon} \cdot \left(\frac{\gamma}{\gamma_k} \right)^2 I_1^2(k_{1m} A) \cdot \left[\frac{R^2}{A^2} \left(\frac{P_1(k_{2m,} R, R)}{P_1(k_{2m}, R, A)} \right)^2 - 1 \right] - (\varepsilon - 1) I_0^2(k_{1m} A) - I_1^2(k_{1m} A) \right\}$$

where $k_{1m} = \dfrac{\omega_m}{c \cdot \beta \cdot \gamma} = k_{2m} \dfrac{\gamma_k}{\gamma}$, with γ being the relativistic gamma- factor, $\gamma_k = 1/\sqrt{\varepsilon \cdot \beta^2 - 1}$, and $P_1(k, R, r) = J_0(kR) N_1(kr) - J_1(kr) N_0(kR)$.
J_m, N_m are the Bessel, and Neumann functions (Bessel equation functions of the 1st and 2nd kinds) respectively. I_m is the modified Bessel function.

If the transverse features need to be included one makes the following alterations:

$\delta(r')/r' \rightarrow B(r')$, and

$$f_m(0) \rightarrow f_{m0} = \int B(r') f_m(r') r' dr', \tag{2.3}$$

where $B(r')$ is the transverse distribution function such that $\int B(r') r' dr' = 1$.

For the transverse Gaussian distribution $B(r') = \dfrac{1}{w_{trans}^2} \exp(-\dfrac{r'^2}{2 \cdot w_{trans}^2})$ with $w_{trans} \leq A/5$ one discovers that the correction to the accelerating gradient is less than or about 0.01 %, and thus the transverse features for tightly focused bunches can be indeed neglected.

The power radiated by a train of e-bunches (computed through the modified Poynting vector [2]) is:

$$P = \sum_m P_m$$

where P_m – the power [Watts] going into/ being taken from the mth mode as the result of interaction with the bunches

$$P_m = \frac{Q_0^2 c \beta}{4 \pi \varepsilon_0 A^2} \cdot \frac{f_m^2(0)}{\alpha_m} \cdot \left(\left(I_c^m \right)^2 + \left(I_s^m \right)^2 \right)$$

where

$$I_C^m = \sum_i q_i \Gamma_C^{m,i} \cdot \cos(\theta_i^m) - \sum_i q_i \Gamma_S^{m,i} \cdot \sin(\theta_i^m)$$

$$I_S^m = \sum_i q_i \Gamma_S^{m,i} \cdot \cos(\theta_i^m) + \sum_i q_i \Gamma_C^{m,i} \cdot \sin(\theta_i^m)$$

with

$$\Gamma_C^{m,i} = \int_{-\infty}^{\infty} dz_0^{'} \cdot F_i(z'_0)\theta_{hv}(z'_0 - z_0)\cos\left(\frac{\omega_m}{c\beta} z'_0\right)$$

$$\Gamma_S^{m,i} = \int_{-\infty}^{\infty} dz'_0 \cdot F_i(z'_0)\theta_{hv}(z'_0 - z_0)\sin\left(\frac{\omega_m}{c\beta} z'_0\right)$$

and $\theta_i^m = \dfrac{\omega_m z_i}{c\beta}$, where z_i is the position of the i^{th} bunch on the z- axis. Note that at any point behind the i^{th} bunch all corresponding coefficients $\Gamma_C^{m,i}$, and $\Gamma_S^{m,i}$ become independent on z_0.

The Γ coefficients determine the structure of the created wake field, and the details of interaction. In our numerical studies we are interested in the wake field excited by bunches of Gaussian shape (with the different tail/head ratio), as well as the triangular (with the different tail/head ratio), and rectangular shape. To provide consistent comparison between different longitudinal shapes we keep the rms-length

$$\sigma_L = 2\sqrt{\langle s^2\rangle - \langle s\rangle^2} \tag{2.4}$$

the same for every distribution.

With the Γ coefficients computed one obtains that at any point behind the very last bunch:

$$(I_c^m)^2 + (I_s^m)^2 = \sum_i \sum_j q_i q_j \cdot (\Gamma_C^{m,i}\Gamma_C^{m,j} + \Gamma_S^{m,i}\Gamma_S^{m,j})\cos(\theta_i^m - \theta_j^m) +$$

$$+ \sum_i \sum_j q_i q_j \cdot (\Gamma_C^{m,i}\Gamma_S^{m,j} - \Gamma_S^{m,i}\Gamma_C^{m,j})\sin(\theta_i^m - \theta_j^m)$$

which clearly demonstrates that the emitted radiation is amplified in a coherent fashion. (The second term of the series vanishes if all bunches have the same longitudinal shape.)

We proceed further by noting that one puts into the structure N driving bunches, having the same shape but with different charges, followed by one test bunch (denoted by $N+1$) whose charge and longitudinal dimensions are different. Then, we have: $\Gamma_C^{m,i} = \Gamma_C^{m,j} = \Gamma_C^m$, $\Gamma_S^{m,i} = \Gamma_S^{m,j} = \Gamma_S^m$, for any $i, j \le N$, and $\Gamma_C^{m,N+1} = \Gamma_{CT}^m$, $\Gamma_S^{m,N+1} = \Gamma_{ST}^m$, $q_{N+1} = q_T$, $\theta_{N+1}^m = \theta_T^m$

Now we introduce the following expressions:

$$\Omega_m = \left[(\Gamma_C^m)^2 + (\Gamma_S^m)^2\right] \cdot \frac{f_{m0}^2}{\alpha_m}, \qquad \Omega_{mT} = \left[(\Gamma_{CT}^m)^2 + (\Gamma_{ST}^m)^2\right] \cdot \frac{f_{m0T}^2}{\alpha_m},$$

$$\Lambda_m = \left[\Gamma_C^m\Gamma_{CT}^m + \Gamma_S^m\Gamma_{ST}^m\right] \cdot \frac{f_{m0}f_{m0T}}{\alpha_m}, \qquad D_m = \left(\Gamma_C^m\Gamma_{ST}^m - \Gamma_{CT}^m\Gamma_S^m\right) \cdot \frac{f_{m0}f_{m0T}}{\alpha_m},$$

$$P_0 = \frac{Q_0^2 c\beta}{4\pi\varepsilon_0 A^2}, \qquad \Psi = \frac{\sum_m \Omega_m}{\sum_m \Omega_{mT}},$$

where the subindex T indicates that the quantity is computed for the test bunch [for f_{m0} see Eq.(2.3)], and rewrite the radiated power in the form:

$$\frac{P}{P_0} = \sum_m \left(\Omega_m \sum_{i,j}^{N,N} q_i q_j \cos(\theta_i^m - \theta_j^m) \right) +$$
$$+ \sum_m \left(q_T^2 \Omega_{mT} + 2 q_T \Lambda_m \sum_i^N q_i \left[\cos(\theta_i^m - \theta_T^m) + \frac{D_m}{\Lambda_m} \sin(\theta_i^m - \theta_T^m) \right] \right)$$

