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Title: Principal Scheme of a Deuteron EDM Ring with a Long Spin Coherence Time. (Cancellation of the Second-Order Perturbations in $\Delta\omega_\alpha$)

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Principal Scheme of a Deuteron EDM Ring With A Long Spin Coherence Time. (Cancellation Of The Second-Order Perturbations In $\Delta\omega_a$)

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1. The basic idea.

The goal of this EDM Note is to show how we can cancel the second-order effects of betatron and synchrotron oscillations violating the main condition of our deuteron EDM experiment ,

$$\omega_a = \frac{e}{m}\left\{ |a| B_V - \left[|a| + \left(\frac{m}{p}\right)^2 \right] \beta E_R \right\} = 0\,, \quad \text{ideal case.} \tag{1.1}$$

The basic idea is the following. Consider, for example, the violation of (1.1) by horizontal betatron oscillations, *x(s)*, *s* is the longitudinal coordinate,

$$x_\beta(s) = A_x \sqrt{\beta_x(s)} \cos\left(\psi_x(s) + \delta_x\right). \tag{1.2}$$

(The subscript β distinguishes free betatron oscillations from closed orbits depending on $\Delta p / p$. We will omit this subscript when it will not lead to the reader's confusion.) In (1.2), I use the Handbook [1] expression for *x(s)*, p.49, section 2.1, formula (2), in which the dimension of A_x is $m^{1/2}$, not m . Any terms linear in x in (1.1) are averaged to zero over time, but the quadratic are not. As a result, we have there the horizontal pitch effects proportional to $\langle x^2 \rangle$ and $\langle \vartheta_x^2 \rangle \equiv \langle (dx/ds)^2 \rangle$, summarized as

$$(\Delta\omega_a)_{\substack{betatron,\\ horizont.}} \equiv \Delta\omega_{a1} = a_1 A_x^2\,. \tag{1.3}$$

The factor a_1 depends on the lattice structure. (1.3) is an incoherent, individual perturbation of ω_a, different for particles having different amplitudes A_x. Therefore, it leads to a steady loss of the beam polarization P in time. Our goal is, of course, to prolong the coherence time as much as possible. So if we want to keep our beam polarized up to, say, 1 minute, while permitting the betatron amplitudes $x_{\max}$ ~2.5cm, then, as we can see from formula (5.20) below, we need to cancel effect (1.3) with accuracy ~ 0.1% . The basic idea analyzed in this Note is to cancel this and other quadratic perturbations with the help of magnetic sextupoles properly distributed along a properly designed EDM ring.

Sextupole #k produces the nonlinear—quadratic—field

$$B_k \equiv B_k''(x^2 - y^2)/2, \; B'' \equiv \partial^2 B/\partial x^2 \tag{1.4}$$

(x, y are horizontal and vertical coordinates; numbering marks both sextupoles and their places in the ring.) This field shifts the betatron equilibrium orbit either inward or outward, changing the balance of fields in (1.1). (On the shift of equilibrium in the simple case of nonlinear oscillators with constant coefficients, see Mechanics, [2], section 28.) This is the main reason for using sextupoles. We can arrange a distribution of the sextupoles along the orbit such that effect (1.3) will be canceled.

It is more or less obvious in advance that, using the same distribution, we can cancel the vertical pitch effect,

$$(\Delta\omega_a)_{\substack{betatron,\\ vertical}} \equiv \Delta\omega_{a2} = a_2 A_y^2 . \tag{1.5}$$

(A_y is defined as in (1.2) with $x \to y$). This is possible because, as already noted, the field (1.4) depends on both *x* and *y* quadratic deviations. The cancellation of (1.3) and (1.5) together is quite similar to the well-known cancellation of both horizontal and vertical chromaticities.

Less obvious, we can also cancel quadratic off-momentum effects violating (1.1),

$$(\Delta\omega_a)_p \equiv \Delta\omega_{a3} = a_3(\Delta p/p)^2 , \tag{1.6}$$

by the same distribution of the sextupoles. We will see that it is certainly impossible in the simple FODO lattice, but it seems possible in a ring with big straight sections, in which we can manipulate the beam's closed orbits. It is important to note that violation (1.6), not so dangerous as (1.3), (1.5), can be controlled rather easily by other means. The effects (1.3), (1.5), and (1.6) are the only types of quadratic perturbations violating condition (1.1).

The quadratic off-momentum effect (1.6) needs clarification. There exists, of course, not only a quadratic, but a linear effect violating (1.1), $(\Delta\omega_a) \propto \Delta p/p \neq 0$. However, in the linear approximation, it is canceled on the average (meaning average in time) by introducing synchrotron oscillations of the particle momenta with the help of the corresponding RF cavities, $\Delta p/p = (\Delta p/p)_0 \cos(\omega_s t + \phi)$. However, synchrotron oscillations are not exactly linear. There are various quadratic terms in the synchrotron equations, different for different particles, and these terms shift the equilibrium momenta of these particles. As a result, the main off-momentum effect violating (1.1) is not zero, and is not linear:

$$\Delta p/p = \langle \Delta p/p \rangle + (\Delta p/p)_0 \cos(\omega_s t + \phi), \tag{1.7}$$

where $\langle \Delta p/p \rangle$ is shifted from zero value by second-order effects proportional to $\langle A_x^2 \rangle$, $\langle A_y^2 \rangle$, and $\langle (\Delta p/p)^2 \rangle$. Such a shift influences all three factors a_1, a_2, and a_3, so we can use it in our design to control (1.1) by manipulating synchrotron equilibrium, i.e., $\langle \Delta p/p \rangle$. The principal possibility of controlling all three factors by sextupoles arises from the fact that the full horizontal oscillations $x(s)$ in (1.4) contain both betatron and synchrotron oscillations,

$$x(s) = x_\beta(s) + D(p,s)\Delta p/p, \tag{1.8}$$

$x_\beta(s)$ from (1.2). (About function *D(p,s)*, see [1], p. 50.) Therefore, on the average,

$$\langle x^2 \rangle = \langle x_\beta^2 \rangle + D^2 \langle (\Delta p/p)^2 \rangle. \tag{1.9}$$

(The uncounted linear term, $2Dx_\beta(p/p)$ plays an important role in the problem of the free betatron oscillations' chromaticity. But it is not our main concern here.) Thus, on the average, field (1.4) of sextupole #k is

$$\langle B_k \rangle = B_k'' \left[\langle x_\beta^2 \rangle_k - \langle y_\beta^2 \rangle_k + D_k{}^2 \langle (\Delta p/p)^2 \rangle \right]. \tag{1.10}$$

In this Note I show that, with the help of the sextupoles, the proposed accuracy of $\omega_a = 0$ is possible in principle. But the calculations for the final design need much more work. For example, I have used here a thin lens approximation, which is not exactly realistic. I have done so because such an approximation is very transparent, almost all effects can be represented analytically, and many preparatory formulas can be verified by [1]. A number of higher-order effects which are needed in order to know a_1, a_2, and a_3 more precisely—with the accuracy 10^{-3} —are not taken into account here. Also, in order not to complicate the main subject of this Note, I have not included the acceleration of particles by the radial electric field into the calculations.

2. Main elements of the lattice of a deuteron EDM ring.

The size of such a ring, Fig. 1, is more or less defined by an assumed magnitude of the radial electric field, E_R, and the desirable momentum of deuterons, *p*. Certainly, we need *p*>0.6 GeV/c. I have assumed the electric field at the equilibrium orbit, E=4 MV/m=1.3333×10^{-2} T, and the equilibrium deuteron momentum, *p*=0.788 GeV/c. This gives me the equilibrium orbit radius inside the BE sections, R=15 m, and the magnetic field there, B=0.2095 T. "BE" means the combination of the vertical magnetic field, B_V, and the radial electric field, E_R. At the ideal equilibrium orbit, condition (1.1) holds, where a=-0.143, m=1.8756 GeV. The related formulas I have used are

$$p = 0.3(B - E/\beta)R, \tag{2.1}$$

$$B = \frac{1}{|a|}\left[|a| + \left(\frac{m}{p}\right)^2\right]\beta E, \tag{2.2}$$

$$E = \frac{1}{0.3R} \times \frac{|a|p\beta}{\beta^2\left[|a| + \left(\frac{m}{p}\right)^2\right] - |a|}, \tag{2.3}$$

$$p = 0.3ER\left[\frac{|a| + \left(\frac{m}{p}\right)^2}{|a|}\beta - \frac{1}{\beta}\right], \tag{2.4}$$

$\beta = v/c = 1/\sqrt{1+(m/p)^2}$, p in GeV/c, B and E in T , R in meters.

