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A Lattice Field Theory Primer

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Outline

- Introduction
- Lattice Fermions
- Calculating masses and matrix elements
- Accessing the chiral limit
- Renormalization of Operators
- Systematic errors

Introduction

Physical observables in QFT calculated in path integral formulation. Schematically,

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{Z} \int \mathrm{d[U]}\, \mathcal{O}(U)\, \exp\left(-i\, g\, S(U)\right)$$

If coupling g is small, expand exponential. $\langle \mathcal{O} \rangle$ is calculated to some prescribed order in the coupling, g^n. Use Feynman diagrams.

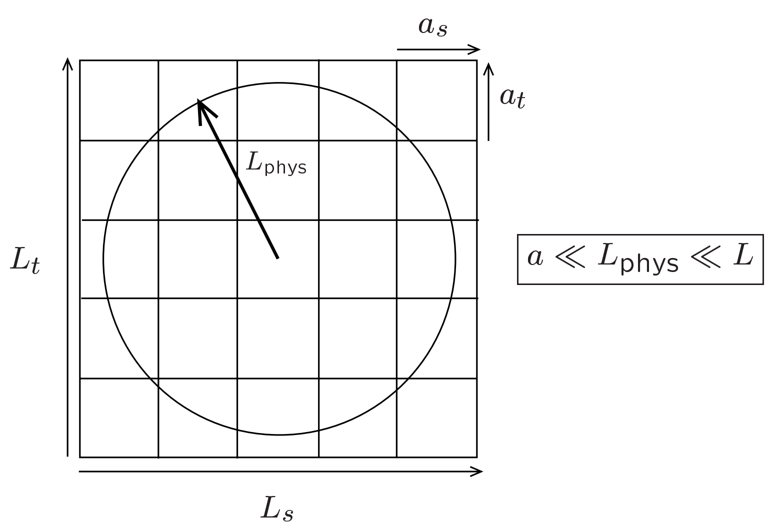
If coupling is not small (low energy QCD) can't expand exponential. Or if bound states required can't use PT. Just do the whole integral. Use lattice/numerical monte-carlo techniques.

Either way, integrals are in general divergent: ∞ number of degrees of freedom (fields) that can take values from $-\infty$ to $+\infty$.

Must make them finite $\longrightarrow$ regularize. Many ways to do this, but must be careful not to destroy symmetries of the original theory (at least they must be recovered when the regulator is removed)

Non-perturbative regularization

Discretize the continuum action on a four-dimensional (Euclidean) space-time lattice with *spacing a*. [K.G. Wilson, 1974]



- Long dist (low energy) physics is insensiitve to a (scaling)
- Path integrals finite: finite number of degrees of freedom (sites)
- Momentum cut-off $p_{\text{max}} \sim 1/a$

Do this in a *gauge invariant* way

Replace continuum vector potential (Gluon fields), $A_\mu = A_\mu^a \lambda^a$ with

$$A_\mu(x) \to U_\mu(x) \;=\; e^{-igaA_\mu(x)}$$

The "link" $U_\mu(x)$ is an element of the group SU(N), with gauge transformation $g(x)$

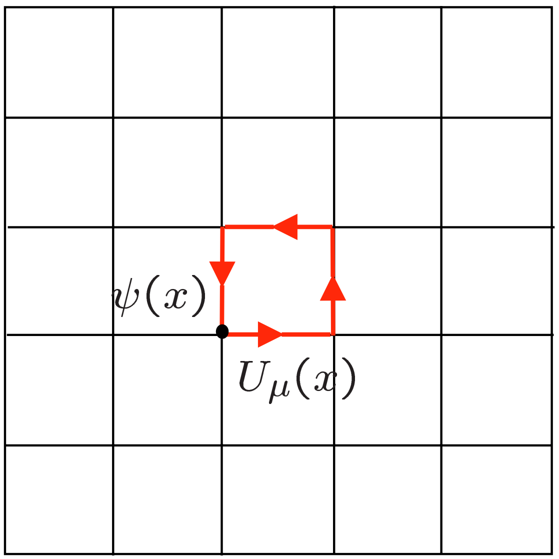
$$U_\mu(x) \to g(x)\, U_\mu(x)\, g(x+\widehat{\mu})^\dagger \qquad U_\mu(x),\, g(x) \,\in\, SU(N)$$

So a path-ordered product of link fields transforms like

$$g(x)\, U_\mu(x)\, U_\mu(x+\widehat{\mu}) \cdots U_\nu(y)\, g(y+\widehat{\nu})^\dagger$$

If the path is a closed loop, *e.g.*

$$g(x)\,U_\mu(x)\,U_\nu(x+\widehat{\mu})\,U_\mu^\dagger(x+\widehat{\nu})\,U_\nu^\dagger(x)\,g(x)^\dagger$$



And we take the trace, it is gauge invariant. Generally true that the trace of (any) closed path-ordered product of links is gauge invariant.

