Fast phase unwrapping algorithm for interferometric applications

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A wide range of interferometric techniques recover phase information that is mathematically wrapped on the interval $(-\pi, \pi]$. Obtaining the true unwrapped phase is a longstanding problem. We present an algorithm that solves the phase unwrapping problem, using a combination of Fourier techniques. The execution time for our algorithm is equivalent to the computation time required for performing eight fast Fourier transforms and is stable against noise and residues present in the wrapped phase. We have extended the algorithm to handle data of arbitrary size. We expect the state of the art of existing interferometric applications, including the possibility for real-time phase recovery, to benefit from our algorithm.

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A host of techniques allow measurement of physical properties based on the retrieval of phase information encoded in an interference pattern. For example, remote sensing techniques based on synthetic aperture radar[1] have a range of ecological and geophysical applications,[2–5] as well as military importance, for instance, in target recognition.[6] Magnetic resonance imaging has great medical importance for mapping internal structures of the human body.[7] Further examples arise from profilometric[8] and interferometric[9,10] techniques that measure mechanical properties (e.g., strain or deformation) of materials. Of particular interest to the authors of this Letter is electron holography,[11] which, when realized in a transmission electron microscope, allows measurement of the phase shift (relative to vacuum) of the electron wave passing through a sample. The recovered electron phase is directly related to the electrostatic and magnetostatic potential distribution in the sample.[12]

A major obstacle that frustrates all the abovementioned techniques is that the recovered phase is mathematically limited to the interval $(-\pi, \pi]$ corresponding to the principal value of the arctangent function. In general, the true phase may range over an interval greater than 2π, in which case the recovered phase contains artificial discontinuities. Unwrapping these discontinuities is a matter of adding an appropriate integer multiple of 2π to each pixel element of the wrapped phase map. In practice, however, the presence of noise and residues complicates effective phase unwrapping, and there is a great current interest in developing algorithms to overcome these difficulties.[13–26] In this Letter we present a fast phase unwrapping algorithm that is largely immune to the presence of noise and residues and does not require strong user input. We present examples culled from electron holography experiments with structure in the wrapped phase (in terms of noise and vortices) analogous to the structure found in phase data recovered by other interferometric techniques.

We begin by considering the true (unwrapped) phase, $\phi(\mathbf{r})$, in terms of the wrapped phase, $\phi_w(\mathbf{r})$, so that $\phi(\mathbf{r}) = \phi_w(\mathbf{r}) + 2\pi n(\mathbf{r})$, where $\mathbf{r}$ refers to pixel position and $n(\mathbf{r})$ is an integer. The goal of our treatment will be to determine $n(\mathbf{r})$. It is common with phase unwrapping algorithms to calculate the difference of neighboring pixels in the wrapped phase and, when it exceeds some threshold value, to take this difference as a phase jump. Apart from scaling, this amounts to taking the partial derivative of the wrapped phase. We follow a similar route by calculating the two-dimensional Laplacian of $\phi(\mathbf{r}) = \phi_w(\mathbf{r}) + 2\pi n(\mathbf{r})$ and solving for $n(\mathbf{r})$. Formulating the problem in terms of the Laplace operator offers a distinct advantage as we will show momentarily. We obtain

$$n(\mathbf{r}) = \frac{1}{2\pi}\nabla_\perp^{-2}[\nabla_\perp^{2}\phi(\mathbf{r}) - \nabla_\perp^{2}\phi_w(\mathbf{r})], \tag{1}$$

where $\nabla_\perp^{2}$ and $\nabla_\perp^{-2}$ are the forward and inverse two-dimensional Laplacian operators, respectively. Equation (1) may be solved by use of fast Fourier techniques for the Laplacian operators[27] given by

$$\nabla_\perp^{2} f(x,y) = -\frac{4\pi^2}{N^2}\,\mathrm{FFT}^{-1}\{(p^2+q^2)\mathrm{FFT}[f(x,y)]\}, \tag{2}$$

$$\nabla_\perp^{-2} g(x,y) = -\frac{N^2}{4\pi^2}\,\mathrm{FFT}^{-1}\left\{\frac{\mathrm{FFT}[g(x,y)]}{(p^2+q^2)}\right\}, \tag{3}$$