The first term

$$\frac{P_N}{P_0} = \sum_m \left(\Omega_m \sum_{i,j}^{N,N} q_i q_j \cos(\theta_i^m - \theta_j^m) \right)$$

is the power radiated by the first N bunches (the power pumped into the wake field by the driving train.) The two last terms, when taken with the minus sign, give the amount of power that goes for the test bunch acceleration, i.e.:

$$\frac{P_T}{P_0} = -q_T \sum_m \left(q_T \Omega_{mT} + 2\Lambda_m \sum_i^N q_i \cos(\theta_i^m - \theta_T^m) + 2D_m \sum_i^N q_i \sin(\theta_i^m - \theta_T^m) \right)$$

Recalling that one places the driving bunches equidistantly with spacing equal to the wake field period, L, and the test bunch follows behind the driving train at the distance $L/2$, i.e.:

$$\theta_i^m = (1-i) L \frac{\omega_m}{c\beta}, \qquad \theta_T^m = \left(1 - N - \frac{1}{2}\right) L \frac{\omega_m}{c\beta},$$

we compute the *enhancement factor* (showing by how much the power radiated from N driving bunches exceeds the power radiated by one bunch with $q_1 = 1$):

$$\xi = \left(\sum_{m,i,j}^{\infty,N,N} q_i q_j \Omega_m \cos\left(\frac{\omega_m L \cdot (i-j)}{c\beta}\right) \right) / \sum_m^{\infty} \Omega_m \tag{2.5}$$

The *structural ratio* (the value of q_T when P_T is maximum) is:

$$\chi_\rho = -\frac{\sum_{m,i}^{\infty,N} \Lambda_m q_i \cos\left(\frac{\omega_m L \cdot (N-i+1/2)}{c\beta}\right) + \sum_{m,i}^{\infty,N} D_m q_i \sin\left(\frac{\omega_m L \cdot (N-i+1/2)}{c\beta}\right)}{\sum_m \Omega_{mT}} \tag{2.6}$$

The *efficiency of energy transfer* from the N- bunch train to a test bunch of charge $Q_T = Q_0 \cdot q_T$ will be:

$$\eta = \frac{P_T}{P_N} = \frac{2 q_T \chi_\rho \sum_m \Omega_{mT} - q_T^2 \sum_m \Omega_{mT}}{\xi \cdot \sum_m \Omega_m} = \frac{2 q_T \chi_\rho - q_T^2}{\xi \cdot \Psi} \tag{2.7}$$

with its maximum value of

$$\eta_{MAX} = \frac{\chi_\rho^2}{\xi \cdot \Psi} \tag{2.8}$$

The average *accelerating gradient*[a] acting on the test bunch:

$$E_{ACC} = \frac{Q_0}{4\pi\varepsilon_0 A^2} \sum_m \Omega_{mT} \cdot \left(2\chi_\rho - q_T\right) \tag{2.9}$$

with its maximum of

$$E_T = \frac{Q_0}{2\pi\varepsilon_0 A^2} \chi_\rho \sum_m \Omega_{mT} \tag{2.10}$$

Note that the acceleration gradient achieves its maximum at $Q_T \to 0$, while the efficiency achieves its maximum at $Q_T = Q_0 \cdot \chi_\rho$. When the efficiency is maximum the accelerating gradient is only $E_T/2$.

After traveling the distance S the energy of an electron from a test bunch will be $W_T = eSE_z + W_0$, where W_0 is the initial energy, assumed to be the same for every test electron; E_z is the field (unchanging in the synchronous approximation) accelerating the given electron [see Eq.(2.2)], and $e = 1.6 \cdot 10^{-19}$C (electron charge). Averaging over the test bunch longitudinal distribution one has $\overline{E}_z = E_{ACC}$, and consequently $\overline{W}_T = eSE_{ACC} + W_0$. The energy spread is then: $\delta W = \frac{\sqrt{\overline{\left(W_T - \overline{W}_T\right)^2}}}{\overline{W}_T} = \frac{\sqrt{\overline{E_z^2} - E_{ACC}^2}}{E_{ACC} + {W_0}/{eS}}$. Now we take into account that usually the energy gained after acceleration is much large than the initial one, i.e. $E_{ACC} >> W_0 /eS$. Also the test bunch charge is relatively small, i.e. $2\chi_\rho >> q_T$, and consequently $E_{ACC} \to E_T$, and computing $\overline{E_z^2}$ one can neglect all sum members of the order of q_T and higher.

Under these condition the *energy spread* of a test bunch becomes:

$$\delta W = \sqrt{\frac{\overline{\chi_\rho^2}}{\chi_\rho^2} - 1}, \tag{2.11}$$

where χ_ρ is given by Eq.(2.6), and $\overline{\chi_\rho^2}$ is defined as:

$$\overline{\chi_\rho^2} \equiv \frac{1}{\left(\sum_m \Omega_{mT}\right)^2} \int_{-\infty}^{\infty}\int_0^A F_T(z) B_T(r) \left(\sum_m \frac{f_{m0} f_m(r)}{\alpha_m} \sum_{i=1}^N q_i \left[\Gamma_C^{m,i} \cos(\psi_m^i) + \Gamma_S^{m,i} \sin(\psi_m^i)\right]\right)^2 rdrdz$$

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[a] Eq.(2.9) allows one to calculate the accelerating gradient about 200 times faster than by the direct integration

where $F_T\ (z)$, and $B_T\ (r)$ are the longitudinal and transverse test bunch distributions respectively; $\psi_m^i = \dfrac{\omega_m \cdot [z - L(N - i + 1/2)]}{c\beta}$, and L is the wake period.

At the location of a test/drive bunch the longitudinal electric field vs. z resembles a hump (see Fig.2) with the foot width

$$2\sigma_{wake} = \sigma_w^+ + \sigma_w^-, \tag{2.12}$$

where values σ_w^+, and σ_w^- are found as the solution of equation:

$$\sum_m \Omega_m \cos\left(\frac{\omega_m}{c\beta}(L \pm \sigma_w)\right) / \sum_m \Omega_m = 0.1$$

The ratio between[b] σ_{wake} and the test bunch rms- length σ_{TB} will determine whether the energy spread is sensitive to changes in a given DWA parameter, or not.

Since all coefficients Ω_m , Ω_{mT} , Λ_m , D_m depend on the bunch shape one can try to use different bunch longitudinal distributions to improve the efficiency, accelerating gradient, and/or energy spread. The efficiency, accelerating gradient, and energy spread depend also on the set of q_i, adjustable by choosing a particular profile for the driving bunch train. This provides an additional degree of freedom for improvements.