Treating the fermions is (naively) straightforward. Transcribe the continuum field $\psi(x)$ to the lattice site x.

$$\psi(x) \;\; \longrightarrow \;\; \psi_x^{\mathsf{latt}}$$

Under a gauge transformation,

$$\begin{aligned} \psi_x^{\mathsf{latt}} \;\; &\longrightarrow \;\; g(x)\,\psi_x^{\mathsf{latt}} \\ \bar{\psi}_x^{\mathsf{latt}} \;\; &\longrightarrow \;\; \bar{\psi}_x^{\mathsf{latt}} g^{-1}(x) \end{aligned}$$

Construct the action.

Work in Euclidean space: analytically continue $t \to i\tau$ (Wick rotate) so metric is

diag(1,1,1,1) and not diag(-1,1,1,1).

Covariant and contravariant indices mean the same thing.

Fermions first.

For a single flavor

$$\begin{aligned} S_f &= \int d^4x\, \bar\psi(x)(\not{D} + m)\psi(x) \\ &\to \bar\psi_x\, M_{xy}\, \psi_y \end{aligned}$$

ψ is now a 12 component vector (3 colors $\times$ 4 spins) at each site on the lattice.

fermion matrix is a 12×12 matrix each pair of n.n. sites

$$M_{xy} = \sum_\mu \gamma_\mu \frac{U_\mu(x)\, \delta_{x+a\hat\mu,y} - U^\dagger_\mu(x-\hat\mu)\, \delta_{x-a\hat\mu,y}}{2} + am\, \delta_{x,y}$$

Factors of the links make the lattice action gauge-invariant.

A large, sparse matrix: $(L^4 \times 12) \times (L^4 \times 12)$. Can invert it in $\mathcal{O}(12 \times L^4)$ operations, not $\mathcal{O}((12 \times L^4)^2)$

Gluon action

$$S_g = \int \mathrm{d}^4x \left(\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\right) \;\rightarrow\; S_g^{latt} = \frac{6}{g^2}\sum_{\text{sites}}\sum_{\mu>\nu}(\mathcal{R}\mathrm{Tr}\Box_{\mu\nu})$$

Which you can check by expanding

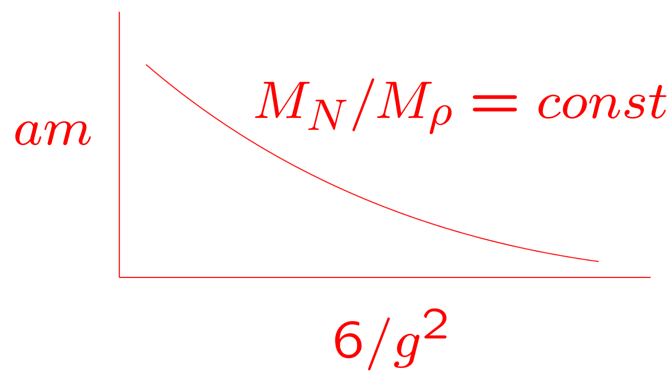
$$\lim_{a\to 0} U_\mu(x) = 1 - igaA_\mu(x) + \cdots$$

and neglecting terms of $\mathcal{O}(a^2)$ and higher.

At this point, the lattice action, $S_f + S_g$, has all the symmetries of the continuum, except Euclidean (Lorentz) invariance which is broken down to (invariance under) the Hypercubic group H(4).

The continuum limit, $a \to 0$ (remove the regulator).

Adjust *bare* coupling, $6/g^2$, and quark mass(es) am to give some observable its physical value, say M_N/M_ρ. Move toward a=0, $g, m \to 0$ keeping M_N/M_ρ fixed. Predict all other (ratios of) physical observables on this (renormalization "group") trajectory.