where (x, y) and (p, q) are real-space and Fourier-space pixel coordinates, respectively, FFT[...] denotes the fast Fourier transform operation, and N is the input image size in pixels. It remains in solving Eq. (1) to determine the Laplacian of the true phase. One can do this[28] by defining the complex quantity $P(\mathbf{r}) = \exp[i\phi_w(\mathbf{r})] = \exp\{i[\phi(\mathbf{r}) - 2\pi n(\mathbf{r})]\} = \exp[i\phi(\mathbf{r})]$ and recognizing that $\mathrm{Im}(1/P\,\nabla_\perp^{2}P) = \nabla_\perp^{2}\phi$, where Im(...) denotes the imaginary part. Hence, $\nabla_\perp^{2}\phi = \cos\phi_w\nabla_\perp^{2}(\sin\phi_w) - \sin\phi_w\nabla_\perp^{2}(\cos\phi_w)$, and we are able to calculate the Laplacian of the unwrapped phase with knowledge of only the wrapped phase. The solution to Eq. (1) is unique once boundary conditions are imposed along with the constraint that $n(\mathbf{r})$ must be an integer.

The constraint that $n(\mathbf{r})$ are integers can be achieved by rounding of the solution obtained from Eq. (1). Applying Fourier techniques to calculate the forward and inverse Laplacians imposes periodic boundary conditions on the solution obtained for $n(\mathbf{r})$, which, generally, is not realistic. Since $n(\mathbf{r})$ are integers, it is

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more realistic that the gradient of $n(\mathbf{r})$ normal to the boundary vanishes. This boundary condition is easily implemented by use of the fast cosine transform rather than the full Fourier transform suggested by Eqs. (2) and (3). Since the fast cosine transform is not always readily available, we present an equivalent means based on symmetrization of the input wrapped phase[29] that allows us to solve Eq. (1) by use of the full Fourier transform, at a cost of increased computation and memory demands.

A synthesized image of dimensions $2N \times 2N$ is created by mirror reflection of the original $N \times N$ image and is used as input to the unwrapping algorithm. The mirrored image is an even function of the pixel coordinates used in the FFT over the two-dimensional plane. The $2N \times 2N$ solution of Eq. (1), therefore, also contains this mirror symmetry, and the $N \times N$ solution for $n(\mathbf{r})$ is obtained by extraction of the original quadrant of the $2N \times 2N$ solution. By construction of an even function from the original phase image in this manner, we rely on the properties of the FFT (appropriate for the whole plane) to give the solution appropriate for the boundary conditions that we wish to impose, i.e., a vanishing normal gradient at the original image boundary.

Although the solution obtained from Eq. (1) is, then, unique, it may not completely unwrap the phase, since the boundary conditions are only approximated, and a rounding step is introduced. We may, however, use the partially unwrapped phase as input to another iteration of the algorithm. With Eqs. (2) and (3) the iterative solution for the unwrapped phase is

$$\phi_{j+1}(x,y) = \phi_j(x,y) + 2\pi \,\mathrm{round}\left[\frac{\phi'(x,y) - \phi_j(x,y)}{2\pi}\right], \tag{4}$$

where the index j refers to iteration step ($j = 0$ is the original wrapped phase), round(...) is the rounding operator, and

$$\begin{aligned}\phi'(x,y) &= \mathrm{FFT}^{-1}\left(\frac{\mathrm{FFT}\{\cos\phi_w \,\mathrm{FFT}^{-1}[(p^2+q^2)\mathrm{FFT}(\sin\phi_w)]\}}{(p^2+q^2)}\right)\\ &\quad -\mathrm{FFT}^{-1}\left(\frac{\mathrm{FFT}\{\sin\phi_w \,\mathrm{FFT}^{-1}[(p^2+q^2)\mathrm{FFT}(\cos\phi_w)]\}}{(p^2+q^2)}\right).\end{aligned} \tag{5}$$

The expression for $\phi'(x,y)$ arises from the term in Eq. (1) involving the Laplacian of the true phase and does not need to be calculated with each iteration. It does, however, represent the vast majority of the computational demands for the algorithm. We note also that $\phi'(x,y)$, by construction, is an estimate of the unwrapped phase, and that the iterative steps in Eq. (4) mean to converge as nearly to $\phi'(x,y)$ as can be accomplished by addition of only integer multiples of 2π. The estimate of the true phase by $\phi'(x,y)$ is exact in the limit that the true phase is second-order differentiable (i.e., free from noise and residues) and that the imposed boundary conditions are correct. Essentially, what this means is that $\phi'(x,y)$ needs only be a fair estimate of the unwrapped phase, i.e., within $\pm\pi$. Since the Fourier operators in Eq. (5) are nonlocal and independent of any unwrapping path, $\phi'(x,y)$ is a good enough estimate if the wrapped data are not abnormally noisy, the density of phase residues is not too high, and the gradient of the phase is not too steep at the boundary. Although these statements are difficult to quantify precisely and are, furthermore, beyond the scope of the Letter, in practice, our algorithm has been found to be extremely robust with respect to noise and vortices when it is applied to electron holography data. We expect our algorithm to be similarly stable when it is applied to wrapped phase data obtained by other inteferometric techniques.

