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Kinetics of Muon Longitudinal Cooling

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Abstract. Kinetics of longitudinal ionization cooling is analysed by use of method of moments. Special attention is given to "supercooling" situation when longitudinal damping coefficient is much more than synchrotron frequency. It is shown that in this case the emittance damping becomes slower, and it should be taken into account in designing of the ionization cooling system.

INDIVIDUAL PARTICLE OSCILLATIONS

For muon muon collider [1] a longitudinal ionization is created by "wedges," which represent the pieces of matter with a transverse gradient of the "ionization losses"; these pieces should be placed at sections where dispersion exisits. Inside such pieces change of Δp (deviation of a muon momentum p from its equalibrium value p_s) is proportional to Δp, which results in longitudinal damping. Equation of longitudinal motion for individual particle in a presence of wedges may be written in the following form:

$$y'' + 2\alpha y' + \Omega_s^2 y = W_y(z). \tag{1}$$

Here $y = z - z_s$ (z is a longitudinal coordinate of particle, z_s is a longitudinal coordinate of equilibrium particle), 2α is a longitudinal damping coefficient, Ω_s is a synchrotron frequency, $W_y(z)$ describes the heating effect due to "knock-on" electrons.

A longitudinal damping coefficient is defined by the following formula:

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$$2\alpha = \frac{1}{mc^2\beta^2\gamma}\left\langle \left(\frac{dE}{dz}\right)_{\text{ion}} \frac{\Psi}{L_0} \right\rangle + \frac{\frac{d}{dz}(p/\Lambda)}{p/\Lambda} - \frac{d}{dp}\left[\left(\frac{dE}{dz}\right)_{\text{ion}} \frac{1}{v}\right] \tag{2}$$

Here: β, γ are relativistic factors; m is a muon mass; sign $\langle\rangle$ means averaging on z; $\left(\frac{dE}{dz}\right)_{\text{ion}}$ are ionization losses of energy per unit of length, depending on a transverse coordinate x; Ψ is the dispersion function. Parameter L_0 is defined by the following expression:

$$L_0 = \frac{\left(\frac{dE}{dz}\right)_{\text{ion}}}{\frac{d}{dx}\left(\frac{dE}{dz}\right)_{\text{ion}}}; \tag{3}$$

Momentum deviation Δp is connected to our variable y by standard expressions:

$$\Lambda \frac{\Delta p}{p} = \frac{dy}{dz} \tag{4}$$

$$\Lambda = \left\langle \frac{\Psi}{R} \right\rangle + \frac{1}{\gamma^2} \tag{5}$$

Emittance in our variables is defined by

$$\epsilon_y^2 = \langle (y')^2 \rangle \langle y^2 \rangle - \langle y y' \rangle^2, \tag{6}$$

and a normalized longitudinal emittance is given by

$$\epsilon_N^2 = \left[\langle (\Delta p)^2 \rangle \langle y^2 \rangle - \langle \Delta p \cdot y \rangle^2\right] \frac{1}{(mc)^2} \tag{7}$$

Using Eq. (4), we find:

$$\epsilon_N = \frac{\beta\gamma}{\Lambda} \epsilon_y \tag{8}$$

The longitudinal (synchrotron) frequency Ω is defined by:

$$\Omega^2 = \frac{2\pi e E_{ac} \cdot \cos\left(\frac{2\pi}{\lambda} z_s\right) \Lambda}{mc^2 \lambda \beta^2 \gamma} \tag{9}$$

Here E_{ac} is an amplitude of the r.f. field, and λ is its wavelength.

Let us consider a case when α and Ω are constant. Eigenvalues of Eq. (1) are defined by:

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \Omega^2} \tag{10}$$

We have two different regimes of operation:

a) $\alpha > \Omega$, non-periodic motion;
b) $\alpha < \Omega$, periodic oscillations.