How do we know this works?
Answer: asymptotic freedom of QCD: non-trivial continuum limit

For sufficiently small g, solution of the QCD β function (physics does not depend on the lattice spacing (regulator)) reads:

$$a\,\Lambda_{QCD} = (g^2\,\gamma_0)^{-\gamma_1/(2\gamma_0^2)}\;\mathrm{exp}\,(-1/(2\gamma_0^2\,g^2)(1+\mathcal{O}(g^2))$$

On lattice then, in the asymptotic scaling regime, all observables scale this way, so in particular, *ratios* of physical observables (*e.g.* M_N/M_ρ, or anythingelse you can think of) are independent of the lattice spacing$\rightarrow$ the renormalization group trajectory.

In practice, the scaling regime is hard to access:

"critical slowing down": as $a \longrightarrow 0$ lattice correlation lengths diverge. Physics is scale invariant. Continuum limit is a 2nd order phase transition.

Instead, simulate at several values of $6/g^2$ (modest lattice spacings) and several quark masses at each lattice spacing.

Extrapolate in quark mass to desired physical point, then extrapolate to $a \rightarrow 0$ in leading discretization error, *i.e.* linear or quadratic in a.

Monte Carlo Simulation

Back to the path integral

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{Z}\int \mathrm{d}[\bar\psi,\psi,\mathrm{U}]\,\mathcal{O}(\bar\psi,\psi,U)\;\exp\left(-i\,S(\bar\psi,\psi,U)\right)$$

Analytically continue (Wick rotate) to Euclidean space-time so the integrand behaves sensibly:

$$\langle \mathcal{O}_E \rangle \;=\; \frac{1}{Z_E}\int \mathrm{d}[\bar\psi,\psi,\mathrm{U}]\,\mathcal{O}_E(\bar\psi,\psi,U)\;\exp\left(-S_E(\bar\psi,\psi,U)\right)$$

(Now drop all "E" subscripts)

Fermion integrals are Gaussian, do them analytically.

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{Z}\int [dU]\mathcal{O}(U)\,\mathrm{det}(\mathrm{M}(U))^{n_f} e^{-S_g(U)}$$

$\det(\mathrm{M}(U))^{n_f} e^{-S_g}$ is an ordinary probability weight: do the integral over gauge fields numerically by Monte Carlo simulation (stat. mech. in d+1 dimensions).

Use *importance sampling* to generate an ensemble of gauge field configurations ($\mathcal{O}(100-1000)$ independent ones):

- 1 configuration = set of link variables over entire lattice
- update algorithm: choose links randomly
- algorithm must satisfy detailed balance and ergodicity

- generate configurations with probability $\det(\mathrm{M}(U))^{n_f} e^{-S_g}$

- Observables become simple averages over configurations.

Simulation with $\det(\mathrm{M}(U))$ (dynamical fermions) is costly.
$\det(\mathrm{M}(U)) = 1$ is the quenched approximation, *i.e.*, no virtual quark loops in the vacuum ($m_q \to \infty$).

Fermion discretizations

(why not naive fermions?)

$$\bar{\psi}\not{D}\psi \;\rightarrow\; \bar{\psi}\gamma_\mu(\psi(x+\hat{\mu}) - \psi(x-\hat{\mu}))/2a$$

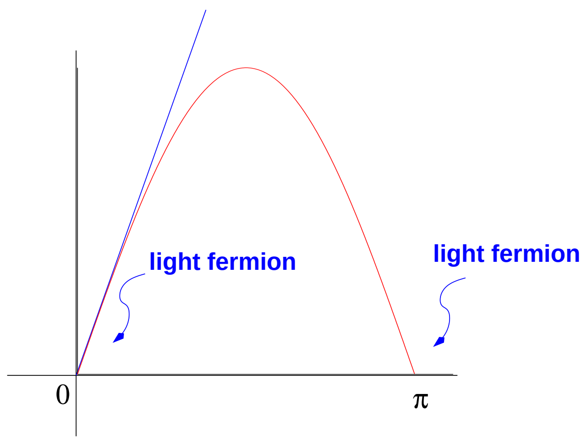
$$G_{latt}(p) \;=\; \frac{i\gamma_\mu \sin(ap_\mu)}{\sum_\mu \sin^2(ap_\mu)} \rightarrow \frac{i\gamma_\mu ap_\mu}{\sum_\mu (ap_\mu)^2}.$$

$G_{latt}(p)$ has a pole at each corner of the Brilloiun zone:
$p^\mu = (\pi/a,0,0,0),\ (0,\pi/a,0,0),\ldots,(\pi/a,\pi/a,\pi/a,\pi/a)$

Lattice theory corresponds to 2^d fermion flavors instead of one.

