Duality of the MDM-transparent RF-E Flipper to the EDM transparent RF Wien-Filter at all Magnetic Storage Rings

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Abstract

Yannis Semertzidis came up with a sensational, and initially skeptically received, finding that the EDM-transparent radiofrequency flipper, operated in the $\mathbf{E}^* = \mathbf{0}$ mode, still generates the EDM signal in all magnetic storage ring. Subsequently, Yuri Orlov has produced a theoretical explanation of Yannis' finding for a model RF-E(B) flipper uniformly distributed along a storage ring . In these notes I discuss to more detail the physics of the RF Wien filter for a "pointlike" flipper much shorter than the ring circumference. My principal point is that the duality between the MDM-transparent RF-E flipper, and the EDM-transparent Wien filter, fully extends to the effect of the flipper on the spin coherence time too.

1 Introduction

Recall that the non-vanishing EDM, $\vec{d}$, gives rise to a precession of the spin of a particle in an electric field. In the rest frame of a particle possessing also a magnetic moment $\vec{\mu}$,

$$\frac{d\vec{S}}{dt^*} = \mu\vec{S} \times \vec{B}^* + d\vec{S} \times \vec{E}^*, \tag{1}$$

where in terms of the lab frame fields

$$\begin{aligned} \vec{E}^* &= \gamma(\vec{E} + \vec{\beta} \times \vec{B})\,, \\ \vec{B}^* &= \gamma(\vec{B} - \vec{\beta} \times \vec{E})\,. \end{aligned} \tag{2}$$

What we care about is a rotation of the spin of a stored particle with respect to its momentum. At $\vec{d} = 0$, the vertical component of the spin of stored particles, $\vec{S}_y \parallel \vec{B}$, is conserved, while the in-plane, horizontal components rotates w.r.t. the momentum around the y-axis at spin tune frequency

$$f_S = G\gamma f_R\,, \tag{3}$$

with the precession angle per turn

$$\theta_S = 2\pi G\gamma\,, \tag{4}$$

and pointing at angle

$$\theta(k) = k\theta_S \tag{5}$$

after k turns.

This remarkably model-independent relationship between the spin tune, f_S, and ring, f_R, frequencies derives from the fact that the spin precesses in precisely the same magnetic field which bends a particle momentum.

A complex motion in a ring makes the actual precession angle per kth turn, θ_k, different from the ideal θ_S by a certain phase slip

$$\theta_k = \theta_S + \delta\theta_k \,. \tag{6}$$

A flipper phase $\theta_F(t) = 2\pi f_F t$ is subject to its own phase slips because of the spread of transit times. The central issue of these notes is whether RF flippers, as new elements in a ring, might shorten the spin coherence time (SCT) and thus impede an accumulation of the EDM signal.

A detailed treatment of the effect of a flipper on SCT is given in Lechrach-Lorenz-Morse-Nikolaev-Rathmann (LLMNR). Specifically, they found a possibility of cancellations of the spin tune and flipper phase slip effects in SCT at judiciously chosen energies. Also, they suggested to eliminate the flipper phase slip effects making use of the flattop excitation of the flipper.

In these notes I demonstrate that: (i) the equations for accumulation of the EDM signal in all magnetic storage ring supplemented by either MDM transparent RF-E flipper of EDM transparent RF Wien filter are identical, (ii) the interplay of the spin tune and flipper phase slips in two setups is exactly the same. To cut it short, Yannis has discovered an exact duality between the MDM transparent RF-E flipper and EDM transparent RF Wien filter.

