Lecture I: SPIN SUM RULE

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Motivation

- Consider a composite system (molecules, atoms, nuclei, nucleon,…) with intrinsic angular momentum S (spin)
- The spin must arise from the dynamics of the underlying constituents.
- Roughly speaking, one may write

  $$S = \Sigma_i s_i + \Sigma_i l_i$$

  the sum of the spin and orbital angular momentum of the constituents. (non-relativistic)
- How do we write down such expression in quantum field theory and what are the complications?

Non-Relativistic and Relativistic Quantum Mechanics

- In non-relativistic quantum mechanics, spin is introduced by hand, not by theoretical consistency
- In relativistic quantum mechanics, spin is a consequence of the **spacetime symmetry**.

  All experimental evidences so far indicate that spacetime is a flat, four-dimensional Minkowski space with signature = (1, –1, –1, –1), (when gravity is ignored)

  $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^i)$
- **Symmetry -> Conservation Law ->Quantum Numbers**

Poincare Group

- The spacetime is invariant under the Poincare transformations
  - Translations

    $x^\mu \longrightarrow x^\mu + a^\mu$,

    4 parameter transformations
  - Lorentz transformations (rotations)

    $x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$

    $\Lambda^\mu{}_\alpha \Lambda^\alpha{}_\nu = g^\mu{}_\nu$ (length is invariant)

    6 parameters: 3 for spatial rotations in (ij) plane;

    3 for Lorentz boosts in i-direction.
- All transfomations form a 10 para. Poincare group
  - Contracted SO(3,2) group

Generators of Transformations

- In quantum mechanical Hilbert space, all Poincare transformations are represented by unitary operators $U(\Lambda, a)$

$$U(\Lambda, a)\, U^{\dagger}(\Lambda, a) = U^{\dagger}(\Lambda, a)\, U(\Lambda, a) = 1$$

- Finite transformations can be built out of infinitesimal ones with

$$\Lambda^{\mu\nu} = I^{\mu\nu} + \omega^{\mu\nu} \quad \text{with} \quad \omega^{\mu\nu} = -\,\omega^{\nu\mu}$$

$$a^{\alpha} = \varepsilon^{\alpha}\,,$$

$$U(\Lambda, a) = I - i\,\omega_{\mu\nu} J^{\mu\nu}/2 + i\,\varepsilon_{\alpha} P^{\alpha}$$

$J^{\mu\nu}$ and P^{α} are *hermitian* operators in the Hilbert space and are the generators of the Poincare transformations.

Physical Observables

- Thus the spacetime symmetry predicts the existence of the following physical observables for *any* physical system,
  - $P^0 = H$ is the hamiltonian of the system,
  - P^i is the linear momentum
  - $J^i = \varepsilon^{ijk} J^{jk}/2$ is the angular momentum
  - $K^i = J^{i0}$ generates Lorentz boosts

K^i is not a familiar observable because we usually choose to diagonalize other observables, like ladder operators in angular momentum algera.

However, in light-cone quantization, states are chosen to have simpler behavior under boosts.

Poincare Algebra

- Poincare generators satisfy the following algebra

$$[P^{\alpha}, P^{\beta}]=0,$$

$$[J^{\mu\nu}, P^{\alpha}] = i(g^{\alpha\nu}P^{\mu} - g^{\alpha\mu}P^{\nu})$$

$$[J^{\mu\nu}, J^{\alpha\beta}] = i(g^{\nu\alpha}J^{\mu\beta} - g^{\mu\alpha}J^{\nu\beta} + g^{\nu\beta}J^{\alpha\mu} - g^{\mu\beta}J^{\alpha\nu})$$

- Any physical system represents a realization of this algebra!

  The states of the system can be classified according to the **representations of this algebra**, independent of whether it is

  an elementary particle (quarks)

  or a composite system (atoms).