These extra fermions are called **doublers**. Appeared because of the inherent periodicity of the lattice.

Minkowski space dispersion relation ($E = |p|$)



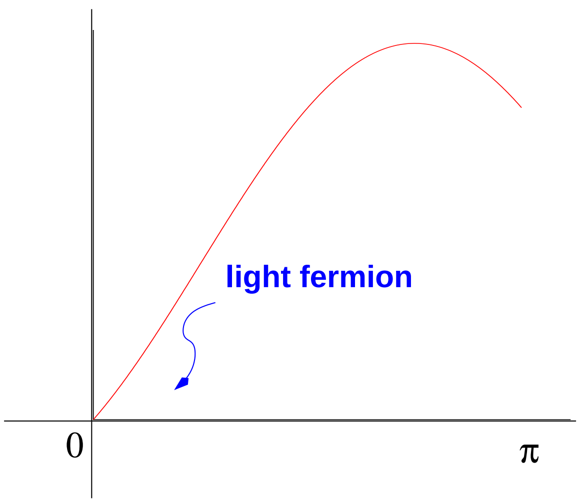
Even worse for the Standard model, the doublers appear in pairs with opposite chirality–theory is **vector-like**(*Nielsen-Niyomiya No-Go theorem*). Deep connection to gauge- invariance.

Must get rid of the doublers.

1. Wilson fermions. Add an irrelevant term to the action

$$\begin{aligned} S_W &= \frac{a}{2}\bar{\psi}\,\partial^2\,\psi \\ &\sim \frac{1}{a}\sum_{\mu} 1 - \cos(p_\mu) \end{aligned}$$

Like a mass term. Doubler mass $\sim 1/a$, and they decouple.



Problems with Wilson Fermions:

- Chiral symmetry (of QCD) is explicity broken, badly broken. (flavor symmetry is still exact, as in the continuum)

- Chiral limit $\neq m_q \rightarrow 0$.

- Complicated **fine tuning** (operator mixing) of observables required for correct chiral behavior.

- Errors are $O(a)$: slow approach to the continuum (can be improved to $O(a^n)$ $n = 2$ now, big job)

All problems solved as $a \rightarrow 0$

2. Kogut-Susskind.

Spin diagonalization. Throw away 3/4 of components: 16 Dirac fermions = 64 componets $\rightarrow$ 16. One component "spinor" on a lattice site.

Exact remnant chiral symmetry, so $m_q \rightarrow 0$ is the chiral limit

Can reconstruct 4 Dirac fermions from components in 2^4 hypercube. In the continuum limit this is a theory of 4 degenerate quarks. For $a \neq 0$ flavor, spin, and space-time symmetries are mixed.

Take fractional power of fermion determinant to simulate real QCD (2+1 flavor).

Problems with Kogut-Susskind fermions

- Have to take fractional powers of the determinant!

- Flavor symmetry is broken: one light pion instead of $16-1=15$

- Relation to continuum operators can be very difficult to work out

- Errors are $O(a^2)$ but are unusually large because of flavor symmetry violation. Again, slow approach to continuum. Can be improved: now the state-of-the-art for dynamical fermion simulations (a^2-tad).

3. **Ginsparg-Wilson fermions**.