2 BMT equation with EDM

Hereafter we only consider pure radial electric field $\vec{E} \perp \vec{B} \perp \vec{\beta}$, i.e., $(\vec{\beta}\cdot\vec{E}) = (\vec{\beta}\cdot\vec{B}) = 0$. With allowance for nonvanishing EDM,

$$d = \eta e/m \,,$$

the BMT equations take the form (hereafter we only consider pure radial electric field $\vec{E}$)

$$\begin{aligned} \frac{d\vec{S}}{dt} &= \vec{\Omega}\times\vec{S}\,, \\ \vec{\Omega} &= -\frac{e}{m}\left\{G\vec{B} + \left(\frac{1}{\beta^2} - G - 1\right)\vec{\beta}\times\vec{E} + \eta\left(\vec{E} + \vec{\beta}\times\vec{B}\right)\right\}\,. \end{aligned} \tag{7}$$

A scale for CP- and P-allowed MDMs is set by a nuclear magneton, μ_N. In order to get a finite EDM one has to pay a price of $\sim 10^{-7}$ for the weak parity violation and extra price of $\sim 10^{-3}$ for CP-violation, which sets a naive scale for a nucleon EDM at $d_N = \eta\mu \sim 10^{-10}\mu_N$. Which also shows that the expected EDM caused spin rotations are about ten orders in magnitude slower than the MDM caused ones, assuming $E \sim B$.

In the ring proper $\vec{E} = 0$. With the above indicated scale for η, interaction of EDM with the permanent motional electric field $\propto \vec{\beta}\times\vec{B}$, gives an entirely negligible inwards or outwards tilt of the stable spin axis in the ring. Still, the effect of the motional electric field in a ring is tricky, as we shall see below.

3 The spin rotation matrices

3.1 The ring

Here $\vec{\beta} \times \vec{B} = -\beta B \vec{e}_x$, and

$$\begin{aligned} \vec{\Omega} &= -\frac{eB}{m\gamma}\{G\gamma\vec{e}_y - \eta\beta\gamma\vec{e}_x\} = \\ &= -f_R\theta_S(\vec{e}_y - \alpha_R\vec{e}_x), , \\ \alpha_R &= \frac{\eta\beta}{G} \ll 1\,. \end{aligned} \tag{8}$$

In terms of the dimensionless $\tau = f_R t$ the equations of motion take the form

$$\frac{dS_x}{d\tau} = -\theta_S S_z \tag{9}$$

$$\frac{dS_z}{d\tau} = \theta_S S_x + \alpha_R\theta_S S_y \tag{10}$$

$$\frac{dS_y}{d\tau} = -\alpha_R\theta_S S_z \tag{11}$$

A full derivation of the corresponding spin rotation matrix is found in Appendix, here we site the result

$$\vec{S}(k+1) = \hat{R}(\alpha_R, \theta_k)\vec{S}(k)$$

$$\hat{R}(\alpha_R, \theta_k) = \begin{vmatrix} \cos\theta_k & -\alpha_R(1-\cos\theta_k) & -\sin\theta_k \\ -\alpha_R(1-\cos\theta_k) & 1 & -\alpha_R\sin\theta_k \\ \sin\theta_k & \alpha_R\sin\theta_k & \cos\theta_k \end{vmatrix}$$

Interaction of the EDM with motional electric field mixes the vertical and horizontal polarizations.

3.2 Perfect RF-E flipper

We define a perfect RF-E flipper such that the flipper is MDM transparent, i.e., $\vec{\Omega} \parallel \vec{E}$, i.e. when the motional magnetic field is canceled for by a magnetic field of a flipper:

$$G\vec{B} = \left(G + 1 - \frac{1}{\beta^2}\right)\vec{\beta} \times \vec{E} \tag{12}$$

Because we are interested in spin precession with respect to the beam momentum, this condition is different from the naive $\vec{B}^* = 0$, it entails

$$\begin{aligned} \vec{\beta} \times \vec{B} &= -\frac{\beta^2}{G} \cdot \left(G + 1 - \frac{1}{\beta^2}\right) \\ \vec{E} + \vec{\beta} \times \vec{B} &= \frac{G+1}{G\gamma^2}\vec{E}\,, \end{aligned} \tag{13}$$

and, consequently, a nonvanishing Lorentz force and an unwanted excitation of radial betatron oscillations, for mor details see Appendix.