Thus,

p=0.788, E=1.3333×10^{-2} T, R=15, B=0.2095, $\beta = 0.3873$, $\beta^2 = 0.15$, $(m/p)^2 = 5.6654$,
$\gamma^2 = 1 + (p/m)^2 = 1.1765$, $\gamma = 1.0847$, $f_c = 0.9306$ MHz, $\frac{1}{2\pi}\frac{\mathrm{e}}{\mathrm{m}}|a|\mathrm{B}_v = 144.34$ kHz (2.5)

Any design of a deuteron EDM ring must obey the following physical conditions:

I. The ideal ring must be symmetric with respect to the clockwise (CW) and counterclockwise (CCW) movements of the deuterons. During the CCW runs, the sign of the magnetic field must be changed not only in the magnetic dipoles, but in all magnetic elements of the lattice shown in Figs. 2, 3. (In particular, a ring version designed for muons in [3], where the magnetic lens currents must be not changed, is not acceptable in the much more precise deuteron

EDM experiment.) The electric field is not changed. Thus, in the "backward CCW movie," the sequence of the ring elements met by the deuterons must be exactly the same as in the "forward CW movie." This is the main condition for cancellation of all three main spin perturbations imitating the deuteron EDM: (a) E_V; (b) Alternating along the orbit $B_L(s)$ and $\Delta\omega_a$(s) perturbations. (c) Similarly alternating B_R(s) and $\Delta\omega_a$(s) perturbations. (b) and (c) are second-order effects caused by sequences of non-commutative perturbations of the spin, as was explained in [4]. It has been formally proved [4] that the CW-CCW cancellation method is applicable to them. (For historical reasons, perturbations (b) and (c) are called "twist" and "saucer.")

Probably, the best lattice from the CW-CCW symmetry point of view would be a circular FODO ring having a single period, for example, $...FO_1F(BE)DO_2D(BE)FO_1F...$, where F and D are focusing and defocusing lenses, O_1 and O_2 are free intervals. (About FODO in general, see Handbook [1], p.60., section 2.2.3.) However, another condition—the cancellation of the second order effects in $\Delta\omega_a$, the subject of this Note—needs more parameters than such a lattice possesses, see IV below.

II. The vertical magnetic field B_V and the radial electric field E_R cannot be separated in space [5]. That is, the (g-2)-cancellation condition (1.1) must be fulfilled as much as possible <u>locally</u>, at every azimuth s, $\omega_a(s) \equiv 0$, and not only on the average, $\langle \omega_a(s) \rangle = 0$ [5]. Indeed, if our lattice were an alternating sequence of the electric and magnetic fields, like $...B_V...E_R...B_V...E_R...$, with $\omega_a = 0$ only on the average, then we would artificially create huge high modes of the $\Delta\omega_a$ -perturbation along the orbit, $\Delta\omega_a(s) \sim \omega_{a0}\cos(k\omega_C s/v)$ ($\omega_{a0} = aeB_V/mc$, k is the mode number), though (1.1) would not be violated on the average. If such an artificially created high mode of $\Delta\omega_a(s)$ were combined with an accidental perturbation of the magnetic field—either longitudinal field $\Delta B = B_L$(s) or radial field $\Delta B = B_R$(s)—of the same high mode along the orbit but a different phase, then we would get a very big "twist" or "saucer." Obviously, similar perturbations will be much-much smaller in the ring with <u>local</u> cancellation of the g-2 rotations.

III. There must be a synchrotron stability of the deuteron momenta, p, that is, the corresponding RF cavities must be installed in the ring. Otherwise, any reasonable initial spread of the particle momenta, Δp, around the ideal momentum would violate condition (1.1). For example, the beam with $(\Delta p/p) \sim 10^{-3}$ is depolarized after 5-10 ms. (We are talking about depolarization in the horizontal plane.) The reason is that in a strong focusing system, which is the most desirable choice for a deuteron EDM ring, the average radius of the orbit only slightly depends on momentum p in formula (2.1): $\langle \Delta R(s)/R(s) \rangle << \Delta p/p$. The term $E/\beta << B$ is small in (2.1). Therefore, according to (2.1), no matter how we construct our strong focusing lattice, we will have violation $\langle \Delta B_V / B_V \rangle \sim +\Delta p/p$ of the condition (1.1) by the off-momentum magnetic field. Simultaneously, the main factor before the electric field in (1.1), $-\beta(m/p)^2$ is also changed, $-\Delta(\beta m^2 p^{-2})/(\beta m^2 p^{-2}) \sim +\Delta p/p$. Thus, these two linear $\Delta p/p$ effects are summarized, and not mutually canceled in (1.1), and only synchrotron stabilization can solve the problem. In the presence of synchrotron stability, in the linear approximation, different momenta of different deuterons, $p = p_0 + \Delta p$, oscillate around the same p_0.

IV. The cancellation of the g-2 rotations proportional to linear deviation $\Delta p/p$ is not sufficient for our accuracy. For example, the horizontal and vertical pitch effects analyzed in this Note will depolarize the beam in the course of some several hundred ms, instead of the designed 10 or more seconds. This will limit our ability to analyze and correct the major systematic errors. We ultimately need to cancel quadratic betatron effects (1.3) and (1.5). This can be easily done even in the simplest FODO ring. But simultaneous cancellation of (1.6) cannot be made in an arbitrary ring. (Without knowing this, I have designed all my previous versions of the deuteron EDM ring as similar to such FODO rings.) For this reason, we need big straight sections, as in Fig. 1—which would be useful also for other purposes. Big straight sections are the identity matrix sections. That is, a particle with horizontal and vertical coordinates $(x, x'; y, y')$ at the entrance of the section is transported in such a way that its coordinates $(x, x'; y, y')$ are exactly the same at the exit of this section.

3. The relevant lattice formulas and parameters.

First of all, we need to specify the B and E fields in the BE section. If the electric plates are infinite straight vertical plates, as we assume here, then the ideal electric field

$$E = E_R(x) = \frac{E_0 R_0}{R} = \frac{E_0}{(1 + x/R_0)}. \tag{3.1}$$

It is not unreasonable to design the ideal magnetic field of the BE sections so that it has similar behavior in the horizontal plane:

$$B(y=0) = B_V(x) = \frac{B_0 R_0}{R} = \frac{B_0}{(1 + x/R_0)}. \tag{3.2}$$

For $y \neq 0$, keeping only terms linear and quadratic in *y*,

$$B_V = \frac{B_0 R_0}{R}\left[1 - \frac{1}{2}\left(\frac{y}{R}\right)^2\right], \quad \text{inside BE,} \tag{3.3}$$

$$B_R = -\frac{B_0 R_0}{R^2} y, \quad \text{inside BE.} \tag{3.4}$$

This choice is rather arbitrary and needs to be compared in future with alternative choices. Different choices produce slightly different factors a_1, a_2, a_3 in formulas (1.3), (1.5), (1.6). The advantage of our choice of BE fields is that for particles with the ideal momentum, condition $\omega_a = 0$ holds for any x (but not for any dx/ds) in the central plane of the BE. The magnetic field index for field ((3.3)-(3.4), $n = -R(\partial B/\partial R)/B = 1$, equals 1 for every x in the central plane. If the particle is moving in the central plane, *y*=0, and in parallel to the ideal orbit, *x*=constant, then in all approximations there are no focusing forces either from E or from B fields because the path length, $dl = ds(1 + x/R)$, is going up while the fields are going down with *x as* $1/(1 + x/R)$, see Fig. 4a. In this case, the fields (3.1), (3.2), averaged over the particle trajectory inside BE's, always equal E_0 and B_0 independently of *x*, so there are no violations of condition (1.1) either.

However, perturbed trajectories are not parallel to the central axis. Correspondingly, there exist effects violating (1.1) and proportional to $\vartheta_x^2 \equiv (dx/ds)^2$. We will take them into account. (We will neglect only the average effects of acceleration in the horizontal E_R-field, which are also proportional to ϑ_x^2.) Nonzero $(dx/ds)^2$ and $(dy/ds)^2$ play the major role in the generalized horizontal and vertical pitch effects considered in the following sections. In particular, the lengthening of the trajectories due to *x* and *y* oscillations, see Fig. 4b

$$\Delta L = L - L_0 = \int_0^{L_0} ds\sqrt{(1 + x/R)^2 + (dx/ds)^2 + (dy/ds)^2} - L_0, \tag{3.5}$$

is the biggest contribution to a_i's. From (3.5), in the second-order approximation,

$$\frac{\Delta L}{L} = \left\langle \frac{x}{R} \right\rangle + \frac{1}{2}\left\langle \vartheta_x^2 \right\rangle + \frac{1}{2}\left\langle \vartheta_y^2 \right\rangle. \tag{3.6}$$

(3.6) is a purely geometrical effect. With respect to betatron oscillations, $\left\langle \vartheta_x^2 \right\rangle$ is proportional to A_x^2, and $\left\langle \vartheta_y^2 \right\rangle$ to A_y^2. In the synchrotron region, $\vartheta_x^2 = (D'(s))^2(\Delta p/p)^2$. Due to various nonlinear terms in the betatron equations, $\langle x/R \rangle$ itself depends on second-order effects,

$$\left\langle \frac{x}{R} \right\rangle = (\alpha_0 + \alpha_1 \frac{\Delta p}{p})\frac{\Delta p}{p} + \alpha_2\left(\frac{\Delta p}{p}\right)^2 + q_x A_x^2 + q_y A_y^2, \tag{3.7}$$

The meaning of α_0, α_1, α_2, q_x, q_y will be clear later in this Note. (3.7) is a purely betatron dynamics effect. In (3.6) and (3.7), $\langle\ \rangle$ means averaging in time high frequency betatron oscillations, while $\Delta p/p$ is considered approximately constant in time. Then (3.6), (3.7) go into the equations for slow synchrotron oscillations. In the next section we will show that $\langle \Delta p/p \rangle$, being averaged over synchrotron oscillations, is shifted by all kinds of quadratic terms from its linear equilibrium $\Delta p/p = 0$, so

$$\left\langle \frac{\Delta p}{p} \right\rangle \propto \text{individual quadratic terms.} \tag{3.8}$$

The following are our lattice parameters accompanied by some useful formulas in the thin lens approximation. A list of some basic parameters is given also in (2.5); the basic ring shape follows from conditions I-IV of the previous section, and the BE fields are given in (3.3), (3.4). We first describe the semicircles, see Fig. 2, and then the straight sections, Fig. 3.