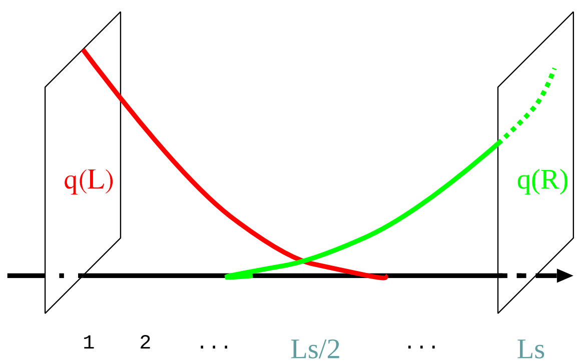
Discovered in 1987 (then forgotten) the most chiral symmetry that a lattice theory can have

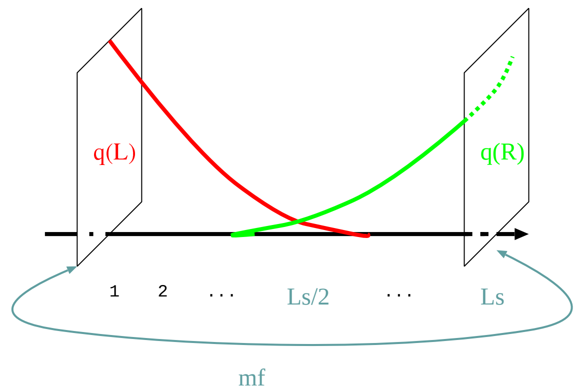
$$\begin{aligned}\gamma_5\, D + D\,\gamma_5 &= a\, D\, R\,\gamma_5\, D \\ \gamma_5\, D^{-1} + D^{-1}\,\gamma_5 &= a\, R\,\gamma_5\end{aligned}$$

Meanwhile, domain wall fermions (DWF) (Kaplan 1992) and later overlap fermions (Neuberger 1997) were discovered. Worked for vector gauge theories. Hasenfratz rediscovered the G-W relation, and it was soon realized that DWF and overlap are examples (with $R = 1/2$).

G-W fermions Remove the doublers while (essentially) preserving full $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry of the continuum **at non-zero lattice spacing**.

We (RBC) use domain wall fermions (Shamir 1993)





Errors are $O(a^2)$

Problems with Ginsparg-Wilson fermions

- Expensive!

- 1st large-scale dynamical fermion simulation done here at BNL (and Columbia University). Light (up and down) quark mass 1/2 to 1 times $m_{strange}$ (need to reduce by 10). Volume is not large ($\sim$ (2fm)^3), and only one lattice spacing.

- Took almost 2 years on our own supercomputer (QCDSP)!

Continuum-like properties $\rightarrow$ approach to continuum is faster

New computer(s) coming: QCDOC ($\times$20 faster, 5 TFlops/sustained)

Masses and Matrix elements from Euclidean space correlation functions.

Consider the pseudo-scalar meson (pion) 2-point correlation function

$$J_5(t) \;=\; \sum_{\vec{x}} \bar{\psi}(x,\,t)\,\gamma_5\,\psi(x,\,t)\,e^{\vec{p}\cdot\vec{x}}$$

Sum over $\vec{x}$ projects onto the state with momentum $\vec{p}$

The zero momentum correlation function reads

$$C(t) \;=\; \sum_{x} \langle 0|\bar{\psi}(x,\,t)\,\gamma_5\,\psi(x,\,t)\,\bar{\psi}(0,0)\,\gamma_5\,\psi(0,0)|0\rangle$$

Wick contract fields into quark propagators

$$C(t) \;=\; \sum_{\vec{x}} \mathrm{Tr}\left[M^{-1}_{0;x,t}\,\gamma_5\,M^{-1}_{x,t;0}\gamma_5\right]$$

What's it good for?

Use time-translation operator $U = \exp(-H\,t)$ and insert a complete set of states (H is the QCD Hamiltonian, and the states are eigenstates of H) (in Euclidean space there is no i in U)

$$
\begin{aligned}
C(t) &= \sum_x \langle 0|e^{H\,t}\bar\psi(x)\,\gamma_5\,\psi(x)\,e^{-H\,t}\bar\psi(0)\,\gamma_5\,\psi(0)|0\rangle \\
&= \sum_x \langle 0|e^{H\,t}\bar\psi(x)\,\gamma_5\,\psi(x)\,e^{-H\,t}\sum_{\mathsf{n}}\frac{|\mathsf{n}\rangle\langle\mathsf{n}|}{2E_{\mathsf{n}}V}\bar\psi(0)\,\gamma_5\,\psi(0)|0\rangle \\
&= \sum_{\mathsf{n}} \langle 0|\bar\psi\,\gamma_5\,\psi\,|\mathsf{n}\rangle\langle\mathsf{n}|\bar\psi\,\gamma_5\,\psi|0\rangle\frac{e^{-E_{\mathsf{n}}\,t}}{2\,E_{\mathsf{n}}V} \\
\lim_{t\to\infty} &= \frac{|\langle 0|\bar\psi\,\gamma_5\,\psi|\pi\rangle|^2}{2\,m_\pi}\,e^{-m_\pi\,t}
\end{aligned}
$$