The excitation function of the RF-E flipper, and of the tilt angle thereof, must be either harmonic,

$$\alpha_k = \alpha_E \cos(2\pi f_F t),$$

or flattop,

$$\alpha = \alpha_E \cdot \mathrm{sign}\{\cos(2\pi f_F t)\},$$

subject to the familiar resonance condition

$$\begin{aligned} f_F &= f_S + K f_R\,, && (14)\\ K &= 0, \pm 1, \pm 2, ... && (15) \end{aligned}$$

where

$$\begin{aligned} \alpha_E &= \frac{G+1}{G\gamma^2} \cdot d \cdot t_F E_F\,, && (16)\\ t_F &= \frac{L_F}{\beta} && (17) \end{aligned}$$

and E_F is an amplitude of the electric field. A length, L_F, of the flipper is assumed to be much shorter than the ring circumference, L_R, and we shall treat flipper as a pointlike element.

The corresponding spin rotation matrix equals

$$\hat{R}_F(\alpha_k) = \begin{vmatrix} 1 & 0 & -0 \\ 0 & \cos\alpha_k & -\sin\alpha_k \\ 0 & \sin\alpha_k & \cos\alpha_k \end{vmatrix}$$

3.3 Perfect RF Wien filter

Here we demand a vanishing Lorentz force,

$$\vec{E} + \vec{\beta} \times \vec{B} = 0$$

Simultaneously, the radial component of $\vec{\Omega}$ would vanish entirely, i.e., such a Wien filter is entirely transparent for the EDM of a particle!

The vertical component of $\vec{\Omega}$ is nonvanishing, though. A simple algebra gives

$$\Omega_{WF} = -\frac{e}{m} \cdot \frac{G+1}{\gamma^2} B = -\frac{e}{m} \cdot \frac{G+1}{\gamma^2 \beta} E(t) \qquad (18)$$

and the spin precession angle per kth pass

$$\psi_k = \Omega_{WF} t_F = \psi_E \cos(2\pi f_F t)\,. \qquad (19)$$

Evidently, for a short flipper,

$$\psi_E \ll \theta_S\,.$$

The corresponding spin rotation matrix equals

$$\hat{R}_F(\psi_k) = \hat{R}(0, \psi_k = \begin{vmatrix} \cos\psi_k & 0 & -\sin\psi_k \\ o & 1 & 0 \\ \sin\psi_k & 0 & \cos\psi_k \end{vmatrix}$$

A central feature of ψ_k is that this precession angle comes from an element which does not bend a particle, but precesses the spin. As such, it is extra to the classic spin tune (3) and gives it a frequency modulation

$$\theta(k) = \theta_S f_R t + \sum_0^k \psi_k \,. \tag{20}$$

4 Spin evolution equations

4.1 Perfect RF-E flipper

In the Lehrach-Lorenz-Morse-Nikolaev-Rathmann the effect of the motional electric field in the ring on the accumulation of the EDM signal has been neglected and rightfully so. A demonstration of this property sets also a stage for a discussion of the RF Wien filter.

Our convention is that the beam is injected in front of the flipper, so that one tern starts with pass through a flipper and then a ring proper

$$\hat{R}(k) = \hat{R}(\alpha_R, \theta_k)\hat{R}_F(\alpha_k) \tag{21}$$

. A simple algebra gives

$$\hat{R}(k) = \left| \begin{array}{ccc} \cos(\theta_k) & -[\alpha_R(1-\cos\theta_k) - \alpha_k \sin\theta_k] & -\sin(\theta_k) \\ -\alpha_R(1-\cos\theta_k) & 1 & -\alpha_R\sin(\theta_k) - \alpha_k\cos\theta_k \\ \sin(\theta_k) & +\alpha_R\sin(\theta_k) + \alpha_k\cos\theta_k & \cos(\theta_k) \end{array} \right|$$

and, in terms of the spin components,

$$\begin{aligned} S_x(k+1) &= S_x(k)\cos(\theta_k) - S_y(k)[\alpha_R(1-\cos\theta_k) - \alpha_k\sin\theta_k] - S_z(k)\sin(\theta_k) & (22)\\ S_z(k+1) &= S_x(k)\sin(\theta_k) + S_y(k)[\alpha_R\sin(\theta_k) + \alpha_k\cos\theta_k] + S_z(k)\cos(\theta_k) & (23)\\ S_y(k+1) &= -S_x(k)\alpha_R(1-\cos\theta_k) + S_y(k) - S_z(0)[\alpha_R\sin(\theta_k) - \alpha_k\cos\theta_k] & (24) \end{aligned}$$