Semicircles

Length of two semicircles, $L_{BE} = 2Nl_{BE} = Nl = 2\pi R = 94.2478m$. (3.9)

Full length of the orbit, $L = L_{BE} + 2L_I = 124.89\text{m}; L_I$ is the length of a straight section. (3.10)

Ratio $L_{BE}/L = 0.7547$. (3.11)

Number of periods (cells), N=24. (N/2= 12 in every semicircle). (3.12)

Length of a half-cell, $l_{BE} \equiv L_{BE}/48$ =1.9635m (The full cell length, $l = 2l_{BE} = 3.93m$). (3.13)

Focal length of a half-lens, f =1.63 $l_{BE} = 3.2\,m$, $\frac{1}{f} = \frac{B' l_{lens}/2}{BR}$, $B' \equiv \partial B_V / \partial x$. (3.14)

Horizontal tune (without big straight sections), $\nu_x = \frac{N}{2\pi} \arccos\left[1 - 2\left(\frac{l_{BE}}{f}\right)^2\right] = 5.04.$ (3.15)

Together with straight sections, $\nu_x + 2$ =7.04:

Note: We are not concerned that this tune is too close to integer 7, because this is not a final design of the ring.

Vertical tune (without str. secs.), ν_y =5.1629, $\cos\frac{2\pi\nu_y}{N} = \cos\frac{2l_{BE}}{R} - 2\left(\frac{R\sin(l_{BE}/R)}{f}\right)^2$. (3.16)

Together with straight sections, $\nu_y + 2$ =7.1629.

$$\beta_x^+ = f\sqrt{\frac{1 + l_{BE}/f}{1 - l_{BE}/f}} = 6.5382\text{m}. \quad (3.17)$$

$$\beta_x^- = f\sqrt{\frac{1 - l_{BE}/f}{1 + l_{BE}/f}} = 1.5662m\,. \quad (3.18)$$

.

$$\beta_x^+ / \beta_x^- = (1 + l_{BE}/f)/(1 - l_{BE}/f) = 4.17$$

$$\gamma_x = \frac{1 + \alpha_x^2(s)}{\beta_x(s)} = \frac{2}{f\sqrt{1 - (l_{BE}/f)^2}} = 0.7915\,m^{-1}. \quad (3.19)$$

Note: α, β, γ here are the Courant-Snyder functions. In our design, γ_x is constant between (and only between) lenses because x is not focusing there, see Edwards and Syphers [5]. p.97, and Handbook [1], pp. 49-50 and under Betatron function.

Horizontal phase advance per cell, $\Delta\psi_x$ =1.321:

Phase advance for a + ↔ − transition, $\Delta\psi_{x+-} = \Delta\psi_x/2 = \arcsin(l_{BE}/f) = 0.6605$. (3.20)

$$D_x^+ = \frac{f^2}{R}\left(1 + \frac{l_{BE}}{2f}\right) = 0.8921\text{m}. \quad (3.21)$$

$$D_x^- = \frac{f^2}{R}\left(1 - \frac{l_{BE}}{2f}\right) = 0.4733m\,. \quad (3.22)$$

$$\left(D_x^+ / D_x^-\right)^2 = \frac{1 + l_{BE}/f + (l_{BE}/2f)^2}{1 - l_{BE}/f + (l_{BE}/2f)^2} = 3.55$$

We see that there is only ±8% difference between $\left(D^+/D^-\right)^2$ and β^+/β^-. $D_x(s)$ is defined here by the linear part of the x-dependence *on* $\Delta p/p$; $x = D_x \Delta p/p$.

. $$a_0 \equiv \langle D_x(s)/R\rangle \frac{L_{BE}}{L} = \frac{1}{R^2}\left(f^2 - l_{BE}^2/12\right)\frac{L_{BE}}{L} = 0.03327 \qquad (3.23)$$

Note: In (3.23), $\langle\ \rangle$ means the average over two semicircles only.

In the following numbers for vertical oscillations, + means, as usual, the places where *x-focusing* quads are placed, and – means places where *x-defocusing* quads are placed. With these notations, we must remember that $\beta_y^+ \approx \beta_x^-$ and $\beta_y^- \approx \beta_x^+$. The formulas used above for *x*-oscillations are not precisely valid for *y*-oscillations, because $n = -RB'/B = 1$ in the BE's.

β_y^+ =1.5229m. (3.24)

β_y^- =6.4298m. (3.25)

$\langle\gamma_y\rangle = 0.7944$

Vertical phase advance per cell, $\Delta\psi_y$ =1.3516

Vertical phase advance for $+\rightarrow-$ transition, $\Delta\psi_{y+-} = \Delta\psi_y/2 = 0.6758$.

(3.27)

Identity matrix straight sections

Number of straight section (N_I)=2

Length of one straight section, $L_I = 15.32m$

Number of free intervals per L_I, $N_s/2 = 8$

Length of a half-cell, $l_I \equiv L_I/8 = 1.915m$

Focal length of a half-lens, $f_I = l_I\sqrt{2} = 2.7082$ m

Phase advance per one interval l_I, $\Delta\psi = \pi/4$

$\beta_{xI}^+ = \beta_x^+ = 6.5382m$.

$\beta_x^- = 1.1218$ m

$$\gamma_{xI} = \frac{2}{f_I\sqrt{1-(l_I/f_I)^2}} = 1.0444m^{-1} \qquad (3.32)$$

Dispersion function in the straight section:

$D_I(0) = 0.8921$ m; $D_I(l_I) = 0.2929D_I(0)$; $D_I(2l_I) = 0$; $D_I(3l_I) = -0.2929D_I(0)$;

$D_I(4l_I) = -D_I(0)$; $D_I(5l_I) = -0.2929D_I(0)$; $D_I(6l_I) = 0$; $D_I(7l_I) = 0.2929D_I(0)$;

$D_I(8l_I) = D_I(0)$

(3.33)

$\beta_{yI}^{\pm} = \beta_{xI}^{\mp}$

$\gamma_{yI} = \gamma_{xI}$

4. Calculation and correction of $a_3(\Delta p/p)^2$.

As noted, the quadratic perturbations (1.3), (1.5), (1.6),

$$\Delta\omega_a = \Delta\omega_{a1} + \Delta\omega_{a2} + \Delta\omega_{a3} \equiv a_1 A_x^2 + a_2 A_y^2 + a_3(\Delta p/p)^2, \tag{4.1}$$

cannot be separated from the perturbation linear in $\Delta p/p$ when sychrotron stabilization holds. What the synchrotron oscillations actually stabilize is the average (in time) period of particle revolutions, T=L/v. If the relevant ring parameters are constant in time, then on the average (in time) all individual T's are the same. This means,

$$\left\langle \frac{\Delta(L/v)}{L_0/v_0} \right\rangle = \left\langle \frac{\Delta L}{L_0} - \frac{\Delta v}{v_0} - \frac{\Delta L}{L_0}\frac{\Delta v}{v_0} + \left(\frac{\Delta v}{v_0}\right)^2 + \ldots \right\rangle = 0. \tag{4.2}$$

This is the only feature of synchrotron oscillations needed for our purpose.

(From now on, we will omit indices "0" if this will not lead to ambiguities.) In this section we consider the case $x_\beta = y_\beta \equiv 0$. In such a case, we have (ignoring cubic and higher-order effects):

$$\left\langle \frac{\Delta L}{L} \right\rangle = \alpha_0 \frac{\Delta p}{p} + (\alpha_1 + \alpha_2 + \alpha_{D'})\left(\frac{\Delta p}{p}\right)^2 , \tag{4.3}$$

$$\left\langle \frac{\Delta v}{v} \right\rangle = \frac{1}{\gamma^2}\left\langle \frac{\Delta p}{p} \right\rangle - \frac{3}{2}\frac{\beta^2}{\gamma^2}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle, \tag{4.4}$$

$$\left\langle \frac{\Delta L}{L}\frac{\Delta v}{v} \right\rangle = \frac{\alpha_0}{\gamma^2}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle. \tag{4.5}$$

The physical difference between second-order compaction factors α_1, α_2 and $\alpha_{D'}$ will be explained shortly. The usual compaction factor α_0 is given in (3.23) above. We see that in order to satisfy (4.2), the individual equilibrium of $\Delta p/p$ is shifted,

$$\left\langle \frac{\Delta p}{p} \right\rangle = \frac{1/\gamma^4 + 3\beta^2/2\gamma^2 - \alpha_0/\gamma^2 + \alpha_{D'} + \alpha_1 + \alpha_2}{1/\gamma^2 - \alpha_0}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle. \tag{4.6}$$

(There are more quadratic terms on the right side of (4.6) in the full expression for $\langle \Delta p/p \rangle$, if we take into account free betatron oscillations. These terms can be considered independently of $\langle (\Delta p/p)^2 \rangle$, as will be done in the following section.) The only factor here that can be changed by our sextupoles (in order to cancel the off-momentum violation of (1.1)) is α_2, and we will show how it can be used.