Fit yields physical particle mass and matrix element.

or the nucleon 3 point correlation function,

$$\langle \chi_N(p',\, t')\, \sum_x e^{\mathrm{i}\vec{q}\cdot\vec{x}} [\bar{\psi}_q(x,t)\, \Gamma_\mu \psi_q(x,t)]\, {\chi^\dagger}_N(p,0)\rangle \;\; \rightarrow$$

$$\sum_{s,s'} \langle 0|\chi_N(p',s')|p',s'\rangle\langle p',s'|\Gamma_\mu(q)|p,s\rangle\langle p,s|\chi^\dagger_N(p,s)|0\rangle \;\; \times$$

$$\frac{e^{-E\,t\,-E'\,(t'-t)}}{2\,E\,2\,E'}$$

where $t' \gg t \gg 0$, $\vec{q} = \vec{p'} - \vec{p}$, and χ_N is the nucleon interpolating operator

Euclidean space continued LSZ reduction formula that relates (the Fourier transform of) Minkowski space Greens functions to S-matrix elements. Exponentials pick them out instead of poles.

This always works for single-particle states (like nucleon matrix elements).

For multi-paritcle states (*i.e.* non-leptonic decays) this is much more difficult

Accessing the chiral limit, $m_q \to 0$

Ideally, adjust the quark masses in our simulations until observables (masses, decay constants, ...) match their physical values

$e.g.$, adjust m_u and m_d until the pseudo-scalar meson mass is $m_\pi = 135$ MeV. Knowing the value of the light quark masses, we can *predict* the proton mass, neutron mass, f_π, *etc*.

Not so simple. The chiral limit, $m \to 0$ is difficult.

- "cost" of quark propagator M^{-1}: #iterations $\sim \frac{1}{m}$

- Compton wavelength of the pion $\frac{1}{m_\pi} \to \infty$ as $m_q \to 0$, so must take $V \to \infty$ to avoid finite volume effects

- Instead, work at unphysical (larger) m_q and extrapolate to the physical regime (chiral limit). Use Chiral Perturbation Theory as a guide.

Chiral Perturbation Theory (S. Weinberg)

Low energy effective field theory of QCD. Systematic expansion in p^2, around $p^2 = 0$ (chiral limit). (Pseudo-) Goldstone bosons are the only degrees of freedom left.

$$\begin{aligned}
\mathcal{L}^{(2)}_{QCD} &= \frac{f^2}{8}\mathrm{tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + \frac{f^2 B_0}{4}\mathrm{tr}[\chi^\dagger \Sigma + \Sigma^\dagger \chi] \\
\Sigma &= \exp\left[\frac{2i\phi^a \lambda^a}{f}\right] \\
\Sigma &\to V_L \Sigma V_R^\dagger \qquad \text{(under a chiral transformation)}
\end{aligned}$$

Σ is the unitary chiral matrix field ($V_{L,R} \in SU(N_f)$), λ^a are proportional to the Gell-Mann matrices with $\mathrm{tr}(\lambda_a \lambda_b) = \delta_{ab}$, ϕ^a are the real pseudoscalar-meson fields, and f is the meson decay constant in the chiral limit. $\chi = (m_u, m_d, m_s)_{\text{diag}}$