At the first regime both modes are damped with different damping rates. For $\alpha \gg \Omega$ a damping rate of "slow" mode is defined by:

$$\lambda_1 = (\alpha - \sqrt{\alpha^2 - \Omega^2}) = -\Omega^2/2\alpha \tag{11}$$

For optimal cooling it is necessary that $\alpha = \alpha_\perp$ where $\alpha_\perp$ is a transverse damping coefficient.

Numerical estimations show that in this case for reasonable amplitudes of the accelerating field $\alpha \gg \Omega$, and a damping rate of "slow" mode is defined by Eq. (11).

For periodic oscillations

$$\lambda_{1,2} = -\alpha \pm i\sqrt{\Omega^2 - \alpha^2} = -\alpha \pm iw_1 \tag{12}$$

We see that both waves are damped with the same damping rate.

Let $y(z=0) = y_0, y'(z=0) = y_0'$; then we can write:

$$\begin{aligned} y &= \frac{\lambda_2\varphi_1 - \lambda_1\varphi_2}{\lambda_2 - \lambda_1} y_0 + \frac{\varphi_2 - \varphi_1}{\lambda_2 - \lambda_1} y_0' \\ y' &= \Omega^2 \frac{\varphi_2 - \varphi_1}{\lambda_2 - \lambda_1} y_0 + \frac{\lambda_2\varphi_2 - \lambda_1\varphi_1}{\lambda_2 - \lambda_1} y_0' \end{aligned} \tag{13}$$

Here $\varphi_{1,2} = \exp(\lambda_{1,2} z)$. For aperiodic case:

$$\frac{\lambda_2\varphi_1 - \lambda_1\varphi_2}{\lambda_2 - \lambda_1} = \exp(-\alpha z)\left[ch\alpha_0\, z + \frac{\alpha}{\alpha_0} sh\alpha_0\, z\right] \tag{14}$$

$$\frac{\varphi_2 - \varphi_1}{\lambda_2 - \lambda_1} = \frac{\alpha \exp(-\alpha z)}{\alpha_0} sh\alpha_0\, z \tag{15}$$

$$\frac{\lambda_2\varphi_2 - \lambda_1\varphi_1}{\lambda_2 - \lambda_1} = \exp(-\alpha z)\left[ch\alpha_0\, z - \frac{\alpha}{\alpha_0}\, sh\alpha_0 z\right] \tag{16}$$

We can obtain similar expressions for periodic case from Eqs. (14)–(16) using change:

$$\left.\begin{aligned} ch\alpha_0\, z &\to \cos w_1 z \\ \frac{sh\alpha_0\, z}{\alpha_0} &= \frac{\sin w_1 z}{w_1} \end{aligned}\right\} w_1 = \sqrt{\Omega^2 - \alpha^2} \tag{17}$$

Using Eqs. (14)–(16) it is easy to calculate an evolution of moments and the emittance by direct calculations. However, in order to take into account the heating it is simpler to use method of moments.

EQUATIONS FOR MOMENTS AND ITS SOLUTION

Using Eq. (1), we derive a system of the first order equations for moments $\langle y^2\rangle$; $\langle y\frac{dy}{dz}\rangle$ and $\langle\left(\frac{dy}{dz}\right)^2\rangle$. Let us define: $\langle y^2\rangle = u$; $\langle y\frac{dy}{dz}\rangle = v$, $\langle\left(\frac{dy}{dz}\right)^2\rangle = t$. Then we can write

$$\frac{du}{dz} = 2v \tag{18}$$

$$\frac{dv}{dz} = t - 2\alpha v - \Omega^2 u \tag{19}$$

$$\frac{dt}{dz} = -2\Omega^2 v - 4\alpha t + W_y \tag{20}$$

$$\frac{d\epsilon_y^2}{dz} = -4\alpha\epsilon_y^2 + tW_y \tag{21}$$

It is interesting to compare Eq. (20) with "standard equation" for $\langle(\Delta p)^2\rangle$ which is usually used in literature [1,2]. Using Eq. (4), we obtain:

$$\frac{d}{dz}\langle(\Delta p)^2\rangle = -4\alpha\langle(\Delta p)^2\rangle - 2\Omega\frac{2\Lambda}{p}\langle y\cdot\Delta p\rangle + W_p \tag{22}$$

We see that this equation differs from the standard one by a presence of the additional second term in RHS. Really, this term does not have affect on equilibrium parameters; however, this term has to be taken into account for calculation of evolution of the moments and the emittance in time.