We need to calculate all $\Delta p/p$ and $(\Delta p/p)^2$ terms violating (1.1). First of all, there exists a factor before E_R equal to $-(e/m)[|d|+m^2/p^2]\beta$ which directly depends on momentum. This coefficient produces

$$(\Delta\omega_a)_{coeff} = \frac{e}{m}|a|B\left\{\frac{1}{\gamma^2}\left(\frac{2E}{|d|\beta B}-1\right)\frac{\Delta p}{p} - \left[\frac{E}{|d|\beta\gamma^4 B}+\frac{3\beta^2}{2\gamma^2}\left(\frac{2E}{|a|\beta B}-1\right)\right]\left(\frac{\Delta p}{p}\right)^2\right\}\frac{L_{BE}}{L}, \quad (4.7)$$

and we know from (4.6) that the term $(\Delta p/p)$ here contains a non-oscillating part proportional to $(\Delta p/p)^2$. Below we will add to (4.7) more terms linear and quadratic in $\Delta p/p$ describing the field perturbations met by moving particles, and will investigate the meaning of $\langle \Delta p/p\rangle$. (The terms not connected to $\Delta p/p$ are considered in the next section.)

The next step is to analyze all effects following from the perturbations of the closed orbit due to $\Delta p/p$, $(\Delta p/p)^2$:

$$x(s) = x_\beta(s) + \tilde{D}(p,s)\frac{\Delta p}{p} + d(s)\left(\frac{\Delta p}{p}\right)^2 . \quad (4.8)$$

Here, by definition, *d(s)* depends only on sextupole fields, so $\alpha_2 = \langle d/R\rangle$. (But we can calculate α_2 without actual calculation of *d(s)*.) Our dispersion function, $\tilde{D}(p,s)$, is different from the usual *D(s)* ,which does not depend on p . In fact, $\tilde{D}(p,s)$ is the (slightly approximate) solution of equation (17) in the Handbook [1] (on p.50), which does not take into account sextupoles. That equation is

$$\text{Eq. (17) of [1]: } D''(p,s) + \left(\frac{1}{R_0^2}+\frac{\partial B/\partial x}{BR_0}\right)\frac{p_0}{p}D(p,s) = \frac{1}{R_0}\frac{p_0}{p} + \frac{D(p,s)}{R_0^2}\frac{\Delta p}{p}. \quad (4.9)$$

The equation for the usual *D(s)* , which leads to our formulas (3.20)-(3.23), (3.33), corresponds to $p = p_0$ and $\Delta p = 0$ in (4.9). $\tilde{D}(p,s)$ is therefore the solution of (4.9), taking into account the next approximation in $\Delta p/p$. The last term in (4.9) is already proportional to $\Delta p/p$; so, with a very small error, we can substitute $\alpha_0 \equiv \langle D(s)/R_0\rangle$ for $D(p,s)/R_0$. Now, remember that in our BE sections,

$$\frac{1}{R_0^2}+\frac{\partial B/\partial x}{BR_0} = 0, \quad \frac{1}{R_0} \neq 0, \quad \text{BE sections.} \quad (4.10)$$

In lenses and free intervals, $R=\infty$, $1/R=0$. Comparing equation (4.9) with its limit $p\to p_0$, $\Delta p\to 0$, we see that we can get $\tilde{D}(p,s)$ from $D(s)$ simply by substituting

$$\frac{1}{f}\to\frac{1}{f}\left(1-\frac{\Delta p}{p}\right),\quad \frac{1}{R}\to\frac{1}{R}\left[1-(1-\alpha_0)\frac{\Delta p}{p}\right];\quad D(s)\to\tilde{D}(p,s). \qquad (4.11)$$

From this, after some algebra, and neglecting $\alpha_0 << 1$ in (4.11), we get

$$\left\langle\frac{\tilde{D}(p,s)}{R_0}\right\rangle=\alpha_0+\alpha_1\frac{\Delta p}{p},\quad \alpha_1=(L_{BE}/L)l_{BE}^2/6R^2=0.0022. \qquad (4.12)$$

We now need to take into account the sextupole fields, see eqs. (1.4), (1.10) above. The shortest way to calculate α_2 is to use formula (2) in [1], p. 263: If a particle passing a very short area $\Delta s=l_{si}$ of the magnetic perturbation, ΔB_i, gets the same kick (angle deflection) θ_i during every revolution, then, on the average, its closed orbit length is changed as

$$\Delta L=\theta_i D(s_i). \qquad (4.13)$$

(This formula is consistent also with the Hamiltonian (44) represented in [1], p. 70.) The kick produced by a perturbation ΔB equals $\theta=-\Delta Bl_s/BR$; in the case of a sextupole, $\Delta B=(B''D^2l_s/2BR)(\Delta p/p)^2$; therefore,

$$\alpha_2=-\sum\frac{B_i''D^3(s_i)l_{si}}{2BRL},\qquad B''\equiv\partial^2B/\partial x^2. \qquad (4.14)$$

In formula (4.6), we also have $\alpha_{D'}$.This factor has no connection with (4.8). It is simply the factor of the lengthening of the particle trajectory due to $x'(s)=D'(s)\Delta p/p$, see formula (3.6), above:

$$(\Delta L/L)_{D'}=\frac{1}{2}\left\langle(dD(s)/ds)^2\right\rangle(\Delta p/p)^2. \qquad (4.15)$$

In a BE section, during the transition from + to - quads, $D'(s)=-D^+/f+s/R$. (See [1], p.60, formula (6) there, with $\sin(\mu/2)=l_{BE}/f$. See also our formula (3.21)). This gives

$$\left(\frac{\Delta L}{L}\right)_{D',BE}=\frac{1}{4}\left[\left(\frac{f}{R}\right)^2+\frac{7}{6}\left(\frac{l_{BE}}{R}\right)^2\right]\frac{L_{BE}}{L}\left(\frac{\Delta p}{p}\right)^2=0.0164\frac{L_{BE}}{L}\left(\frac{\Delta p}{p}\right)^2. \qquad (4.16)$$

In a straight section, between lenses #k and #(k+1), $D'=(D_{k+1}-D_k)/l_s$, take numbers from (3.33) and around it. This gives

$$\left(\frac{\Delta L}{L}\right)_{D',I}=0.508\frac{l_s}{L}\left(\frac{\Delta p}{p}\right)^2. \qquad (4.17)$$

With $L_{BE}/L=0.7547$, $l_I/L=1.5333\times10^{-2}$,

$$\alpha_{D'} = 0.0202. \tag{4.18}$$

Now let us estimate the contribution of $D(s)\Delta p/p$ to $\Delta\omega_a$ through the direct influence of the average magnetic and electric fields. As usual, we assume that without perturbations (1.3), (1.5), (1.6), condition (1.1) is satisfied. With our choice of fields inside the BE sections, (3.1)-(3.4), condition (1.1) does not depend, inside BE's, on a particle horizontal coordinate, in this case on $x = D(s)\Delta p/p$ +quadratic term. But it depends on the particle velocity, in this case on $\vartheta_p = dx/ds = (dD/ds)(\Delta p/p)$ + (non-essential in this case) $(\Delta p/p)^2$ terms. First, there is the horizontal acceleration, which we do not take into account here. Second, the perturbed velocity is not exactly perpendicular to $\vec{E}$, so there is a factor $\cos(dx/ds) = 1 - 0.5(dx/ds)^2$. A similar effect in horizontal betatron oscillations is explained in detail in the next section. This gives

$$(\Delta\omega_a)_{D'E} = +\frac{e}{m}\left[|a| + \left(\frac{m}{p}\right)^2\right]\beta E_R \cdot \frac{1}{2}\left\langle (D')^2 \right\rangle_{BE} \left\langle (\Delta p/p)^2 \right\rangle \cdot \frac{L_{BE}}{L} \tag{4.19}$$

We already know, see (4.13), (4.14), that

$$\frac{1}{2}\left\langle (D')^2 \right\rangle_{BE} = \frac{1}{4}\left[\left(\frac{f}{R}\right)^2 + \frac{7}{6}\left(\frac{l_{BE}}{R}\right)^2\right] = 0.0164. \tag{4.20}$$