To lowest order

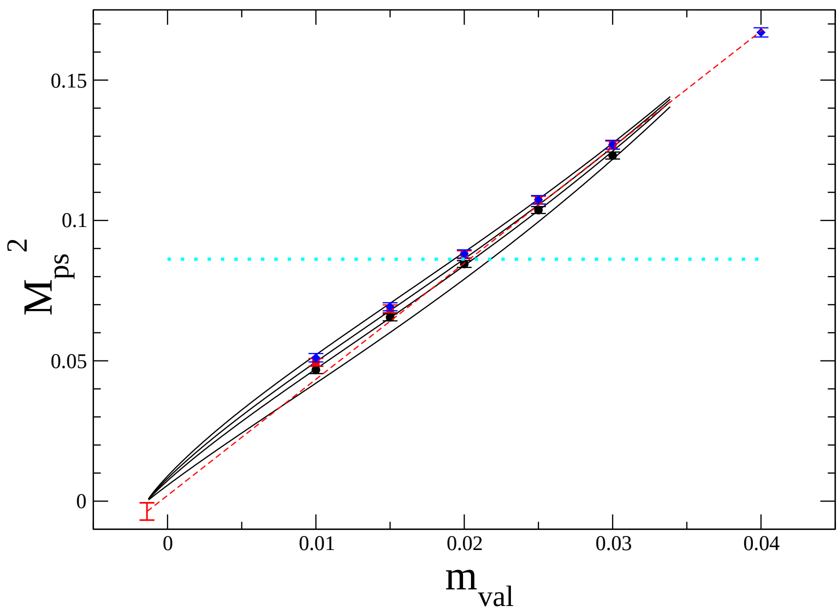
$$
\begin{aligned}
m_\pi^2 &= B_0(m_u + m_d) \\
m_K^2 &= B_0(m_d + m_s) \\
&\cdots
\end{aligned}
$$

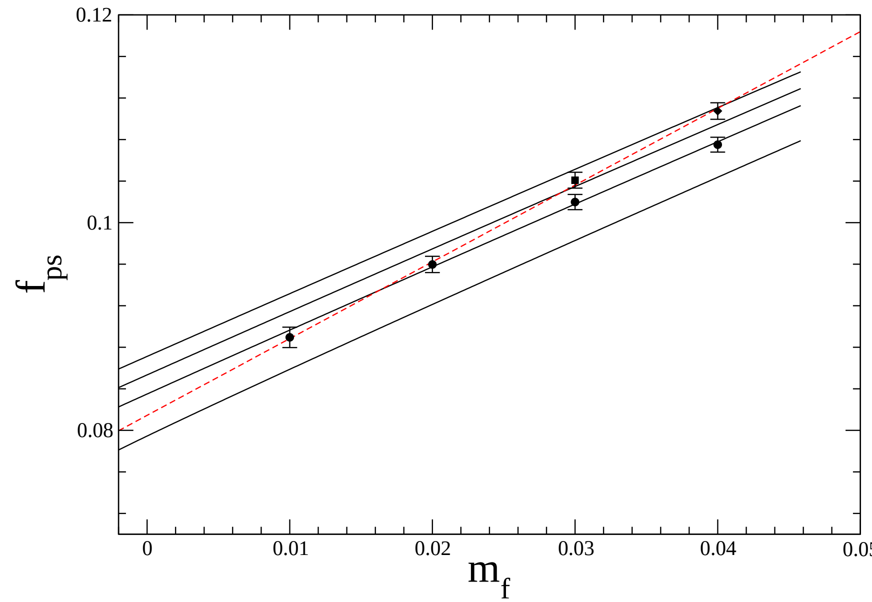
At this order, we can work with mesons made from *degenerate* quarks, so the quark masses corresponding to the physical mesons are

$$
\begin{aligned}
m_l &= \frac{m_u + m_d}{2} \\
m_s/2 &= \frac{m_d + m_s}{2}
\end{aligned}
$$

Can go to higher order in χPT ($\mathcal{O}(p^4)$)

RBC $n_f = 2$ dynamical quark simulation:





$f_K/f_\pi = 1.194(12)$ (statistical error only)

Operator Renormalization

In lattice QCD calculations, we often calculate matrix elements of local operators generated by an Operator Product Expansion (OPE) of a non-local operator (usually a product of two currents). *e.g.* DIS, or non-leptonic Weak decay of hadrons.