It is sufficient to treat the EDM effect to α_E accuracy. Following the LLMNR routine, we define

$$S_z(k) + iS_y(k) = Y(k)e^{-i\theta(k)}$$

and find the evolution of the envelope, $Y(k)$, of the horizontal polarization

$$Y(k+1) = Y(k) + 2\alpha_R S_y(k)\sin(\frac{1}{2}\theta_k)exp^{-i(\theta(k)-\frac{1}{2}\theta_k])} + S_y(k)\alpha_E\cos\theta_F(k)e^{-i\theta(k)}\,. \tag{25}$$

Because α_R is constant, the corresponding oscillating contribution from the motional electric field averages out, while

$$\cos\theta_F(k)e^{-i\theta(k)} \Longrightarrow \cos\theta_F(k)\cdot\cos\theta(k) \Longrightarrow \frac{1}{2}\cos(\theta_F(k) - \theta(k))$$

and this equation takes precisely the form of the LLMNR equation

$$Y(k+1) = Y(k) + \frac{1}{2}\alpha_E S_y(k)\cos(\theta_F(k) - \theta(k)) \tag{26}$$

which furnishes a formal proof that, in the setup with perfect RF-E flipper, the motional electric filed of the ring does not affect the long time accumulation of the EDM signal.

4.2 Perfect RF Wien filter

In this case $\hat{R}(k) = \hat{R}(\alpha_R, \theta_k)\hat{R}(0, \psi_k)$ and

$$\hat{R}(k) = \left| \begin{array}{ccc} \cos(\theta_k + \psi_k) & -\alpha_R(1 - \cos\theta_k) & -\sin(\theta_k + \psi_k) \\ \alpha_R\{\cos(\theta_k + \psi_k) - \cos\psi_k\} & 1 & -\alpha_R(\sin(\theta_s + \psi_k) - \sin\psi_k)\} \\ \sin(\theta_k + \psi_k) & \alpha_R \sin(\theta_k) & \cos(\theta_k + \psi_k) \end{array} \right|$$

which in terms of the spin components boils down to

$$\begin{aligned} S_x(k+1) &= S_x(k)\cos(\theta_k + \psi_k) - S_y(k)\alpha_R(1 - \cos\theta_k) - S_z(k)\sin(\theta_k + \psi_k) && (27) \\ S_z(k+1) &= S_x(k)\sin(\theta_k + \psi_k) + S_y(k)\alpha_R\sin(\theta_k) + S_z(k)\cos(\theta_k + \psi_k) && (28) \\ S_y(k+1) &= S_x(k)\alpha_R\{\cos(\theta_k + \psi_k) - \cos\psi_k\} + S_y(k) - \\ &- S_z(k)\alpha_R(\sin(\theta_s + \psi_k) - \sin\psi_k)\} && (29) \end{aligned}$$

The envelope of the horizontal spin, $Y(k)$, defined with allowance for the Wien filter contribution, $\psi(k)$, to the spin precession,

$$S_z + iS_x = Y(k)e^{-i[\theta(k)+\psi(k)]}\,. \qquad (30)$$

satisfies an equation

$$Y(k+1) \;=\; Y(k) + 2\alpha_R \sin\frac{1}{2}\theta_S S_y(k)\exp\{i[\theta(k) + \psi(k) + \frac{1}{2}\theta_k]\} \qquad (31)$$