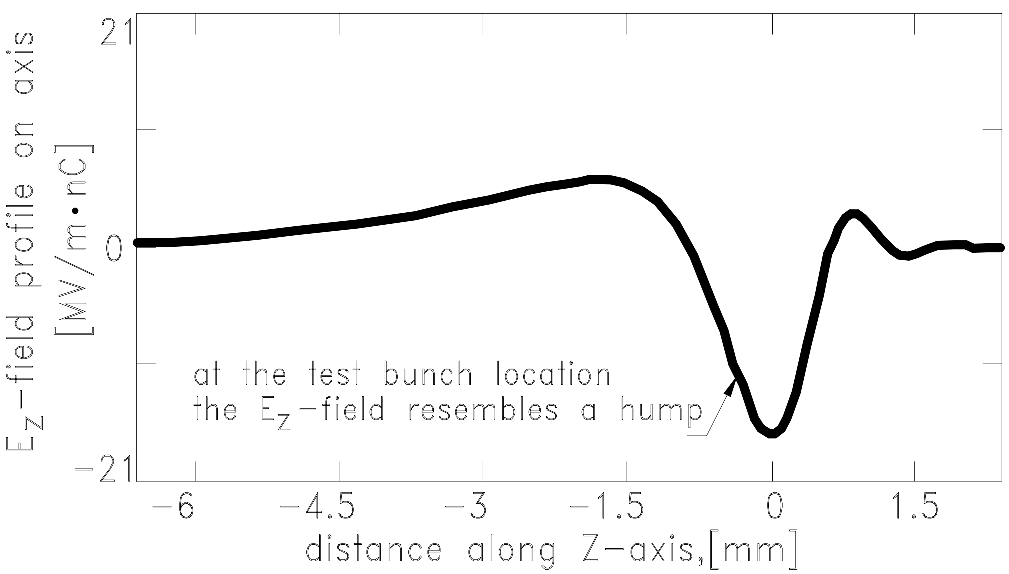

FIG.2 The longitudinal electric field vs. Z (at the axis of structure)

[b] σ_{wake} enters also into the equation on synchronism between drive bunches and wake field and is being referred to later as the coherent length

3. NUMERICAL STUDIES

A. Performance vs. bunch shape, and drive train profile

Using the drive charge Q_D as a source of the wake fields in a *MM- DWA* one asks what its longitudinal distribution shall be to provide the best efficiency and accelerating gradient acting on a collinearly- located test bunch. Since there are limitations on the maximum bunch charge at a given energy due to space charge effects, one cannot put all drive charge into a single bunch, and the need for considering a multi-bunch drive train appears. The charge Q_i of the i^{th} bunch, thus, can be anywhere within the range from *0* to $Q_{\max}$, where $Q_{\max}$ is specified by the space charge effects, arising due to the transverse fields. However, space charge effects themselves do not specify the particular value of every Q_i, nor put any upper limit on the number of bunches N in a multi- bunch drive train ($1 \le i \le N$, and the minimum of N will be defined, of course, as $Q_D / Q_{\max}$). Studying the details of the interaction between a multi- bunch drive train and pursuing the achievement of maximum efficiency and accelerating gradient, one can find the set of values $Q_i = Q_i^{opt}$, and some particular value of N such that the efficiency, and gradient are maximized. Note then that all Q_i^{opt} are defined by the longitudinal forces. To find Q_i^{opt} values and the corresponding N one solves the problem in which one sorts out among many possible sets of Q_i, with $1 \le i \le N$, where N can be, in principle, infinitely large, and the constraint that $\sum_{i=1}^{N} Q_i = Q_D$ is superimposed (total drive charge is preserved). If the solution is such that $Q_i^{opt} \le Q_{\max}$ for every i, and $N > Q_D / Q_{\max}$, it should be used to profile the driving train to optimize the *MM- DWA*.

After the particular driving train profile is found to provide the best performance, one can proceed further by asking what the distribution of a charge within every single drive bunch should be to, again, maximize the efficiency, and gradient. We can reshape the bunch (we used Gaussian (G), Triangular (Δ), and Rectangular(R) shapes), and changed the ratio[c] of head/tail lengths σ_1 / σ_2), and changed the rms-length (σ_L for a drive bunch, and/or σ_{TB} for a test bunch), while preserving the bunch charge, since it is already fixed by virtue of the previous problem.

In the supporting examples we take $R = 19.31$ *mm,* and $\varepsilon = 9.65$ unless otherwise specified. The accelerating gradient acting on a test bunch or energy losses experienced by a drive bunch will be given per *1 nC* of the drive bunch, e.g. *MV/m·nC*. The radiated power will also be given per *1 nC* of a drive bunch (e.g. MW/nC^2), unless otherwise specified.

Our conclusions are drawn under the condition $\sigma_{TB} \le \sigma_{wake}/4$, since the energy spread is tolerable when this condition is satisfied, and are summarized below:

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[c] For any rectangular bunch (in the formal notation) one always has $\sigma_2 = 0$, and thus, $\sigma_1 / \sigma_2 = \infty$ and is unchanged.

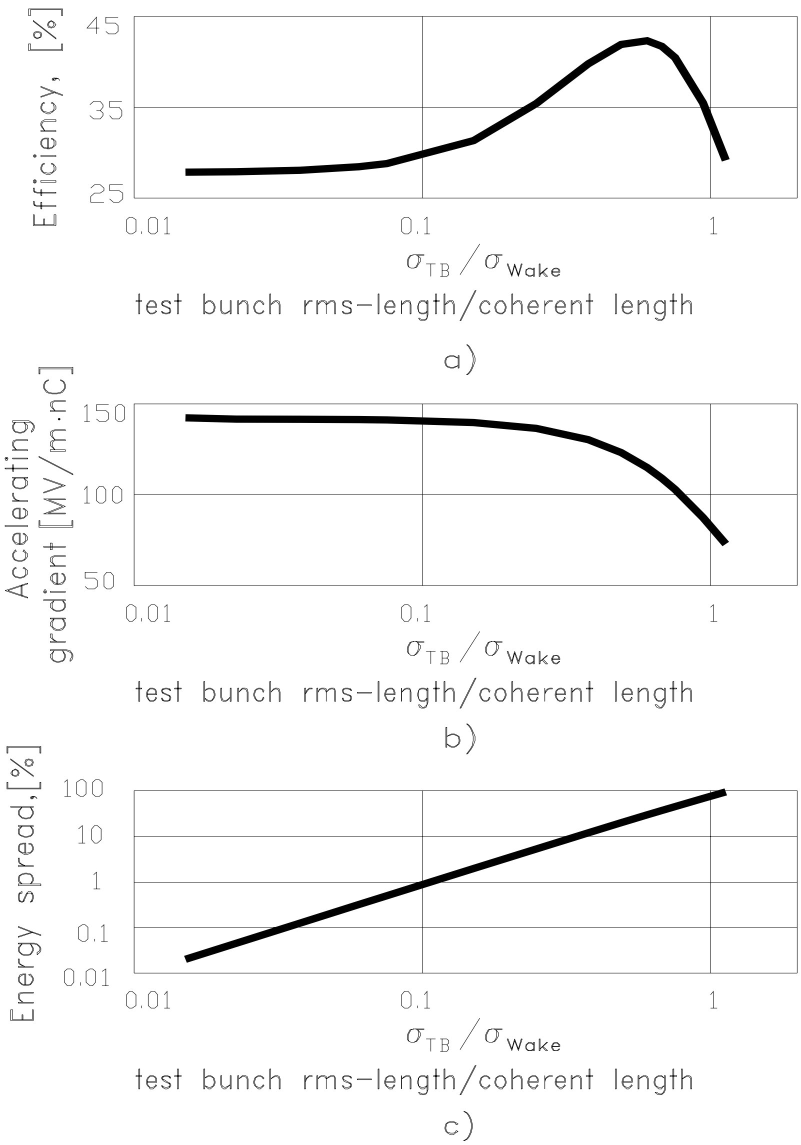

FIG.3. (a) Efficiency vs. the ratio σ_{TB} /σ_{wake} (test rms-length/ coherent length). (b) Accelerating gradient [MV/m·nC of the driving bunch] vs. σ_{TB} /σ_{wake}. (c) Energy spread vs. σ_{TB} /σ_{wake}. Inner radius *A = 1.0 mm*, drive profile = sin- like, $\hat{\alpha} = 1.7$, *N*= 20 bunches in the train, drive bunch rms- width σ_L = 1.6 *psec*, Gaussian, head/tail ratio σ_1 /σ_2 = 1:8, wake period *L* = 0.21590 *m*, coherent length σ_{wake} = 2.1 *psec*. (In (c) the spread is given for a Gaussian test bunch with σ_1 /σ_2 =1).