Thus,

$$(\Delta\omega_a)_{D'E} = \frac{e}{m}|a| B_V \cdot 0.0164 \frac{L_{BE}}{L}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle. \tag{4.21}$$

As for the magnetic field effects, both linear and quadratic in $\Delta p/p$, that violate (1.1), they can be produced only by quadrupoles and sextupoles. Let us first calculate the perturbed B-fields met by a particle passing the quadrupoles. Inside the two semicircles, in our lattice of equal alternating gradients

$$\left(\frac{\langle \Delta B \rangle}{B_V}\right)_{p,quad} = \frac{1}{B_V}\left(\frac{\partial B}{\partial x}\right)^+ \frac{l_{lens}}{2 l_{BE}}\left(\tilde{D}^+ - \tilde{D}^-\right)\frac{L_{BE}}{L}\frac{\Delta p}{p}\ , \tag{4.22}$$

where ΔB_V refers to quads, while B_V to the ideal field inside BE's. l_{BE} is the half-cell length, l_{lens} is the full-quad length. This gives

$$\left(\frac{\langle \Delta B_V \rangle}{B_V}\right)_{p,quad} = \frac{1}{B_V}\left(\frac{\partial B}{\partial x}\right)^+ \frac{l_{lens} f}{2R}\frac{L_{BE}}{L}\frac{\Delta p}{p}. \tag{4.23}$$

Finally, using (4.11) and the designed definition of the focal length, see (3.14), we get

$$\left(\frac{\langle \Delta B_V \rangle}{B_V}\right)_{p,quad} = \frac{L_{BE}}{L}\left(1+\alpha_0 \frac{\Delta p}{p}\right)\frac{\Delta p}{p}, \quad (4.24)$$

$$(\Delta\omega_a)_{p,quad} = \frac{e}{m}|d| B \frac{L_{BE}}{L}\left(1+\alpha_0\frac{\Delta p}{p}\right)\frac{\Delta p}{p}. \quad (4.25)$$

In the frame of our assumptions, there are no other contributions to violations of (1.1) by off-momentum terms. Indeed, straight sections' quadrupoles cannot produce any effect, because the D-function is oscillating there. And it can be proved (and confirmed by numerical calculations) that sextupoles do not change the average field. The proof is the following.

Taking into account only first- and second-order perturbations, the equation of motion inside the BE's is

$$x'' = \frac{1}{R_0}\frac{p_0}{p}\frac{\Delta p}{p} + \frac{1}{R_0}\frac{x}{R_0}\frac{\Delta p}{p}, \quad \text{inside BE.} \quad (4.26)$$

In the same approximation, the equation for the ring parts outside the BE's can be written as

$$x'' + \frac{B_V(s)}{B_V R_0}\frac{p_0}{p} = 0, \quad \text{outside BE.} \quad (4.27)$$

The sum of the averages, $\langle x'' \rangle_{inside} + \langle x'' \rangle_{out} = 0$ because it represents the average along the full orbit, and we are talking here about the closed orbits. Therefore,

$$\left(\frac{\langle B_V(outside) \rangle}{B_V}\right) = \left(1+\alpha_0\frac{\Delta p}{p}\right)\frac{\Delta p}{p}, \quad (4.28)$$

which is the result (4.24) for *quads alone*. ($B_V(outside) \equiv \Delta B_V$, since the designed fields of all lenses equal zero at $\Delta p / p = 0$.) Therefore, the sextupoles' contribution equals exactly zero.

Now, gathering all contributions together, using the formulas and numbers represented here, ($L_{BE}/L = 0.7547$, relativistic $\gamma^2 = 1.1765$,. $\alpha_0 = 0.03327$, etc.), we have:

$$(\Delta\omega_a)_p = \frac{e}{m}|d| B_V \frac{L_{BE}}{L}\left\{\left(1+\frac{1}{\gamma^2}\left(\frac{2E}{|a|\beta B}-1\right)\right)\left\langle\frac{\Delta p}{p}\right\rangle + \left[0.0164+\alpha_0-\left(\frac{E}{|a|\beta\gamma^4 B}+\frac{3\beta^2}{2\gamma^2}\left(\frac{2E}{|d|\beta B}-1\right)\right)\right]\left\langle\left(\frac{\Delta p}{p}\right)^2\right\rangle\right\}$$

(4.29)

$$\left(\Delta\omega_a\right)_p = a_3\left(\frac{\Delta p}{p}\right)^2, \quad a_3 = \frac{e}{m}|a|B_V\left[0.9881 + 1.9437\alpha_2\right]. \qquad (4.30)$$

To eliminate a_3, we need

$$\alpha_2 = -\sum \frac{B_i'' D_i^3 l_{si}}{2BR}\frac{1}{L} = -0.51. \qquad (4.31)$$

For an estimation, assume that we can use only 10 sextupoles placed next to + quads, $l_s = 0.5m$, $D^+ = 0.8921m$, B=0.2095T, R=15m, L=124.89m. Then we need $B'' = 90m^{-2}$. ANL uses sextupoles with $B'' = 415m^{-2}$ (for Advanced Proton Source, see [1] , p. 443, table 2).

5. Calculation and correction of the $a_1 A_x^2$ and $a_2 A_y^2$ terms.

We now discuss the (generalized) horizontal and vertical pitch effects. The horizontal pitch effect has not been previously noted. (In my EDM Note #10, I considered only cases $x_\beta = dx_\beta / ds = 0$, that is $A_x^2 = 0$.)

There are four physically different effects leading to the dependence of $\Delta\omega_a$ on A_x^2, A_y^2.

1. The first effect is the result of the combination of trajectory lengthening, $\Delta L/L$, due to $\vartheta_x^2 \equiv (dx_\beta / ds)^2$ and $\vartheta_y^2 \equiv (dy/ds)^2$, see formula (3.6), and the synchrotron stability leading to dependence of $\langle \Delta p/p \rangle$ on this lengthening. The effect can be calculated immediately.

$$\left(\frac{\Delta L}{L}\right)_{x'_\beta, y'} = \frac{1}{2}\left(\langle\vartheta_x^2\rangle + \langle\vartheta_y^2\rangle\right) = \frac{1}{4}\frac{1}{s}\int_0^s ds'\left(\gamma_x(s')A_x^2 + \gamma_y(s')A_y^2\right), \qquad (5.1)$$

where $\gamma = (1+\alpha^2)/\beta$ is one of the three Courant-Snyder parameters. In (5.1) $s \to \infty$. Formula (5.1) follows from formulas (2), (3) of [1], p.49,

$$x(s) = A_x\sqrt{\beta_x(s)}\cos\left(\psi_x(s) + \delta_x\right), \; x'(s) = -\frac{A_x}{\sqrt{\beta_x(s)}}\left[\alpha_x(s)\cos\left(\psi_x(s)+\delta_x\right) + \sin\left(\psi_x(s)+\delta_x\right)\right] \quad (5.2)$$

and analogously for *y*. A, δ are constants,

$$A^2 = x_{\max}^2(s)/\beta(s) = (x_{\max}^2(s))_{\max}/\beta_{\max}. \qquad (5.3)$$

So if, for example, $\left(x_{\max}(s)\right)_{\max}$ =2.5cm, and $\beta_{\max}$ =6.5m, then A^2=0.9654×10^{-4}m. In our lattice, $\beta_{\max} = \beta^+$.

In (5.1), γ_x =constant between quadrupoles (but not inside quadrupoles), see Edwards and Syphers [6], p.97. We have (with γ-values given in (3.19), (3.32) above),

$$\left(\frac{\Delta L}{L}\right)_{x'_\beta} = \left(\frac{\Delta L}{L}\right)_{x'_\beta BE} + \left(\frac{\Delta L}{L}\right)_{x'_\beta I} = \frac{A_x^2}{4}\left[\gamma_{x,BE}\frac{L_{BE}}{L} + \gamma_{x,I}\frac{L_I}{L}\right] = 0.2134\,A_x^2. \tag{5.4}$$

This goes into $\langle \Delta p/p \rangle$,

$$\left\langle \frac{\Delta p}{p} \right\rangle_{x'_\beta} = \frac{(\Delta L/L)_{x'_\beta}}{1/\gamma^2 - \alpha_0} = 0.2613 A_x^2. \tag{5.5}$$

This, in turn, goes into

$$(\Delta\omega_a)_{x'_\beta} = \frac{e}{m}|a|B_V \frac{L_{BE}}{L}\left[1 + \frac{1}{\gamma^2}\left(\frac{2E}{|a|\beta B} - 1\right)\right]\left\langle \frac{\Delta p}{p} \right\rangle = \frac{e}{m}|a|B_V \cdot 0.4147 A_x^2. \tag{5.6}$$

$\gamma_y(s)$ has exactly the same value as γ_x=1.0444 in the straight sections. It is slightly different, and not exactly constant, in the BE's, because vertical oscillations are focused there (the field index n=1). In the linear approximation, in which γ_y is defined,

$$y'' + \frac{1}{R^2}y = 0, \quad \text{inside BE's, linear approximation.} \tag{5.7}$$