We do this out of necessity since the physical processes can not be calculated purely perturbatively or non-perturbatively.

$$\mathcal{A}^{\text{phys}} = \sum_n C_n(\mu) \, \langle f|\mathcal{O}_n(\mu)|i\rangle$$

$\mathcal{A}^{\text{phys}}$ and *states* do not depend on scale μ

Define finite, renormalized operator at scale μ

$$\mathcal{O}(\mu) \ = \ Z_{\mathcal{O}}(a\mu)\mathcal{O}(a)$$

$Z_{\mathcal{O}}(a\mu)$ can be computed:

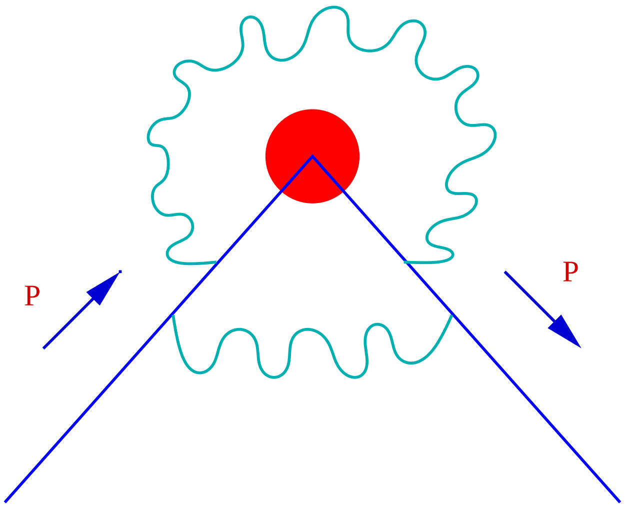
- In lattice perturbation theory
- Non-perturbatively (RI-MOM) (mimic perturbation theory $\sim$ very high order perturbative calculation)
- perturbative matching to $\overline{\mathrm{MS}}$, or whatever scheme is used to compute $C_n(\mu)$

Lattice complications:

Broken symmetries (Lorentz, chiral symmetry, flavor, ...) $\Longrightarrow$ operator mixing

Non-perturbative renormalization (NPR) *required* when mixing with lower dimensional operators occurs. These are power divergent in the lattice spacing $a^{-(d-d')}$ instead of the usual logarithmic divergence $\log(a\mu)$ (... domain wall fermions)

To calculate $Z_{\mathcal{O}}$ compute Landau gauge off-shell matrix elements of $\mathcal{O}(a)$ between quark and/or gluon states

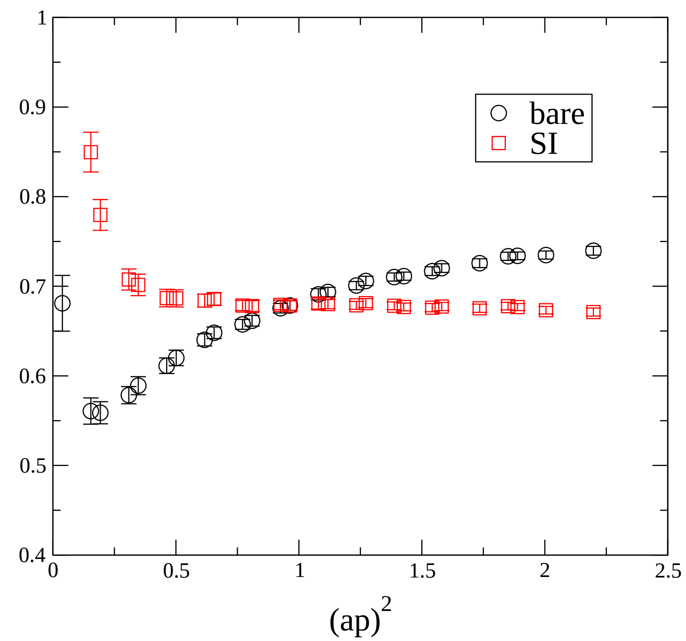


$$TrV_{\mathcal{O}}(p^2)\Gamma\Big|_{p^2=\mu^2}\frac{Z_{\mathcal{O}}}{Z_q} = 1$$

- $V_{\mathcal{O}}(p^2)$ the amputated vertex constructed from the full non-pert quark propagator
- Γ a projector

This defines the MOM scheme. Extrapolate to $m_f \to 0$ and we have the RI scheme (Regularization Independent).

Martinelli et.al. Nuc.Phys.B445 81 (1995)



$Z_s(\mu^2)$ $(\bar{\psi}\psi)$ renormalization factor, and divided by 3-loop perturbative running.
RBC (2001).

Statistical and Systematic errors

- Finite sample of configurations: statistical errors
- Finite volume
- non-zero lattice spacing
- chiral limit
- quenched approximation

Lattice Gauge Theory provides a first principles framework to solve QCD, with (in principal) arbitrary precision