(3.1.1) η_{MAX} and E_T strongly depend on the rms-length σ_{TB} of a test bunch. The changes introduced by variation in the test bunch shape and changes in σ_1 /σ_2 are almost negligible: $\Delta\eta/\eta_{\mathrm{MAX}} \le \pm 0.023$, $\Delta E_T / E_T \le \pm 0.008$. For instance $\eta_{\mathrm{MAX}} \approx 50$ % when one uses *N= 10* driving bunches, and thus $\Delta\eta \approx 1$ %. In particular, the changes due to variations in σ_1 /σ_2 are: $\Delta\eta/\eta_{\mathrm{MAX}} \le \pm 0.005$, $\Delta E_T / E_T \le \pm 0.005$. The energy spread δW strongly depends on the rms- length σ_{TB} of a test bunch, as well as its shape. Rectangular bunches have the minimal energy spread, and for narrow bunches ($\sigma_{TB} \leq \sigma_{wake}/40$) the energy spread of a Gaussian bunch with $\sigma_1 = \sigma_2$ is almost as good as that of a rectangular one. However, δW is large for Gaussian bunches with $\sigma_1 \geq 3\sigma_2$ or vice versa. For instance, δW of a Gaussian bunch with $\sigma_1 >> \sigma_2$ is larger by about 80, 70, and 60% than δW of a rectangular bunch[d] if $\sigma_{TB} \approx 0.25$, 0.03, and 0.015 of σ_{wake}, respectively.

Figure 3 demonstrates the efficiency, accelerating gradient, and energy spread behavior when one varies the test bunch rms-length. The narrower test bunch has the higher accelerating gradient.

(3.1.2) Although η_{MAX}, ξ, and E_T, strongly depend upon the total rms-length σ_L of a driving bunch, they are weakly dependent on the variations in the drive bunch shape. When the test bunch shape is changed from Gaussian to triangular, and to rectangular (G-Δ-R), and σ_1/σ_2 is varied too the maximum variations are (see the Tab. #1):

Table 1[*]

Inner Radius	$\Delta\eta/\eta_{\mathrm{MAX}}$, %	$\Delta\xi/\xi$, %	$\Delta E_T/E_T$, %
$A \leq 0.5$ mm	+/- 6.2	+/- 1.6	+/- 5.1
0.5 mm $< A \leq 1.5$ mm	+/- 7.3	+/- 2.0	+/- 5.4

[*] $N \geq 5$

The maximum in η_{MAX} and E_T happens when one uses the Gaussian shaped bunch with $\sigma_1 >> \sigma_2$. The improvement can be up to $\Delta\eta/\eta = 0.14$ in the efficiency, and $\Delta E_T / E_T \approx 0.10$ in the gradient. Practically, instead of using bunches with $\sigma_1 >> \sigma_2$ one can limit oneself by utilizing bunches with[e] $\sigma_1 \geq (3\div4)\sigma_2$, since

$E_T(\sigma_1 \geq [3\div4]\sigma_2) \approx 0.98 \cdot E_T(\sigma_1 >> \sigma_2)$, and the same small difference is in the efficiency. The more narrow the driving bunches are, the higher will be the radiated power, and accelerating gradient (see Fig. 4, and 5)

The energy spread δW is minimized when one uses triangular drive bunches with $\sigma_1 << \sigma_2$, and maximized if Gaussian drive bunches with $\sigma_1 >> \sigma_2$ are used. The difference $\Delta\delta W/\delta W$ is essentially determined by the test bunch, and can be up to 50% (of the higher value) at $\sigma_{TB} \approx 0.25\sigma_{wake}$, and become less than 25 % at $\sigma_{TB} \leq 0.04\sigma_{wake}$. However, the Gaussian drive bunches with $\sigma_1 << \sigma_2$ provide the energy spread almost as low as that from the triangular, $\sigma_1 << \sigma_2$ drive bunches (the difference is about 10% of the higher value).

The accelerating gradient after a train of Gaussian, $\sigma_1 << \sigma_2$ drive bunches is higher by about 7% than that after the train of triangular (Δ),$\sigma_1 << \sigma_2$ drive bunches; but is lower by about 5 % than the gradient after the Gaussian, $\sigma_1 >> \sigma_2$ drive bunch train. Consequently, one might recommend Gaussian drive bunches with a short head, and a long tail to compromise between the test bunch energy spread, and the accelerating gradient.

[d] I.e. the difference $\Delta\delta W/\delta W$ = 80, 70, and 60% (of the lower value)

[e] The symbol ÷ is being used to indicate the range of values

(3.1.3) Everywhere in this statement ε = const, and the outer radius, *R,* is adjusted so that the wake period *L* is preserved. (*A* is the inner radius, σ_L is the rms- length of a drive bunch, σ_{TB} is the rms- length of a test bunch)

(a) The numerical study shows that[f] if $A \to A/\widetilde{K}$, $\sigma_L \to \sigma_L/\widetilde{K}$, and $\sigma_{TB} \to \sigma_{TB}/\widetilde{K}$ then the following quantities are unchanged:

$\sum_m \Omega_m = const$, ξ =*const*, $\sum_m \Omega_{mT} = const$, and χ_ρ = *const*, or equivalently the maximum efficiency, maximum accelerating gradient, and radiated power behave as:

$\eta_{\mathrm{MAX}} = const$,

$E_T \to E_T \cdot \widetilde{K}^2$,

$P \to P \cdot \widetilde{K}^2$.

(b) If $A \to A/\widetilde{K}$, $\sigma_L \to \sigma_L/\widetilde{K}$, and $\sigma_{TB} \to \sigma_{TB}/\widetilde{K}$, where $\widetilde{K} \le 5$, then the energy spread, and coherent length behave as:

$\delta W \approx const$, (with a range of $\Delta\delta W/\delta W \le \pm 1$ %)

$\sigma_{wake} \to \sigma_{wake}/\widetilde{K}$, (with a range of $\dfrac{\Delta\left(\sigma_{wake}/\widetilde{K}\right)}{\sigma_{wake}/\widetilde{K}} \le \pm 0.5$ %)

(c) The enhancement factor, ξ, has a maximum at some σ_L^{opt}. If $A \to A/\widetilde{K}$, then $\sigma_L^{opt} \to \sigma_L^{opt}/\widetilde{K}$.

(d) When $\sigma_L \ge \sigma_L^{opt}$ and grows, the efficiency, η_{MAX}, quickly drops. If $\sigma_L \le \sigma_L^{opt}$ the efficiency changes negligibly; for instance, $\left|\eta_{MAX}(\sigma_L^{opt}) - \eta_{MAX}(0)\right| \le 0.5\%$ if $N \ge 10$.