We get

$$\left(\frac{\Delta L}{L}\right)_{y'} = 0.2139 A_y^2, \quad (\Delta\omega_a)_{y'} = \frac{e}{m}|a|B_V \cdot 0.4157 A_y^2. \tag{5.8}$$

2. The second effect also depends on betatron angles, ϑ_x, ϑ_y. Due to these angles, magnetic and electric fields met by a spin passing a BE section differ from their designed values. With respect to the electric field, the average vector product $\vec{v} \times \vec{E}$ is changed. (Such an effect was not dangerous in our (g-2) ring only because, in the g-2 experiment, the equilibrium electric field equalled zero.) As for the magnetic field, the effect is the usual F. Farley pitch. The best way to understand both electric and magnetic field perturbations in this case is to analyze the J.D. Jackson formula (11.171), [7], p.550:

$$\frac{d}{dt}\left(\frac{\vec{v}}{v}\cdot\vec{s}\right) = -\frac{e}{mc}\vec{s}_{perp}\cdot\left[\left(\frac{g-2}{2}\right)\frac{\vec{v}}{v}\times\vec{B} + \left(\frac{g\beta}{2} - \frac{1}{\beta}\right)\vec{E}\right], \tag{5.9}$$

where $\vec{v}$ is velocity, $\beta = v/c$, $\vec{s}$ is the rest frame spin, $\vec{s}_{perp}$ is a part of the spin vector perpendicular to $\vec{v}$. We consider the case of $\vec{B} = \vec{B}_V$ perpendicular, and $\vec{E} = \vec{E}_R$ parallel to the ideal orbit plane. $\vec{v}\times\vec{B}$ is also parallel to the plane; therefore, only the component of $\vec{s}_{perp}$ lying in the plane contributes to rotation of the spin relative to velocity. It is easy to see, with a little algebra, that if $\vec{v}$ is also lying in the plane and is perpendicular to $\vec{E}$, and condition (1.1) is satisfied, then there is no (g-2)-rotation; the square brackets in (5.8) equal zero. (To see this, use the identity $g\beta/2-1/\beta \equiv a\beta-\beta/\gamma^2\beta^2$.) But if the velocity is not lying in the ideal plane, then in (5.9),

$$\left|\vec{v}\times\vec{B}\right| = vB\cos(dy/ds),$$

(5.10)

while the angles between $\vec{s}_{perp}$ (lying, remember, in the ideal plane) and $\vec{v}\times\vec{E}$, and between $\vec{s}_{perp}$ and $\vec{E}$ are not changed. The component s_{perp} lying in the horizontal plane is also not changed. But if $\vec{v}$ deviates from the ideal orbit in the horizontal plane, then in (5.9),

$$\vec{s}_{perp}\cdot\vec{E} = s_{perp}E\cos(dx/ds), \qquad (5.11)$$

while the magnetic term is not changed. Therefore,

$$\left(\Delta\omega_a\right)_{x'E,y'B} = \frac{e}{m}|a|B_V\cdot\frac{L_{BE}}{L}\frac{1}{2}\left[\left\langle\vartheta_x^2\right\rangle-\left\langle\vartheta_y^2\right\rangle\right] = \frac{e}{m}|a|B_V\frac{L_{BE}}{L}\frac{1}{4}\left(\left\langle\gamma_x\right\rangle A_x^2-\left\langle\gamma_y\right\rangle A_y^2\right) \qquad (5.12)$$

$$= \frac{e}{m}|a|B_V\cdot\left(0.1493A_x^2-0.1499A_y^2\right). \qquad (5.13)$$

3. Inside BE's, the vertical magnetic field depends on a particle's vertical position, see (3.3) above. This immediately gives us

$$\left(\Delta\omega_a\right)_{y^2} = \frac{e}{m}|a|B_V\left(\frac{\Delta B_V}{B_V}\right)_{y^2} = -\frac{e}{m}|a|B_V\cdot\frac{1}{4}\frac{L_{BE}}{L}\frac{\left\langle\beta_y\right\rangle_{BE}}{R^2}A_y^2 = -\frac{e}{m}|a|B_V\cdot 0.00291A_y^2. \qquad (5.14)$$

($\left\langle\beta_y\right\rangle_{BE} = 3.507$.) The effect is very small by comparison with effects 1 and 2. In addition to (5.14), the field perturbation $(\Delta B/B)_{y^2}$ slightly shifts the equilibrium horizontal orbit; we will neglect this effect.

4. We obviously need to compensate effects proportional to $A_{x,y}^2$ because with $A_{x,y}^2 \sim 10^{-4}$ and $\left(\Delta p/p\right)^2 \sim 10^{-6}$, the violation of condition (1.1) by betatron oscillations is more than one order larger than the violation by the momentum spread. So the fourth effect is the effect of sextupoles used for these compensations. When the particle performing betatron oscillations passes sextupole #i periodically, it periodically gets a horizontal angle deflection equal to

$$(\theta_x)_i = -\frac{B_i''\left(x^2-y^2\right)_i l_{si}}{2BR}. \qquad (5.15)$$

On the average,

$$\left\langle(\theta_x)_i\right\rangle = -\frac{B''(\beta_x A_x^2-\beta_y A_y^2)_i\, l_{si}}{4BR}. \qquad (5.16)$$

According to [1], p.263, formula (2), this periodic deflection shifts the horizontal equilibrium, so

$$\left(\frac{\Delta L}{L}\right)_q = -\sum \frac{B_i' D_i (\beta_{xi} A_x^2 - \beta_{yi} A_y^2) l_{si}}{4BRL} . \qquad (5.17)$$

Because of the synchrotron stability, (5.17) leads to the shift of the momentum equilibrium,

$$\left\langle \frac{\Delta p}{p} \right\rangle_q = \frac{(\Delta L / L)_q}{1/\gamma^2 - \alpha_0} = -\frac{1}{(1/\gamma^2 - \alpha_0)} \sum \frac{B_i' D_i (\beta_{xi} A_x^2 - \beta_{yi} A_y^2) l_{si}}{4BRL} . \qquad (5.18)$$

And, finally, this leads to the corresponding violation of condition (1.1). Using the term proportional to $\langle \Delta p / p \rangle$ in our formula (4.26), we get

$$(\Delta\omega_a)_q = -0.9719 \sum \frac{B_i'' D_i \beta_{xi} l_{si}}{2BRL} A_x^2 + 0.9719 \sum \frac{B_i' D_i \beta_{yi} l_{si}}{2BRL} A_y^2 . \qquad (5.19)$$

After gathering all betatron terms, we have

$$(\Delta\omega_a)_{xq} \equiv a_1 A_x^2 = \left(0.564 - 0.9719 \sum \frac{B_k' D_k \beta_{xk} l_{sk}}{2BRL} \right) A_x^2 , \qquad (5.20)$$

$$(\Delta\omega_a)_{yk} \equiv a_2 A_y^2 = \left(0.2629 + 0.9719 \sum \frac{B_k'' D_k \beta_{yk} l_{sk}}{2BRL} \right) A_y^2 . \qquad (5.21)$$

6. Conclusion.

Thus, the situation is the following. To cancel a_1, a_2, a_3, we need to satisfy three conditions:

$$1. \sum \frac{B_k' D_k \beta_{xk} l_{sk}}{2BRL} = 0.58 , \qquad a_1 = 0.$$

$$2. \sum \frac{B_k' D_k \beta_{yk} l_{sk}}{2BRL} = -0.27 , \qquad a_2 = 0.$$

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$$3. \sum \frac{B_k' D_k^3 l_{sk}}{2BRL} = 0.51 , \qquad a_3 = 0.$$

It is instructive to compare these conditions with the conditions for the betatron chromaticity cancellation:

$$\sum \frac{B_k' D_k \beta_{xk} l_{sk}}{2BRL} \approx 0.43 , \quad \xi_x = 0.$$

$$\sum \frac{B_k' D_k \beta_{yk} l_{sk}}{2BRL} \approx -0.43 \quad \xi_y = 0.$$

We see, first, that when we satisfy our conditions 1 and 2, we simultaneously reduce the x-chromaticity to -26%, and the y-chromaticity to 37% of their original magnitudes. And second, since the cancellation of chromaticities is a routine part of the storage ring's operations, there is no doubt that we can satisfy conditions 1 and 2.

It is much more difficult to satisfy condition 3 of cancellation of $(\Delta p/p)^2$. In the FODO (semicircle) part of our ring, $D^2(s) \sim 0.12\beta_x(s)$, and for this reason cancellations 1 and 2 require eight times smaller sextupole fields than cancellation 3. That means that we may easily cancel 1 and 2 almost without influencing 3, but satisfying all three conditions requires some tricks. We need to design areas where the D-function is 3-4 times bigger than usual.

Taking this into account, it seems much simpler to squeeze $\Delta p/p$ by some factor 3 (we will not need more) by using well-known synchrotron methods, not sextupoles.

If $(\Delta p/p)^2 = 10^{-6}$, $x_{max} = y_{max} = 2.5$ cm, then the coherence time ~10 s can be achieved by satisfying 1 and 2 only, with accuracy ~1%. To get coherence time ~1 min, it is sufficient to squeeze $\Delta p/p$ by a factor of three, and to satisfy 1 and 2 with accuracy 0.15%.