The account of phase unwrapping algorithms is both long and active, and there are many different algorithms that use many different approaches and techniques, including elements also used by our algorithm.[30] Our approach, however, combines the elements outlined above in a distinct manner. In particular, we do not formulate the unwrapping solution as a least-squares minimization problem as is commonly done. This allows us to avoid any assumption, implicitly or explicitly, about the noise statistics of the wrapped data. (This is important because the least-squares solution depends on assumed statistics.[31]) The other key approach of our algorithm is that Fourier transforms are used not only to solve the differential equation but also to calculate the derivatives, rather than calculating them by finite differences. This allows us to avoid nearest-neighbor operations that are sensitive to noise and vortices. The result is an algorithm with a significantly higher ratio of effectiveness per computational demands than existing algorithms, and we believe that this is its primary strength. We do not expect our algorithm to supplant the most sophisticated algorithms. Rather, we expect it to offer a balance between effectiveness and simplicity that is unavailable at present and that this combination may open possibilities that are not currently feasible for a variety of inteferometric techniques.

Figure 1a is an experimental wrapped phase from electron holography experiments carried out on $Nd_2Fe_{14}B$ hard magnets, in which vortices in the wrapped data are abundant. We show our unwrapping results in Fig. 1b, which took eight iterations. The image size was 204×184 pixels. For comparison, we show in Fig. 1c the unwrapping results obtained by a simpler and faster, but path-dependent, phase unwrapping algorithm. The path-dependent method incorrectly unwraps the phase near the vortices and propagates the error through the rest of the unwrapped phase. Figure 2a is another example of wrapped phase data (from electron holography of $Nd_2Fe_{14}B$) that, along with Fig. 1a, were chosen to emulate phase data with structure similar to, for example, synthetic aperture radar or profilometer techniques. Figure 2b is the unwrapped phase corresponding to Fig. 2a after three iterations of our algorithm. The image size in Fig. 2a is 204×184

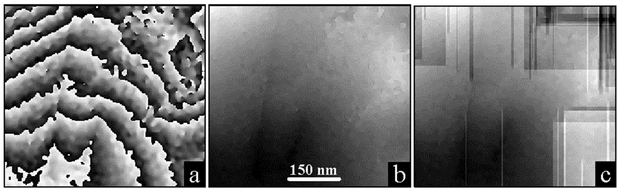

Fig. 1. Experimental wrapped phase (a) from holography experiments carried out on $Nd_2Fe_{14}B$ hard magnets. The unwrapped phase obtained by (b) our algorithm and (c) a path-dependent method.

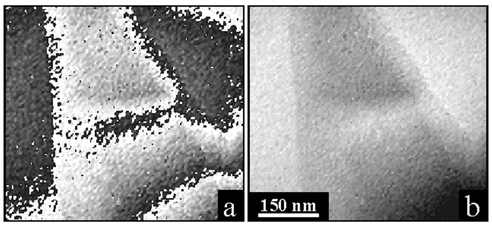

Fig. 2. Wrapped phase (a) from holography experiments carried out on $Nd_2Fe_{14}B$ hard magnets. The unwrapped phase (b) obtained by our algorithm (three iterations).

pixels. We note that, as presented, our algorithm demands that the input image size be a power of 2, since fast Fourier transforms are used. This requirement may be relaxed, as demonstrated by Figs. 1b and 2b, if the region to be unwrapped is padded with its boundary values to the nearest power of 2 in size.

One advantage of our phase unwrapping algorithm is its simplicity to program and implement. This provides the opportunity for research to focus on more important issues than the cumbersome task of unwrapping phase data. It also allows existing interferometric techniques to be integrated more easily with other techniques, for example, model calculation or simulation. The algorithm is fast enough that real-time application and *in situ* studies are feasible. Furthermore, there is no loss of measurement integrity, in the sense that our algorithm adds only integer multiples of 2π to the wrapped data, and one does not need to deal with artifacts associated with the unwrapping process. Our algorithm is reliable, which should increase the performance of automated testing and analysis, and aids in visualizing the phase, which is invaluable for medical diagnostic applications. Although additional work is needed to assess precisely the quantitative limitations of our algorithm with respect to noise, vortices, and boundary conditions, we suppose that improvements or, perhaps more importantly, consistency checks may be devised to allow the most sophisticated applications to benefit from it. For example, one can imagine advances in adaptive optics applications in which computational speed is crucial or integration of this algorithm with high-performance optical devices and applications of holographic information storage and retrieval.[32]

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