If one plots $P/\widetilde{K}^2$ (scaled radiated power) vs. $\sigma_L \cdot \widetilde{K}$ (scaled rms- length) [see Fig. 4], or $E_T/\widetilde{K}^2$ (scaled accelerating gradient) vs. $\sigma_L \cdot \widetilde{K}$ [see Fig.5] with the coefficient $\widetilde{K}$ computed as the radius ratio, i.e. $\widetilde{K} = A / A_0$ (where A_0 is some chosen reference radius), one makes all curves corresponding to the different radii *A* coincide. This means that under the transformation $A \to A/\widetilde{K}$, $\sigma_L \to \sigma_L/\widetilde{K}$, $\sigma_{TB} \to \sigma_{TB}/\widetilde{K}$, and *L* = *const* (by adjusting *R)* one has: $P \to P \cdot \widetilde{K}^2$, and $E_T \to E_T \cdot \widetilde{K}^2$ as was stated in *(3.1.3)*

[f] The symbol → is being used to indicate that one value (left-hand side) has been changed to another (right-hand side)

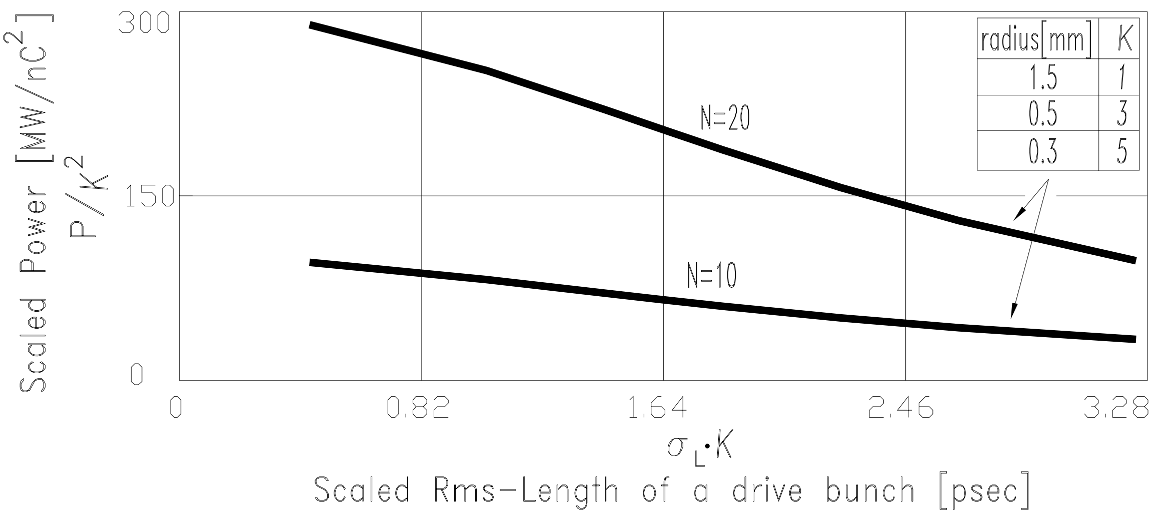

FIG.4. Scaled radiated power $P/\widetilde{K}^2$ vs. scaled rms-length $\sigma_L \cdot \widetilde{K}$ for two uniform drive trains with N = *10*, and N= *20* bunches. The curves are given for three radii A = 1.5, 0.5, and 0.3 *mm*. The radius ratio is $\widetilde{K} = (1:3:5)$. One sees that all curves corresponding to different radii A (but the same N) coincide. In all examples the head/tail ratio σ_1 / σ_2 = 4/1, wake period L = *0.21027 m*.

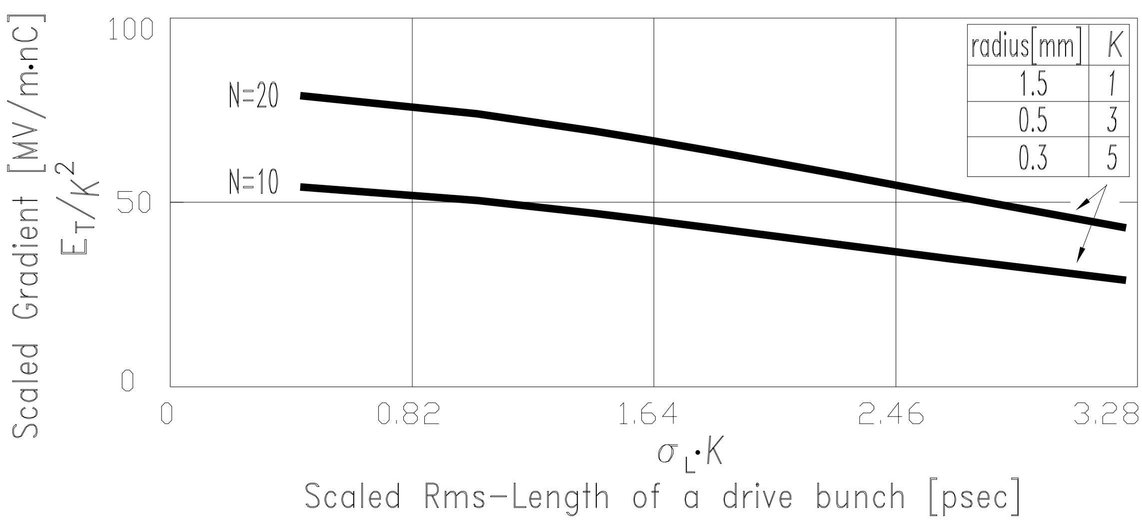

FIG.5. Scaled accelerating gradient $E_T/\widetilde{K}^2$ vs. scaled rms length $\sigma_L \cdot \widetilde{K}$ for two uniform drive trains with N = *10*, and N= *20* bunches, and the test bunch rms-length $\sigma_{TB} = \sigma_L /3$. The curves are given for three radii A = 1.5, 0.5, and 0.3 mm. The radius ratio is $\widetilde{K} = (1:3:5)$. One observes that all curves corresponding to different radii A (but the same N) coincide. In all examples the head/tail ratio σ_1 / σ_2 = 4/1, wake period L = *0.21027 m*.

Figure 6 shows that the enhancement factor always has its maximum at some value of the rms- length, σ_L^{opt}; and under the transformation $A \to A/\widetilde{\mathrm{K}}$, and L = *const,* this maximum behaves as $\sigma_L^{opt} \to \sigma_L^{opt} / \widetilde{\mathrm{K}}$. For instance, with A= 1.5 *mm* (ε = 9.65, and L = *0.21027 m*) one finds $\sigma_{L1}^{opt} \approx 1.640$ *psec.* Changing to A = 0.5 *mm* it will be $\sigma_{L2}^{opt} \approx 0.546$ *psec*, and for A = 0.3 *mm* one will find $\sigma_{L3}^{opt} \approx 0.328$ *psec.* One sees that ($\sigma_{L1}^{opt} : \sigma_{L2}^{opt} : \sigma_{L3}^{opt}$) = (1.5 : 0.5 : 0.3).