Now we make use of an explicit form

$$\psi(k) = \psi_E \sum_0^k \cos k\theta_F = \psi_E \cdot \frac{1}{2\sin\frac{1}{2}\theta_F}\left(\sin\frac{1}{2}\theta_F + \sin(\theta_F(k) + \frac{1}{2}\theta_F)\right)\,, \qquad (32)$$

expand in $\psi(k) \ll 1$ and, following the LLMNR procedure, suppress all rapidly oscillating terms

$$\begin{aligned} e^{+i[\theta(k)+\psi(k)+\frac{1}{2}\theta_k]} &= (1 + i\psi(k))e^{-i(\theta(k)+\frac{1}{2}\theta_S)} \Longrightarrow \\ &\Longrightarrow -\psi_E\frac{1}{2\sin\frac{1}{2}\theta_S}\sin(\theta_F(k) + \frac{1}{2})\theta_F)\sin(\theta(k) + \frac{1}{2}\theta_S) \Longrightarrow \\ &\Longrightarrow \psi_E\frac{1}{4\sin\frac{1}{2}\theta_S}\cos\{\theta_F(k) - \theta(k)\} && (33) \end{aligned}$$

which gives the final equation for the buildup of the horizontal spin

$$Y(k+1) \;=\; Y(k) + \frac{1}{2}\alpha_R\psi_E S_y(k)\cos\{\theta(k) - \theta_F(k)\}\,. \qquad (34)$$

which is an exact counterpart of the LLMNR equation (26).

The same similarity holds also for the evolution of $S_y x(k)$ in the two setups.

5 Duality between the ideal RF-E flipper and RF Wien flipper

The Wien filter evolution equation (34) and the LLMNR evolution equation for RF-E flipper (26) look very much the same. As a matter of fact, they are just identical.

For both setups a resonance condition is the same:

$$\begin{aligned} \theta_F &= \theta_S + 2\pi K \\ f_F &= f_S + f_R K \end{aligned} \tag{35}$$

Recall that for a perfect RF-E flipper

$$\alpha_E = \frac{G+1}{G\gamma^2} \cdot \frac{\eta e}{m} \cdot E_F \tau_F \,, \tag{36}$$

and it is easy to check that

$$\alpha_R \theta_E \equiv \alpha_E \tag{37}$$

This accomplishes a proof that pure magnetic storage rings, supplemented with the MDM-transparent perfect RF-E flipper, and the EDM-transparent RF Wien filter, do accumulate the EDM signal at identical rates.

Finally, an interplay of the spin tune phase slip and of the flipper phase slip is identical for both setups. Just recall that with allowance for the beam momentum spread the spin precession angle acquires the phase slip

$$\theta(k) = k\theta_S + \Delta(k) \,.$$

Exactly the same momentum spread gives the transit time slip, which translates into

$$\theta(k) = k\theta_F + \frac{f_F}{f_S} \cdot \frac{\eta}{\beta^2} \cdot \Delta(k) \,.$$

so that

$$\theta(k) - \theta_F(k) = k(\theta_S - \theta_F) + \left(1 - \frac{f_F}{f_S} \cdot \frac{\eta}{\beta^2}\right) \Delta(k)$$

and a condition for the decoherence-free spin rotation by a flipper

$$C = 1 - \frac{f_F}{f_S} \cdot \frac{\eta}{\beta^2} = 0$$

is precisely the same for both setups, which consequently would have identical SCT properties. The elimination of the flipper phase slip effects by going to the flattop excitation of the flipper would work for both setups.

An educated guess is that whatever a strength of the vertical in-phase magnetic field in the flipper , it can be viewed as a superposition of the perfect RF-E flipper and a perfect RF Wien filter, so that at a fixed amplitude of the overall RF electric field, the EDM signal would stay put. The point that the RF Wien filter does not excite unwanted betatron oscillations, makes it a setup of the choice.

One caveat is that above duality is exact only at a nominal energy. Off the nominal energy, the RF Wien filter is no longer the EDM transparent one, there will under- or over-compensated radial electric field $\propto \delta p/p$. Because the spin precession and flipper phase slips are also $\propto \delta p/p$, the imperfection of the Wien flipper might interfere in a nontrivial way with the phase slip corrections. Such corrections might prove different for two setups, which requires further scrutiny.

As discussed in LLMNR, the effects of the RF-E EDM-flipper with pure radial electric can readily be modeled experimentally by RF-B MDM-flipper with pure radial magnetic field. However, there doesn't seem to be any way of modeling the RF Wien filter rather than the Wien filter itself.