Let us consider a case when α, Ω are constant. Then our system of four linear equations with constant coefficients can be solved analytically.

Equilibrium solution of the system can be found, if we assume that all derivatives are equal to zero. Then (for α, Ω, W_y are constant) we obtain

$$\langle y^2\rangle_{eq} = W_y/4\alpha\Omega^2 \tag{23}$$

$$\left\langle\left(\frac{dy}{dz}\right)^2\right\rangle_{eq} = W_y/4\alpha \tag{24}$$

$$\epsilon_y^{eq} = W_y/4\alpha\Omega \tag{25}$$

We see that for $\Omega = 0$ an equilibrium solution is absent.

If at an initial point the beam has a longitudinal crossover, initial conditions may be written as follows:

$$\left.\begin{array}{ll} u(z=0) = u_0; & v(z=0) = 0; \\ t(z=0) = t_0; & \epsilon(z=0) = \epsilon_0 \end{array}\right\} \tag{26}$$

We can solve our system with these initial conditions by use of Laplace transform (see Appendix).

In a case of "superdamping" ($\alpha \gg \Omega$) it is possible to obtain comparatively simple asymptotic expressions for all interesting parameters. Let us assume that $x = \alpha/\Omega$, $\tau = \Omega z$, and that the initial beam is "matched," i.e., $t_0 = \Omega^2 u_0$. Then, using Eqs. (45)–(47) we obtain (for $exp(-2x\tau) \ll 1$):

$$\frac{u_{fr}}{u_0} = \frac{1}{2} exp(-\tau/x) \frac{x}{\sqrt{x^2-1}} \left[\frac{x}{\sqrt{x^2-1}} - 1 \right] \tag{27}$$

$$\frac{v_{fr}}{u_0} = -\frac{1}{2} \Omega exp(-\tau/x) \frac{x}{x^2-1} \tag{28}$$

$$\frac{t_{fr}}{u_0} = \frac{1}{2} \Omega^2 exp(-\tau/x) \frac{x}{\sqrt{x^2-1}} \left[\frac{x}{\sqrt{x^2-1}} - 1 \right] \tag{29}$$

where u_{fr}, v_{fr} and t_{fr} correspond to free oscillations (for $w_y = 0$). It is easy to see, that

$$u_{fr} \cdot t_{fr} - v_{fr}^2 = 0 \tag{30}$$

Thus, we observe the following mechanism of the longitudinal free emittance damping in "superdamping" regime: a) the beam size is damped very slowly and practically doesn't change (see Eq. (27)); b) the beam spread on momenta is fast decreased to asymptotic value:

$$\frac{t}{t_0} = \left[\frac{1}{2} \frac{x}{\sqrt{x^2-1}} \left(\frac{x}{\sqrt{x^2-1}} - 1 \right) \right]^{1/2} \tag{31}$$

and then it is diminished very slowly.

Attenuation of the emittance is due to the "crossing" moment term $\left(\langle y \frac{dy}{dz} \rangle \right)$.

In the region of slowly changed t, value of ϵ_y^2 is determined to high accuracy by the following formula:

$$\epsilon_y^2 \simeq \frac{u(z) \cdot W_y}{4\alpha} \tag{32}$$

Thus, in order to find $u(z)$ we must add to u_{fr} an expression for u_{heat}. Using Eq. (48), we find:

$$\frac{u_{\text{heat}}}{u_{eq}} \simeq 1 - \frac{x \cdot \left(x + \sqrt{x^2-1} \right)}{(x^2-1)2} exp(-\tau/x) \tag{33}$$

Using Eq. (27) and (32)–(33) we obtain the final result:

$$\epsilon_y \simeq \epsilon_{eq} \left[1 + \left(\frac{u_0}{u_{eq}} - 1 \right) exp\left(-\frac{\tau}{x} \right) \right]^{1/2} \tag{34}$$