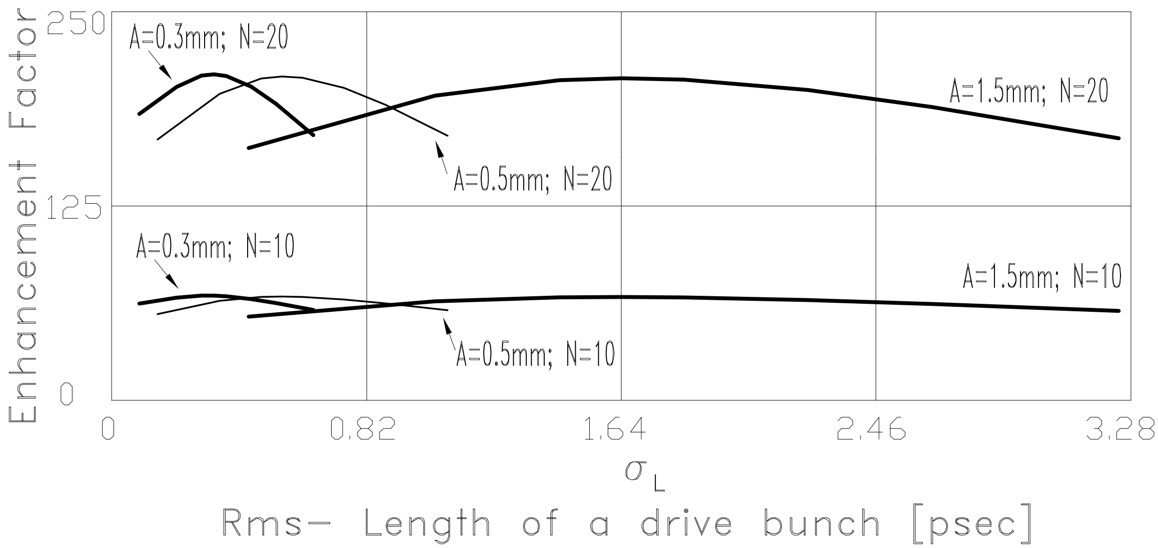

FIG.6. Enhancement Factor ξ vs. drive bunch rms-length σ_L for uniform drive trains with N = *10*, and N= *20* bunches. The curves are given for three radii A = 1.5, 0.5, and 0.3 *mm*. One finds that maxima occur at $\sigma_{L1}^{opt} \approx 1.640$, $\sigma_{L2}^{opt} \approx 0.546$, and $\sigma_{L3}^{opt} \approx$ 0.328 *psec*. One finds that the ratio of these σ_L^{opt} to each other are exactly the same as the ratio of radii, i.e. ($\sigma_{L1}^{opt} : \sigma_{L2}^{opt} : \sigma_{L3}^{opt}$) = (1.5 : 0.5 : 0.3). (In all examples the head/tail ratio σ_1 / σ_2 = 4/1)

Figure 7 demonstrates that the efficiency changes slowly if $\sigma_L \le \sigma_L^{opt}$.

With regard to ***(3.1.1)*** - ***(3.1.3)*** we point out that in investigating the dependence of DWA performances one should always adjust the test bunch position relative to the drive train so that the efficiency, and gradient are at their maximum values. The test bunch position is given by: $z_T = L \cdot (1 - N - 1/2) + \Delta z_k$, where L is the period[g] of the wake field, N is the total number of driving bunches, $|\Delta z_k| \ll L$ and is determined by maximization of the accelerating gradient.

[g] To determine L one maximizes the enhancement factor ξ [see Eq.(2.5)]

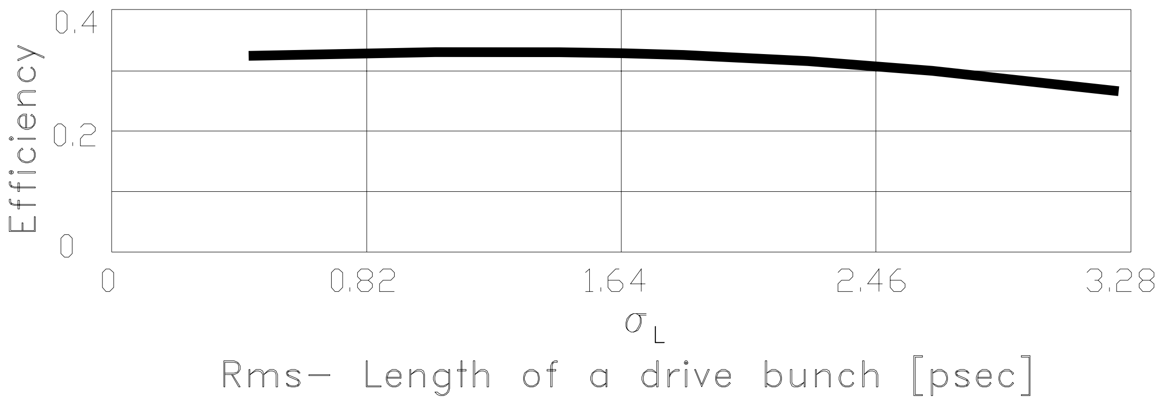

FIG.7. Efficiency vs. rms-length for the uniform *N*= *10* bunch drive train. One finds that for the rms – lengths less than $\sigma_{L1}^{opt} \approx$ 1.640 *psec* the efficiency has almost the same value (this example is for head/tail ratio σ_1 / σ_2 = 4/1, *A* = *1.5 mm*, σ_{TB} = 0.147 *psec*)

(3.1.4) The energy spread does not depend on the drive train profile, assuming that drive bunches can have different changes, but are of the same shape. However, it was found that the shortest possible drive train with the highest possible charge in every single bunch gives the best in efficiency and accelerating gradient, assuming the total drive charge is preserved. Obviously, the simplest realization of a shortest drive train with the highest charge in every bunch is a uniform train. Also to maximize the *MM-DWA* performance one should put a test bunch as close to a drive train as possible, i.e. at $z_T \approx L\cdot(1 - N - 1/2)$.

Adding drive bunches to a driving train, one will have the efficiency, and accelerating gradient computed according to Eqs.(2.8), and (2.10). However, one can also try to redistribute the given drive charge between some number of identical bunches. In this case one has $\sum_{i=1}^{N} Q_i = const$, or equivalently [see Eq.(2.1)] the sum $\sum_{i=1}^{N} q_i = \widetilde{N}$ for any N. Under this condition the efficiency $\widetilde{\eta}_{MAX}$, accelerating gradient $\widetilde{E}_T$, and energy spread $\delta\widetilde{W}$ become:

$$\widetilde{\eta}_{MAX} = \eta_{MAX}\,, \qquad \widetilde{E}_T = E_T \cdot \left(\widetilde{N} / N\right), \qquad \delta\widetilde{W} = \delta W$$

[the expressions for η_{MAX}, E_T, and δW are given by Eqs.(2.8) , (2.10), and (2.11)]

One observes that the efficiency and energy spread do not depend on $\widetilde{N}$. Figure 8 presents the typical example of behavior of the efficiency, accelerating gradient, and energy spread when the total charge of a drive train is preserved at $\sum_{i=1}^{N} Q_i = \widetilde{N} \cdot Q_0$, or accumulates as $N\cdot Q_0$. The efficiency and accelerating gradient achieved after the train with $N = \widetilde{N}$ driving bunches are always lower than the efficiency and gradient after shorter drive trains ($N < \tilde{N}$), but with the total charge $\tilde{N} \cdot Q_0$. The best performance will be achieved if one is able to put all the drive charge into a single bunch.