Appendix: Technicalities of derivations and miscellanea

Cooling the RF-E-excited radial betatron oscillations

An obvious drawback of an ideal RF-E flipper is an oscillating Lorentz force which excites the radial betatron oscillations. Because the excitation function of these betatron oscillations is exactly known, they can readily be cooled by a second flipper with a separation corresponding to the horizontal betatron phase shift $\theta_B = \pm\pi$, run with the phase shift,

$$\Delta\theta_F = -\frac{G\gamma}{Q_x}\theta_B\,, \tag{38}$$

to compensate for tame of flight from first to second flipper. Still arranging for such a cancellations would demand special ring optics and involve a ring section with large betatron oscillations.

Derivation of the ring spin rotation matrix

In terms of the dimensionless $\tau = f_R t$ the equations of motion take the form

$$\frac{dS_x}{d\tau} = -\theta_S S_z \tag{39}$$

$$\frac{dS_z}{d\tau} = \theta_S S_x + \alpha_R \theta_S S_y \tag{40}$$

$$\frac{dS_y}{d\tau} = -\alpha_R \theta_S S_z \tag{41}$$

Their solution for a single pass of a ring proceeds as follows:

Upon introduction of a convenient complex $Z = S_z + iS_x$ the equations (39) and (40) combine into

$$\frac{dZ}{d\tau} = -i\theta_S Z + \alpha_R \theta_S S_y \tag{42}$$

$$Z = Z(0)e^{-i\theta_S\tau}$$

which describes the idle rotation of the horizontal spin:

$$\begin{aligned} Z(\tau) &= Z(0)e^{-i\theta_S\tau} \\ S_x(\tau) &= S_x(0)\cos(\theta_S\tau) - S_z(0)\sin(\theta_S\tau)\,, \\ S_x(\tau) &= S_x(0)\sin(\theta_S\tau) + S_z(0)\cos(\theta_S\tau)\,. \end{aligned} \tag{43}$$

while the vertical polarization is conserved, S_y =const.

To the next order in α_R we search for a solution

$$Z = Ye^{-i\theta_S\tau}$$

subject to a boundary condition $Y(0) = S_z(0) + iS_x(0)$, which yields an equation

$$\frac{dY}{d\tau} = \alpha_R \theta_S S_y e^{i\theta_S\tau} \tag{44}$$

which must be solved with S_y taken to the lowest order approximation:

$$Y(\tau) = Y(0) - i\alpha_R S_y(0)(e^{i\theta_S \tau} - 1)$$

Simple algebra gives the following result at $\tau = 1$, i.e., the relevant spin rotation matrix of the horizontal spin components per single turn, :

$$\begin{aligned} S_x(1) &= S_x(0)\cos\theta_S - S_y(0)\alpha_R(1-\cos\theta_S) - S_z(0)\sin\theta_s \\ S_z(1) &= S_x(0)\sin\theta_S + S_y(0)\alpha_R\sin\theta_S + S_z(0)\cos\theta_s \end{aligned} \tag{45}$$

Next we solve (41) with the idle precession solution for S_z, and find the third row of the spin rotation matrix per singe turn:

$$\begin{aligned} S_y(1) &= S_y(0) - \alpha_R\theta_S \int_0^1 d\tau S_z(\tau) = \\ &= -S_x(0)\alpha_R(1-\cos\theta_S) + S_y(0) - S_z(0)\alpha_R\sin\theta_s \end{aligned} \tag{46}$$