Usually $\frac{u_0}{u_{eq}} = \frac{\epsilon_0}{\epsilon_{eq}} \gg 1$, and with good accuracy we can write:

$$\epsilon_y \simeq (\epsilon_{eq} \cdot \epsilon_0)^{1/2} exp(-\tau/2x) \tag{35}$$

It is easy to see that for normalized emittances Eq. (35) is also valid. For normalized equilibrium emittance we have the following expression:

$$\epsilon_{eq} = \frac{1}{4\alpha \cdot \Omega} \cdot \frac{d}{dz} \frac{\langle(\Delta p)^2\rangle}{(mc)} = \frac{1}{4\alpha \cdot \Omega \beta^2 (mc^2)} \frac{d}{dz} \langle(\Delta E)^2\rangle \tag{36}$$

Here

$$\frac{d}{dz}\langle(\Delta E)^2\rangle = K_s \gamma^2 \left(1 - \frac{\beta^2}{2}\right) \tag{37}$$

$$K_s = 4\pi (r_e m_e c^2)^2 \frac{\rho Z}{A} \tag{38}$$

Here r_e, m_e - classical radius and mass of electron; m - muon mass ; ρ, Z, A are density, charge and atomic number of the wedge material.

DISCUSSION

We see that kinetics of the longitudinal ionization cooling for strong damping coefficient have a lot of special features:

- Longitudinal oscillations of individual particles have nonperiodic character with different damping rate of two modes of oscillations.
- During the cooling longitudinal beam becomes "unmatched."
- Longitudinal beam size is changing very slowly, and therefore time necessary to reach the equilibrium emittance is increased with an enhancement of the longitudinal damping rate.

It is clear that these special features of longitudinal cooling should be taken into account when we look for the optimal characteristics of the cooling system.

APPENDIX

Let us apply Laplace transform to Eqs. (19)–(21) with inital conditions (26). Thus we obtain

$$\left.\begin{array}{c} pU - 2V = u_0 \\ \Omega^2 U + (p + 2\alpha)V - T = 0 \\ 2\Omega^2 V + (p + 4\alpha)T = \frac{w}{p} + t_0 \end{array}\right\} \tag{39}$$

For emittance we have the following equation

$$(p+4\alpha)\Phi = \Phi_0 + wU(p) \tag{40}$$

Solving this linear system, we have:

$$U(p) = \frac{2(\Omega^2 u_0 + t_0) + u_0(p+2\alpha)(p+4\alpha)}{\Delta_0(p)} + \frac{2w}{p\Delta_0(p)} \tag{41}$$

$$V(p) = \frac{p(t_0 - \Omega^2 u_0) - 4\alpha u_0 \Omega^2}{\Delta_0(p)} + \frac{w}{\Delta_0(p)} \tag{42}$$

$$T(p) = \frac{2\Omega^4 uo + t_0[p(p+2\alpha) + 2\Omega^2]}{\Delta_0(p)} + \frac{w\,[p(p+2\alpha)+2\Omega^2]}{p\Delta_0(p)} \tag{43}$$

Denominator $\Delta_0(p)$ is given by $\Delta_0(p) = (p+2\alpha)(p^2 + 4\alpha p + 4\Omega^2)$.
The first terms in Eqs. (41)–(43) describe evolution of free oscillations, the second ones - evolution due to heating. Substituting Eq. (42) in Eq. (41), we obtain:

$$\Phi = \frac{\Phi_0}{p+4\alpha} + \frac{w\,[2(\Omega^2 u_0 + t_0) + u_0(p+2\alpha)(p+4\alpha)]}{\Delta_0(p)(p+4\alpha)} + \frac{2w^2}{p(p+4\alpha)\Delta_0(p)} \tag{44}$$