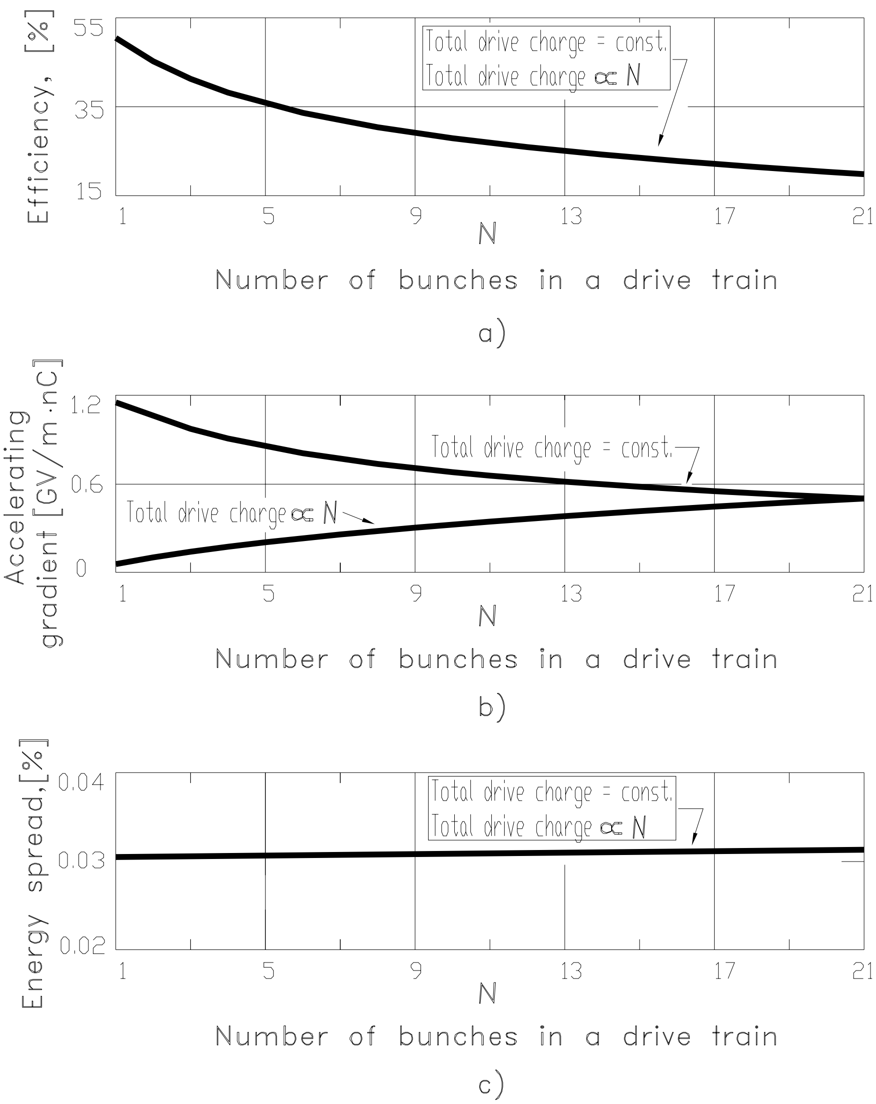

FIG.8. The DWA performance vs. *N*, when the total drive charge = *$21Q_0$* (constant for every *N*), or accumulates as *$N \cdot Q_0$*. Inner Radius *A* = 0.5 *mm*, drive profile = uniform, drive bunch rms-length σ_L = 0.8 *psec*, Gaussian, head/tail ratio σ_1 / σ_2 = 1:4, wake period *L* = 0.22154 *m*, test bunch rms-length σ_{TB} = 0.02 *psec* (In (c) the spread is given for a Gaussian test bunch with σ_1 / σ_2 =1).

The efficiency is reduced when the number of driving bunches, *N*, is increased because **a)** more driving bunches are situated further from a test bunch; **b)** when the distance between a driving bunch and the test bunch is increased, the driving bunch contributes less to the test bunch acceleration.

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B. *MM- DWA* performance vs. the inner radius *A*, chosen wake field period *L*, outer radius *R*, and the dielectric constant ε

The dependence of accelerating gradient on the radius, *A*, of the channel in the dielectric material directly follows from ***(3.1.3).*** Namely, if one reduces $A \to A/\widetilde{\mathrm{K}}$ accompanied by reduction in the bunch sizes $\sigma_L \to \sigma_L/\widetilde{\mathrm{K}}$, and $\sigma_{TB} \to \sigma_{TB}/\widetilde{\mathrm{K}}$ then the accelerating gradient scales as $E_T \to E_T \cdot \widetilde{K}^2$. Thus, changing from *A = 1.5 mm* to *A = 0.5 mm* one increases the gradient by factor of 9. The reduction of bunch length is also essential: for example, if instead of a drive bunch with the rms-length of optimum size, one uses a drive bunch twice longitudinally oversized, the accelerating gradient is reduced by 40%. The efficiency also improves when one reduces the channel radius. For instance, going from *A = 1.5 mm* to *A = 0.5 mm* one improves the efficiency by $\Delta\eta/\eta \approx$ 15%, and by going from *A = 1.5 mm* to *A = 0.3 mm* one has improvement of about $\Delta\eta/\eta \approx$ 35%.

There are two parameters to change the wake period, *L*: the outer radius, *R,* of the metal casing, and the electric permeability, *ε*, of dielectric material. At first we will study how the *MM- DWA* performance behaves when one changes *L* by adjusting *R,* and keeping *ε = const.* Then we will proceed to studying the performance for the opposite case: *R =const.,* while ε is changed to increase/decrease the wake period *L.*

Table 2[э]

	1)	2)	3)	Deviations, [%]
R, [mm]	18.35	9.42	4.95	
L, [m]	0.210248	0.105193	0.052607	
η_{MAX}, %	29.5	29.2	29.3	1.0
E_T, [MV/m/nC]	599.0	596.5	595.5	0.7
δW, [%]	3.18	3.20	3.23	1.6
E^{20}_{loss}, [MeV/m/nC]	492.4	492.5	491.0	0.3
σ_{wake}, [psec]	0.833	0.83	0.826	0.8
M[R]	1	.264	0.073	

[э] Inner radius *A = 0.5 mm, ε =9.65*; Gaussian drive bunch with σ_L = 0.546 *psec* and the head/tail ratio σ_1 / σ_2 =4/1; Gaussian test bunch with σ_{TB} = 0.182 *psec*, σ_1 / σ_2 =1; E^{20}_{loss} - energy losses by the *i* = 20th bunch in a uniform drive train.