Constructing A Representation

- One needs to find a complete set of commuting observables.
  - Start with the Abelian subalgbra P^μ.
  - **Mass**: Casimir operator $P^2 = P^\mu P_\mu = m^2$
  - **Little group**: for a massive particle, we can choose $p^\mu_0 = (m, 0, 0, 0)$ particle rest frame
    - The little group is the SO(3) rotation
    - The generators of the little group is the angular momentum operator J^i.
    - label states with different $J^2 = s(s+1)$ and $J_z = m_s$: $s = 0, 1/2, 1, 3/2, \ldots$ ❾ the **spin** states

Pauli-Lubanski Vector

- Generalizing the angular momentum operator to arbitrary frame (Pauli-Lubanski),

  $$W^\mu = -\varepsilon^{\mu\alpha\beta\gamma} J_{\alpha\beta} P_\gamma / 2\sqrt{P^2}$$

  – In the particle rest frame

  $$W^\mu = (0, J^i) \quad \text{and } W_\mu P^\mu = 0 \quad \text{(true for any frame)}$$

  – $[W^\mu, P^\nu] = 0$,

  W^μ can be diagonalized at the same time as P^ν

  – $[W^\mu, W^\nu] = i\, \varepsilon^{\mu\nu\alpha\beta} W_\alpha P_\beta / 2\sqrt{P^2}$

  W-generators form a group (little group) for a fixed p^μ.

Frame-Independent Spin Quantum Number

- There must be a Casmir operator which defines the spin as a frame-independent property, just like mass:

  $W^2 = W^\mu W_\mu$

  It has quantum numbers $-s(s+1)$ with $s = 0, 1/2, 1, \ldots$

- For a particle with non-zero momentum, W^μ also involves the Lorentz boost generators.
  - In that sense, the content of the spin of a particle seem different in the different frame?

Magnetic Quantum Number

- The magnetic quantum number m_s depends on the quantization axis
  - Choose in the rest frame the 3-vector s^i.

$$\vec{s}\cdot\vec{J}\left|p^{\mu}{}_{0}sm_s\right\rangle = m_s\left|p^{\mu}{}_{0}sm_s\right\rangle$$

  - This can be generalized to any momentum

$$-s^{\mu}W_{\mu}\left|p^{\mu}sm_s\right\rangle = m_s\left|p^{\mu}sm_s\right\rangle$$

where s^{μ} is a polarization vector reduces to $(0,s^i)$ in the rest frame.

$s^{\mu}p_{\mu} = 0$ true in any frame.

Choices of the Polarization Vector

- Three most popular choices:
  1. $s^i = (0,0,1)$: *S. Weinberg*
  2. s^i arbitrary: *Bjorken & Drell*

     In both cases, $s^\mu W_\mu$ involves boost operators
  3. $s^\mu = (|p|/m, p^0 p^i/m|p|)$: *Jacob and Wick*

     magnetic quantum number is called helicity: λ

$$h = -s^\mu W_\mu = \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}$$

the helicity operator is independent of the boost operators and is independent of the velocity of the particle!

Helicity (spin) Sum Rule

- Consider a particle moving in the z-direction with helicity λ

$$J_z \left| m p_z s \lambda \right\rangle = \lambda \left| m p_z s \lambda \right\rangle$$

Or we can write

$$\lambda = \left\langle m p_z s \lambda \right| J_z \left| m p_z s \lambda \right\rangle$$

- If the angular momentum operator J_z can be written as a sum of contributions $\Sigma_i J_{zi}$, we have a helicity sum rule

$$\lambda = \sum_i \lambda_i = \sum_i \left\langle m p_z s \lambda \right| J_{zi} \left| m p_z s \lambda \right\rangle$$

Constructing Generators of Poincare Group

- Reps of the Poincare group can be used to classify elementary particles (quarks, leptons, gauge bosons, etc.)
- Particle creation and annihilation operators can be used to construct quantum fields forming representations of the Lorentz group (scalars, spinors, vectors, ect.)
- Quantum fields facilitate constructions of lagragian densities and actions.
- Continuous Spacetime symmetries of the actions lead to conserved charges by Noether's theorem