Fig. 1 The ring.

Fig. 2 One small period of the FODO part.

Fig. 3 Straight section. The half (focusing) lens is 1/f but the whole lens should be 2/f instead of 1/f as is erroneously written.

Figs. 4 a, b. The effects of trajectory lengthening.

Fig. 5 β_x, β_y, D, D^2, γ_x in the semicircle FODO part of the ring.

Fig. 6 β_x, β_y, D, D^2, γ_x in the straight section.

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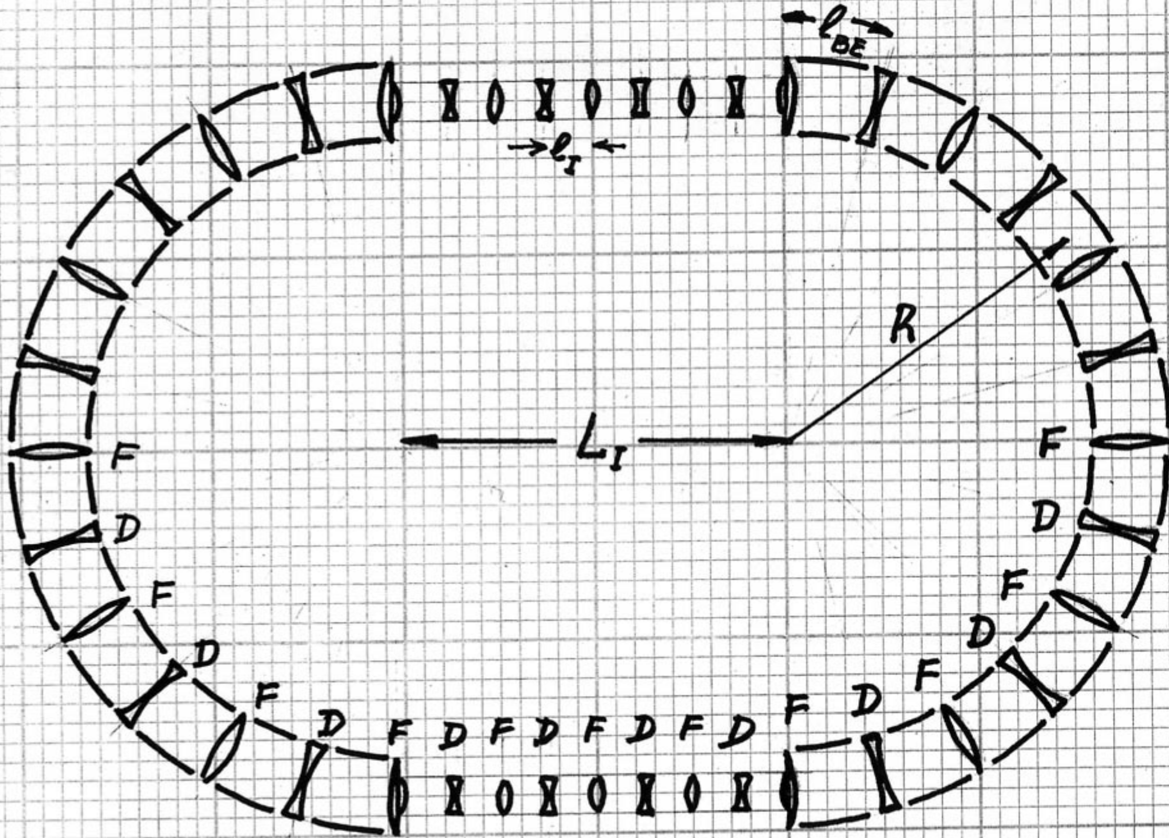


Fig. 1 The ring.

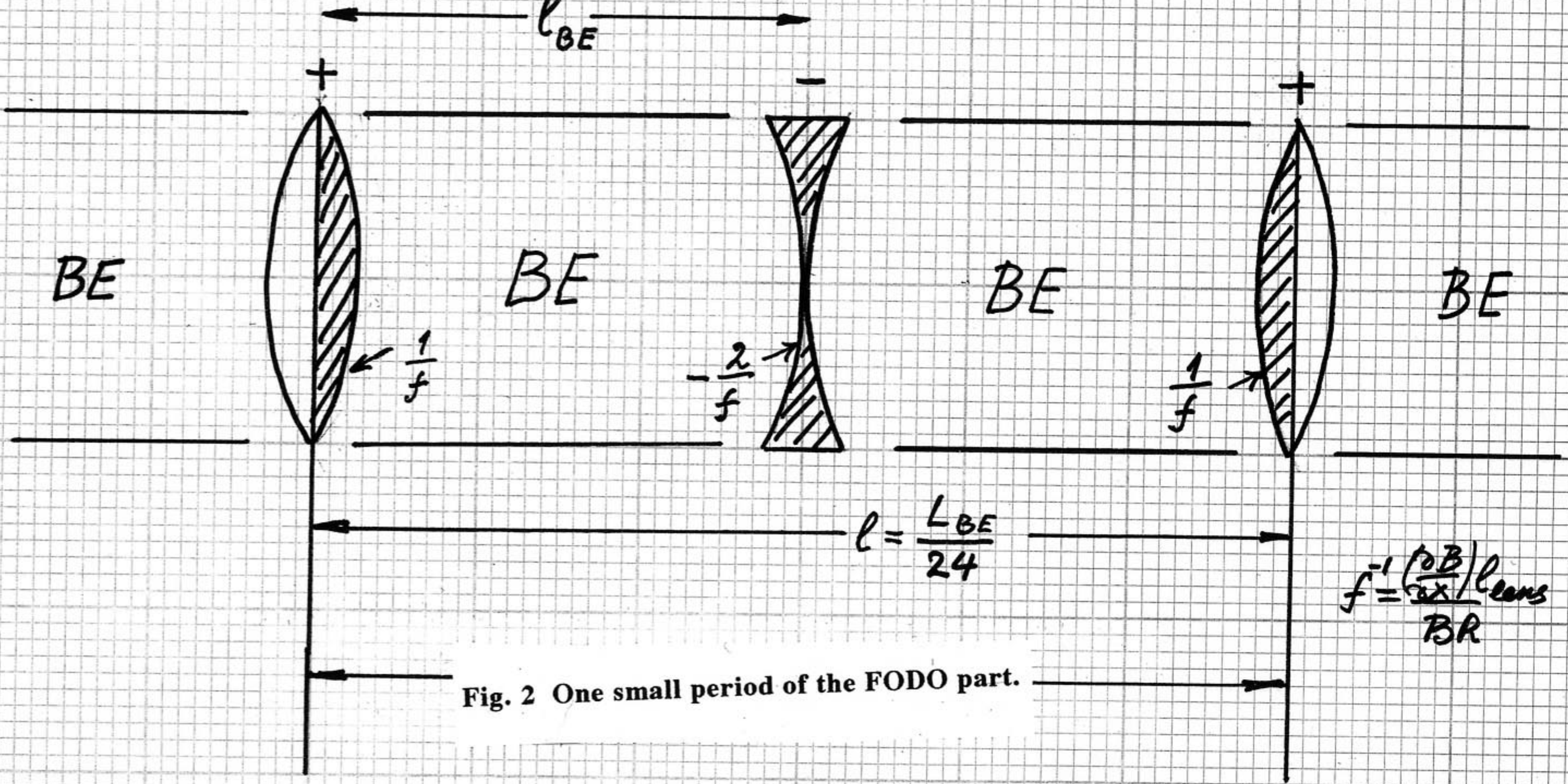


Fig. 2 One small period of the FODO part.