Here U, V, T and Φ are, correspondingly, Laplace transforms for u, v, t and ϵ^2. The first term describes a damping of the initial emittance, the second term describes the emittance growth due to synergism between free oscillations and the heating, the last term describes the emittance growth due to heating in an absence of an initial phase volume.
Using the backward Laplace transform, we obtain for terms, describing a damping of free oscillations:

$$u_{fr}(z) = exp(-2\alpha z)\left\{u_0\left(ch2\alpha_0 z + \frac{\alpha}{\alpha_0} sh2\alpha_0 z\right) + \frac{\Omega^2 u_0 + t_0}{2\alpha_0^2}\,(ch2\alpha_0 z - 1)\right\} \tag{45}$$

$$v_{fr}(z) = exp(-2\alpha z)\left\{\frac{\alpha(t_0 + \Omega^2 u_0)}{2\alpha_0^2}(1 - ch2\alpha_0 z) + \frac{t_0 - \Omega^2 u_0}{2\alpha_0} sh2\alpha_0 z\right\} \tag{46}$$

$$t_{fr}(z) = exp(-2\alpha z)\left\{\frac{\Omega^2(t_0 + \Omega^2 u_0)}{2\alpha_0^2}(ch2\alpha_0 z - 1) + t_0\left(ch2\alpha_0 z - \frac{\alpha}{\alpha_0} sh2\alpha_0 z\right)\right\} \tag{47}$$

For the second terms (due to heating) we obtain:

$$\frac{u_{\text{heat}}(z)}{u_{eq}} = 1 + \frac{\Omega^2}{\alpha_0^2} exp(-2\alpha z) - \frac{\alpha exp(-2\alpha z)}{\alpha_0^2} \cdot (\alpha_0 sh2\alpha_0 z + \alpha ch2\alpha_0 z) \quad (48)$$

$$v_{\text{heat}}(z) = \frac{w}{4\alpha_0^2} exp(-2\alpha z)(ch2\alpha_0 z - 1) \quad (49)$$

$$\frac{t_{\text{heat}}(z)}{t_{eq}} = 1 + exp(-2\alpha z) sh2\alpha_0 z \left(1 - \frac{\alpha}{\alpha_0}\right) - exp(-2\alpha z) \frac{\alpha^2 ch2\alpha_0 z - \Omega^2}{\alpha_0^2} \quad (50)$$

It is easy to see, that $\frac{u_{\text{heat}}}{u_{eq}}$ and $\frac{t_{\text{heat}}}{t_{eq}} \to 1$ for $z \to \infty$; $v_{\text{heat}} \to 0$ for $z \to \infty$. Expresion for $\epsilon^2(z)$ have the more complicated form:

$$\begin{gathered} \epsilon^2_{(z)} = \epsilon_0^2 exp(-4\alpha z) + \tfrac{w(\Omega^2 u_0 + t_0)}{4} \\ \Big\{ -\tfrac{1}{4\alpha} \left[\tfrac{exp(-4\alpha z)}{\Omega^2} + \tfrac{exp(-2\alpha z)}{\alpha_0^2} \right] + \tfrac{exp(-2\alpha z)}{2\alpha_0^2 \Omega^2} [\alpha ch2\alpha_0 z - \alpha_0 sh2\alpha_0 z] + \\ \tfrac{w u_0}{\alpha_0} exp(-2\alpha z) sh2\alpha_o z + \\ \epsilon^2_{eq} \left[1 + exp(-4\alpha z) - 2exp(-2\alpha z) \tfrac{\alpha^2 ch2\alpha_0 z - \Omega^2}{\alpha_0^2} \right] \Big\} \end{gathered} \quad (51)$$

Here $\alpha_0 = \sqrt{\alpha^2 - \Omega^2}$. Expressions for oscillation regime can be found by substitutions using:

$$\left.\begin{array}{ccc} ch2\alpha_0 z & \to & \cos 2w_1 z \\ \dfrac{sh2\alpha_0 z}{\alpha_0} & \to & \dfrac{\sin 2w_1 z}{w_1} \end{array}\right\} w_1 = \sqrt{\Omega^2 - \alpha^2} \quad (52)$$

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