A Spin-1/2 Free Particle

- Lagragian density

$$L = \overline{\psi}\left(i\partial - m\right)\psi$$

- Under infinitesimal translation $x^\mu \to x^\mu + a^\mu$

$$\psi(x) \dashrightarrow (1 - a^\mu \partial_\mu)\,\psi(x)$$

The action $S = \int L d^4x$ is invariant. The conserved current is the energy-momentum (stress-energy) tensor

$$T^{\mu\nu}{}_C = \frac{\partial L}{\partial \partial_\mu \phi_\alpha}\partial^\nu \phi_\alpha - g^{\mu\nu} L$$

$$= \frac{1}{2}\left(\overline{\psi}\gamma^\mu i\vec{\partial}^\nu \psi + \overline{\psi}\gamma^\mu i\overleftarrow{\partial}^\nu \psi\right)$$

- The conserved charge is $P^\mu = \int d^3x T^{0\mu}$

Angular Momentum Density

- Under Infinitesimal rotation $x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$, the action is also invariant,

$$\psi(x) \longrightarrow S(\Lambda)\psi(\Lambda^{-1}x)$$
$$= \left(1 - \frac{i}{2}\omega_{\mu\nu} j^{\mu\nu}\right)\psi(x)$$

$$j^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2} + \left(x^\mu i\partial^\nu - x^\nu i\partial^\mu\right)$$

- The symmetry leads to the following conserved current (angular momentum density)

$$M^{\alpha\mu\nu}{}_c = x^\mu T^{\alpha\nu}{}_c - x^\nu T^{\alpha\mu}{}_c + \frac{\partial L}{\partial\partial_\alpha\phi_\gamma}\left(-i\Sigma^{\mu\nu}\right)_{\gamma\beta}\phi^\beta$$
$$= \overline{\psi}\gamma^\alpha\left[\sigma^{\mu\nu}/2 + \left(x^\mu i\partial^\nu - x^\nu i\partial^\mu\right)\right]\psi$$

- The conserved charges are $J^{\mu\nu} = \int d^3x M^{0\mu\nu}$

Gauge Theories

- The *canonical* energy-momentum and angular-momentum densities are not gauge-invariant under gauge transformations.

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^2 - F^{\mu\alpha} \partial^\nu A_\alpha$$

- Although Noether's theorem guarantees the existence of a conserved charge, conserved current is not unique,

$$j^\mu = j^\mu{}_c + \partial_\nu X^{[\mu\nu]}$$

X is called a superpotential.

- One can use this freedom to obtain an energy-momentum tensor which is symmetric in their indices and gauge invariant (couple to gravity).

An “improved” Energy-Momentum Tensor

Belinfante Improvement: make the energy-momentum tensor symmetric in the two indices. It can be shown

$$\frac{\partial L}{\partial \partial^{\mu}\phi^{\alpha}}\partial^{\nu}\phi^{\alpha} - \frac{\partial L}{\partial \partial^{\nu}\phi^{\alpha}}\partial^{\mu}\phi^{\alpha} = \partial_{\beta}\lambda^{\beta\mu\nu}$$

$$\lambda^{\beta\mu\nu} = \frac{\partial L}{\partial \partial_{\beta}\phi^{\alpha}}\left(\Sigma^{\mu\nu}\right)_{\alpha\alpha'}\phi^{\alpha'}$$

Thus,

Antisymmetric in beta and mu

$$T^{\mu\nu}{}_{B} = T^{\mu\nu}{}_{C} + \partial_{\beta}X^{\beta\mu\nu}$$

Adding a symmetric piece

$$X^{\beta\mu\nu} = \frac{1}{2}\left[-\lambda^{\beta\mu\nu} + \lambda^{\mu\beta\nu} + \lambda^{\nu\beta\mu}\right]$$

Cancel out the antisymmetric part

Quantum Chromodynamics

- A fundamental theory of strong interactions
- Building blocks
  - Spin-1/2 quarks $\psi_{\alpha if}$
    of three colors i=1,2,3 &
    of six different flavors: f=u,d,s,c,b,t
  - Spin-1 massless gluons $A_{\mu a}$
    of eight colors a=1,…,8
- An SU(3) gauge theory

$$L_{\mathrm{QCD}} = \overline{\Psi}_{if}(i\partial\!\!\!/ - m_{qf})\Psi_{if} - \frac{1}{4}F_{\mu\nu a}F^{\mu\nu a} - g_s\overline{\Psi}_{if}A\!\!\!/_{ij}\Psi_{jf}$$

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - g_s f_{abc}A_{b\mu}A_{c\nu}$$