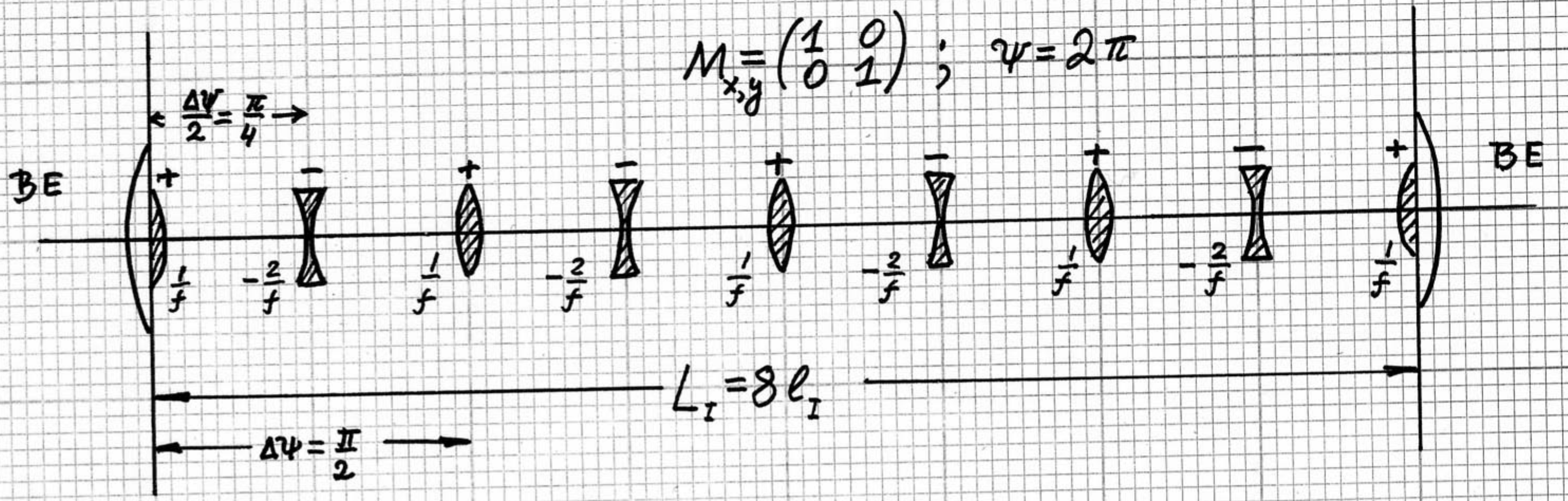
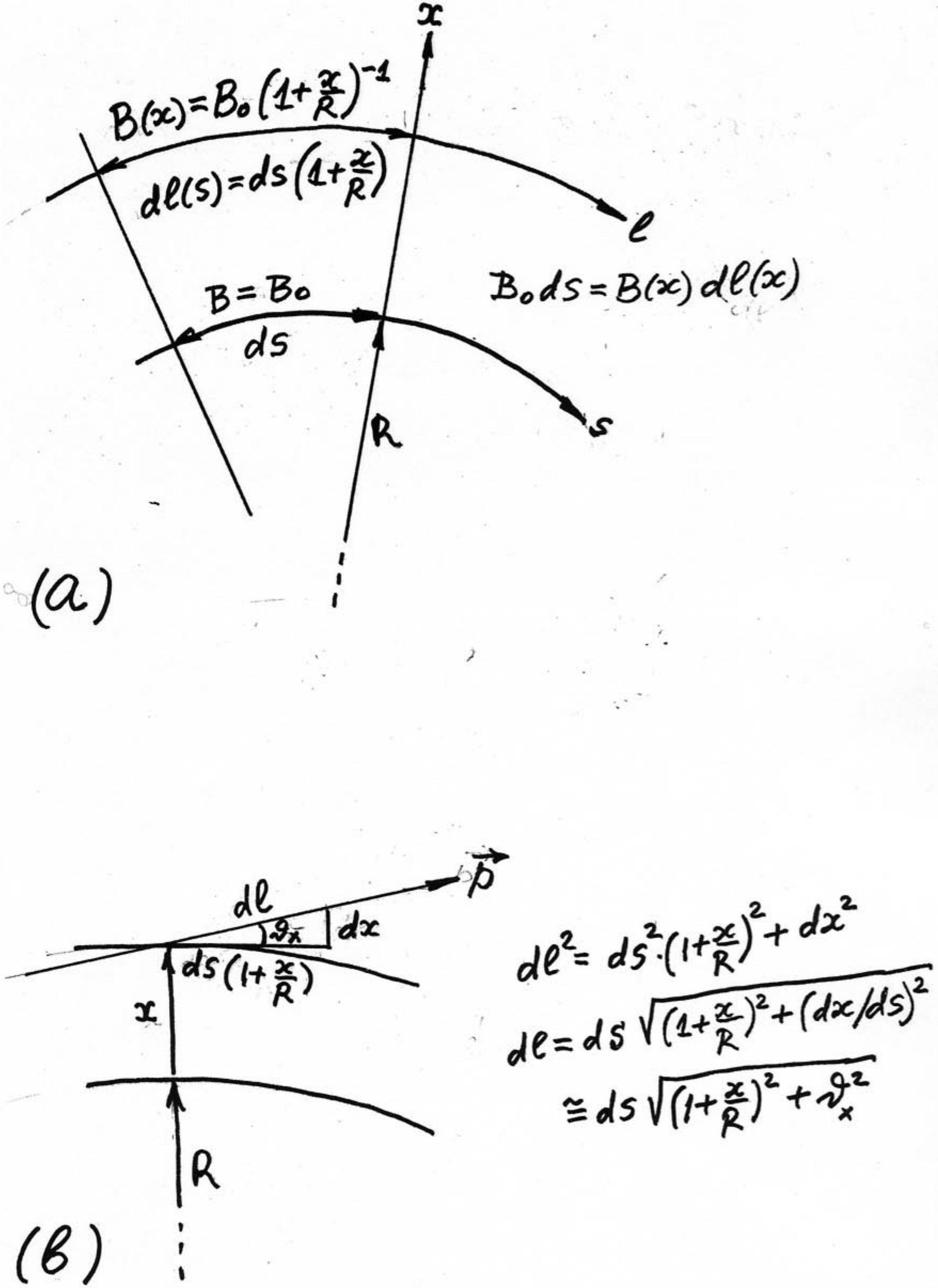


Fig. 3 Straight section.



Figs. 4 a, b. The effects of trajectory lengthening.

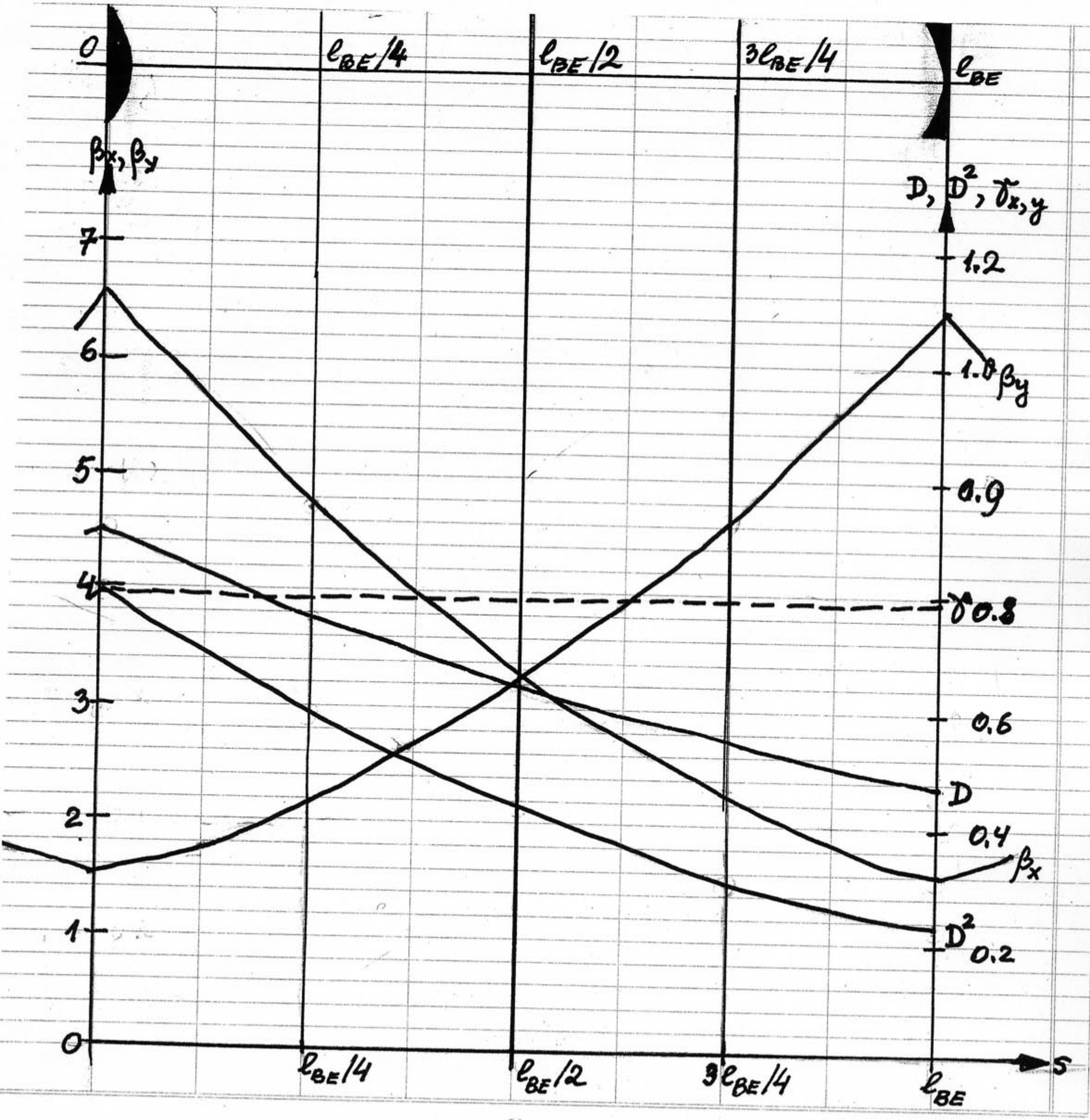


Fig. 5 β_x, β_y, D, D^2, γ_x **in the semicircle FODO part of the ring.**

Fig. 6 β_x, β_y, D, D^2, γ_x in the straight section.