Rapidly oscillation driving force and slow rotation

In all cases our interest is EDM-driven slow rotation of the spin on the background of rapid spin precession, which introduce functions oscillation with the spin tine frequency. In all cases the envelope, $Y(k)$, of the horizontal spin and the vertical spin $S_y(k)$ are two very slow variables of k, and we can go differential in k:

$$\frac{d}{dk}\vec{X}(k) = \hat{R}(\theta(k), \psi(k))\vec{X}(k) \tag{47}$$

where $\vec{X} = (\vec{Y}, S_y)$. A formal solution is $\vec{X}(k) = \hat{U}(k)\vec{X}(o)$, where $\hat{U}(k)$ is the k-ordered evolution operator (the $\hat{R}$s take at different time do not commute),

$$\hat{U}(k) = \mathrm{T}_k\left\{\exp\left[\int_0^k dn\hat{R}(\theta(n), \psi(n))\right]\right\}\,, \tag{48}$$

which makes it obvious that all components in $\hat{R}(\theta(n), \psi(n))$, which oscillate rapidly with the spin tune and flipper frequences, in the long run would oscillate with fixed amplitude and wouldn't generate any sustained spin rotation. The only relevant terms in the exponent are those which depend on $\{\theta_F(k) - \theta(k)\}$, what vanishes under the resonance condition.

Filtering out rapid oscillations from evolution of S_y with RF Wien filter

A derivation of the evolution of S_y is a bit more tedious, as the effect of ψ_k in the overall rotation matrix enters alongside the frequency modulation of the spin tune. With real valued Y(k) we have

$$S_x(k) = Y(k)\sin(\theta(k) + \psi(k)) = Y(k)\left\{\sin\theta(k) + \psi(k))\cos\theta(k)\right\}\,, \tag{49}$$

$$S_z(k) = Y(k)\cos(\theta(k) + \psi(k)) = Y(k)\left\{\cos\theta(k) - \psi(k)\sin\theta(k)\right\}\,. \tag{50}$$

The relevant expansion for the elements of the rotation matrix is

$$\cos(\theta_k + \psi_k) - \cos\psi_k = -(1 - \cos\theta_s) - \psi_k \cos\theta_S\,, \tag{51}$$
$$\sin(\theta_s + \psi_k) - \sin\psi_k = \sin\theta_s - \psi_k(1 - \cos\theta_s) \tag{52}$$

Here it is convenient to represent $\psi(k)$ as

$$\psi(k) = \frac{1}{2}\psi_E\{1 + \cos\theta_F(k) + \cot\frac{1}{2}\theta_k \sin\theta_F(k)\} \tag{53}$$

and upon the elimination of rapidly oscillations terms,

$$\psi(k)\cos\theta(k) \Longrightarrow \frac{1}{2}\psi_E \cos\theta(k)\cos\theta_F(k)\,, \tag{54}$$
$$\psi(k)\sin\theta(k) \Longrightarrow \frac{1}{2}\psi_E \cot\frac{1}{2}\theta_S \sin\theta(k)\sin\theta_F(k)\,, \tag{55}$$
$$\psi_k \sin\theta(k) \Longrightarrow 0\,, \tag{56}$$
$$\psi_k \cos\theta(k) \Longrightarrow \theta_E \cos\theta(k)\cos\theta_F(k)\,. \tag{57}$$

we find the evolution equation for S_y will read

$$S_y(k+1) = S_y(k) - \frac{1}{2}\alpha_R \psi_E S_y(k)\cos\{\theta(k) - \theta_F(k)\}\,. \tag{58}$$

In view of the identity (36) it coincides with the LLMNR equation for perfect RF-E flipper.

Injection of misaligned spin

So far we discussed the case of real valued $Y(0) = S_z(0) + iS_x(0)$. As explained in LLMNR, in this case the EDM-driven rotation of the spin proceeds in the co-rotating plane with the vertical y-axis as its rotation axis. The co-rotating plane, and the accumulated horizontal spin thereof, point at a median angle

$$\theta_M = \theta(k) + \psi(k)\,. \tag{59}$$

The co-rotating plane rotates with the non-uniform frequency.

As a matter of fact, a complex valued $Y(k)$, i.e., the initial condition with $S_x(0)$ is well possible. The evolution term in the r.h.s. of equation (34) is real valued and the imaginary component of $Y(k)$ does not change with time. As explained in LLMNR, the $S_x(0) \neq 0$ component of the misaligned spin stays put and orthogonal to the co-rotating plane and does not participate in the EDM-caused rotation of the spin which couples S_y and ReY.