Improved Energy-Momentum Tensor and Angular Momentum Densty of QCD

- The QCD energy-momentum tensor can be decomposed into quark and gluon contributions

$$T^{\mu\nu}{}_{\mathrm{B}} = T^{\mu\nu}{}_{q} + T^{\mu\nu}{}_{g}$$

$$T^{\mu\nu}{}_{q} = \sum_{f} \overline{\psi}_{f} \gamma^{\mu} i D^{\nu} \psi_{f},$$

$$T^{\mu\nu}{}_{g} = (1/4) g^{\mu\nu} F^{2} - F^{\mu\alpha} F_{\alpha}{}^{\nu}$$

the total contribution is conserved, but not the individual part

- Improved angular momentum density

$$M^{\alpha\mu\nu}{}_{B} = T^{\alpha\mu}{}_{B} x^{\nu} - T^{\alpha\nu}{}_{B} x^{\mu}$$

Angular momentum Operator of QCD

- Angular momentum is a spatial moment of the momentum density T^{0i}

$$\vec{J} = \int \vec{r} \times \vec{T} d^3 \vec{r}$$

thus it depends on the stress-energy tensor!

- Correspondingly the QCD angular momentum can be decomposed as

$$\vec{J} = \vec{J}_q + \vec{J}_g$$
$$\vec{J}_q = \int \psi^+ \vec{\Sigma} / 2 \psi + \int \psi^+ \vec{r} \times i \vec{D} \psi$$
$$\vec{J}_g = \int \vec{r} \times \left(\vec{E} \times \vec{B} \right)$$

However, the individual terms do not transform like the spatial components of four-vectors!

Nucleon Helicity Sum Rule

- Consider a proton in its helicity eigenstate. If it is moving along the z-direction

$$1/2 = \langle p | J_z | p \rangle$$

- Inserting the QCD expression for the angular momentum operator, we have

$$1/2 = J_q(\mu) + J_g(\mu)$$
$$J_q(\mu) = \Delta\Sigma(\mu)/2 + L_q(\mu)$$

- $\Delta\Sigma$ is measurable through polarized deep-inelastic scattering.

  But, how to get J_q and J_g?

Lorentz Transformation

- The individual contributions are invariant under a subclass of Lorentz transformations
  - Boost along the z-direction
  - Rotation around the z-direction
- Consider the matrix element

$$\left\langle p \left| \int d^4x M^{\mu\alpha\beta} \right| p \right\rangle = \bar{U}(p)\Big[\frac{i}{2}A\left(p^\mu[\gamma^\alpha,\gamma^\beta] + \gamma^\mu[\gamma^\beta p^\alpha - \gamma^\alpha p^\beta]\right) + B\left(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha\right)\Big]U(p)(2\pi)^4\delta^4(0) + \ldots$$

Taking $\mu = 0$, $\alpha = 1$, and $\beta = 2$, we have,

$$\langle p+|J^z|p+\rangle / \langle p+|p+\rangle = A \ .$$

Thus A must be $1/2$.

Lorentz Invariance

Now suppose $M^{\mu\alpha\beta} = \sum_i M_i^{\mu\alpha\beta}$. The Lorentz symmetry yields,

$$\left\langle p \left| \int d^4x M_i^{\mu\alpha\beta} \right| p \right\rangle = \bar{U}(p)\Big[\frac{i}{2}A_i(\mu)p^\mu[\gamma^\alpha,\gamma^\beta] + B_i(\mu)\gamma^\mu\left(\gamma^\beta p^\alpha - \gamma^\alpha p^\beta\right) + C_i(\mu)\left(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha\right)\Big]U(p)(2\pi)^4\delta^4(0) + \ldots$$

where $A_i(\mu)$, $B_i(\mu)$ and $C_i(\mu)$ are scalar constants. Then the consistency between Eqs. (1 and (20) yields,

$$\sum_i A_i(\mu) = A = \frac{1}{2} \ . \qquad (2$$

From this, one has

$$\langle p+|J_i^z|p+\rangle / \langle p+|p+\rangle = A_i(\mu) \ ,$$

Independent of the momentum p!

The sum rule is valid in the rest frame and infinite momentum frame