Changing the outer radius *R,* but preserving the dielectric constant ε, we find that the accelerating gradient does not change: $E_T(\sigma_L, \sigma_{TB}, A, \varepsilon, R) = E_T(\sigma_L, \sigma_{TB}, A, \varepsilon)$, and the energy losses experienced by every consecutive bunch in a drive train do not change either: $E^i_{loss}(\sigma_L, A, \varepsilon, R) = E^i_{loss}(\sigma_L, A, \varepsilon)$. Also the coherent length remains the same: $\sigma_{wake}(\sigma_L, A, \varepsilon, R) = \sigma_{wake}(\sigma_L, A, \varepsilon)$. An example is given in the Tab. #2.

The overall length of a *MM- DWA* is determined by the accelerating gradient, and thus is independent of changes in *R*. The length of the stage (when a DWA is divided to several stages such that after each stage the drive bunches which fall out of synchronism with the wake field are kicked out, and fresh highly energetic bunches are introduced) is determined by σ_{wake}, and the set of values E^i_{loss}, and thus is independent on *R*. Since, both the overall length, and the stage length do not change, the number of stages remains the same. Consequently, the amount of supporting hardware (such as kicker- magnets, steering, and focusing magnets, etc) remains unchanged if one changes *R*. The only obvious advantage of using the lower *R* is the reduction in the weight of dielectric material. The weight coefficient showing by how much the dielectric mass changes is computed as:

$$M[R] = \frac{R^2 - A^2}{R_1^2 - A^2} \cdot \frac{E_T(R_1, \varepsilon_1)}{E_T(R, \varepsilon)} \tag{2.13}$$

For instance, if $A = 0.5$ *mm*, $\varepsilon = 9.65$, and one goes from $R = 18.35$ *mm* to $R = 4.95$ *mm* (the wake period becomes 4 times shorter) the dielectric weight is reduced by almost 14 times.

Thus, one concludes, that, in general, there are no advantages, or disadvantages in choosing some particular *R*: the number of required elements, total length, and construction expense should stay the same.

The changes in the outer radius, *R,* do not provide any means to reduce the *MM-DWA* cost, but what about ε? We start with an example presented in the Tab. # 3.

First of all, one sees that with higher dielectric constants the efficiency drastically improves. Growth up to 2 times in η_{MAX} is possible when one changes from $\varepsilon \approx 10$ to $\varepsilon \approx 150$. By going to the higher ε one reduces the accelerating gradient, which results in the increment in the *MM- DWA* length if one is planning to preserve the final energy gain. However, the energy losses E^i_{loss} become smaller too, which results in the lengthening of the stage. At the higher ε the number of required sections (stages) is reduced. For instance, changing from $\varepsilon \approx 3 \div 10$ to $\varepsilon \approx 150$ one can reduce the number of stages, and, consequently, the number of kickers, steering, and focusing magnets, and the number of vacuum channels by ≈ 20 %. Thus, some cost reduction is possible when one employs high dielectric constant materials.

Table 3♠

	1)	2)	3)	4)
ε	140.40	35.82	9.65	3.00
L, [m]	0.209370	0.104857	0.052627	0.025782
η_{MAX}, %	34.2	30.0	23.6	19.3
E_T, [MV/m/nC]	107	140	146	131
δW, [%]	1.66	1.17	0.74	0.65
E_{loss}^{20}, [MeV/m/nC]	76.0	108	121	110
σ_{wake}, [psec]	1.58	1.60	1.64	1.63
M[R]	1.30	1	1	1.12
LINAC Length♥	130 %	100 %	100 %	112 %
Stage Length♦ S, *[m]*	2.40	1.70	1.52	1.67
Number of Sections	82 %	90 %	100 %	102 %

♠ Inner radius $A = 1.0\ mm$, outer radius $R = 5.43\ mm$; drive bunch with $\sigma_L = 1.09\ psec$ and $\sigma_1 / \sigma_2 = 1/4$, test bunch with $\sigma_{TB} = 0.182\ psec$; E_{loss}^{20} - energy losses by the $i = 20^{th}$ bunch in a uniform drive train; for M[R] see Eq.(2.13); ♥ to achieve the same final energy; ♦ for the initial energy $W_i^0 = 730\ MeV$, and drive bunch charge = -4 nC

2.5. SUMMARY and DISCUSSION

We have developed and presented a general formalism to describe the interaction between the electron bunches and their wake fields, each having an arbitrarily chosen longitudinal and transverse particle distribution, combined in a train of any desired charge profile. We have analyzed numerous cases and have drawn quantitative conclusions about the behavior of the *MM- DWA* performance: (a) the amount of power radiated by drive bunches into the wake field; (b) efficiency of energy transfer from a drive train to a test bunch; (c) accelerating gradient acting on the test bunch; and (d) energy spread of the test bunch, as affected by changes in the structure, bunch dimensions, and type of a dielectric material. Among other things, we have demonstrated that the efficiency and accelerating gradient are negligibly affected by the longitudinal shape of a drive/test bunch as long as the bunch rms- length is preserved. We have found an important scaling law that provides a straightforward way to connect changes in the DWA performance with changes in the DWA parameters.

We did not consider, however, transverse forces arising due to the radial electric, and azimuthal magnetic fields [see Eq.(2.2)]. The net transverse force acting on some portion of the test/drive bunch is defocusing. Depending on the bunch charge and dimensions, and the time of interaction between the bunch and wake field, the transverse bunch size may grow sufficiently so that the transverse features should be taken into account (see Eq.(2.3)]. Moreover, such bunch defocusing may lead to significant losses of charge on the surface of the dielectric. Defocusing then must be suppressed by the external focusing. The extensive study of transverse forces and their impact on beam dynamics in rectangular structures has been conducted by several authors [4][5][6][7]. An approach to consider the transverse forces in circular structures has been developed in the course of our work; however, it is beyond the scope of this article.

In MM- DWAs using nondispersive dielectrics (i.e. with ε independent of frequency), the number of drive bunches should be limited by $N \approx 20$ because the efficiency becomes low if $N > 20$ [see Fig.8.a.]

An improvement in the efficiency can be achieved by going to the higher dielectric constant ε. A growth of up to 2 times in η_{MAX} is possible when one changes from $\varepsilon \approx 10$ to $\varepsilon \approx 150$. However, the energy spread grows by the same factor of 2, or slightly more. The best tradeoff between the efficiency and energy spread seems to be at $\varepsilon \approx 30$.

An improvement in the accelerating gradient is achieved through the reduction of radius A of the inner channel, accompanied also by corresponding reduction in the longitudinal sizes of drive/test bunches σ_L, and σ_{TB}. While the bunch shape is not crucial at all, the reduction of bunch rms-length is essential. A twice-oversized drive bunch, for instance, causes a reduction in the accelerating gradient by about 40 %.

The energy spread is lowered by reduction of the test bunch rms- length. In the structures with the inner radius of the order of a few millimeters (0.5 ÷ 2 *mm*) where the wake field is excited by drive bunches of the order of a few or less picoseconds ($\sigma_L \le 2$ *psec*), test bunches with the rms- length of a few hundred femtoseconds ($\sigma_{TB} \approx 300 \div 100$ *fsec*) will have the energy spread between $10^{-2} \div 10^{-3}$. The energy spread ~10^{-3} or less is achieved for test bunches with the rms- length of several tens of femtoseconds ($\sigma_{TB} \le 80$ *fsec*).

A mass reduction of a dielectric material can be done by changing the outer radius *R*. Such change has little effect on the accelerating gradient, and other related performance quantities.

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