Shear viscosity and thermal fluctuations in heavy-ion collisions

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Fluctuation-dissipation and the importance of noise

- Thermal noise in heavy-ion collisions
- Where does thermal noise come from?

Fluctuation-dissipation and the importance of noise

- The relations between Green functions

Fluctuating hydrodynamical variables in BES-II

- The shape of second-order noise
- The variance of momentum eccentricity of central Au+Au $\sqrt{s_{NN}}$=19.6 GeV

Conclusions

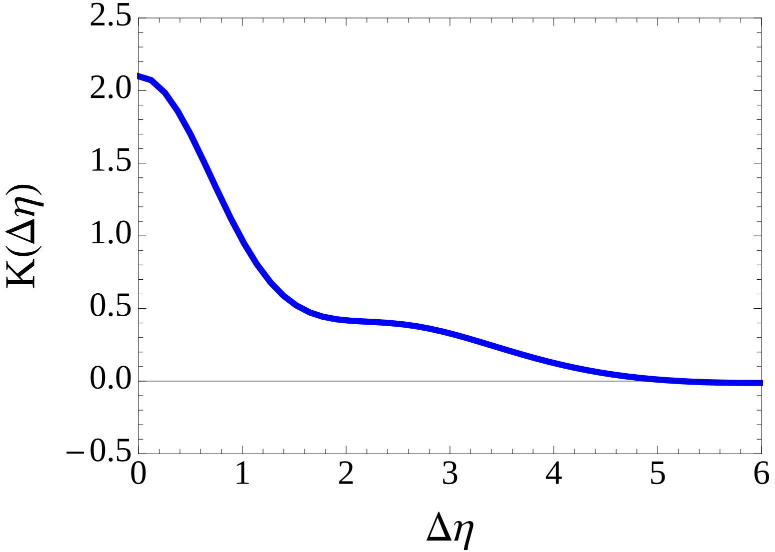
Thermal noise in heavy-ion collisions

Kapusta, Müller, Stephanov: Thermal noise exists in viscous relativistic fluids, represented by $S^{\mu\nu}_{\text{heat}}$ and $S^{\mu\nu}_{\text{visc.}}$ (in the Eckart frame) and $S^{\mu\nu}$ and I^{μ} in the (Landau-Lifshitz frame).

$K(\Delta\eta) \propto \left\langle \frac{dN}{d\eta}(\eta + \Delta\eta)\frac{dN}{d\eta}(\eta)\right\rangle - \left\langle \frac{dN}{d\eta} \right\rangle^2$, the two-particle correlation as a function of rapidity gap, has a contribution from thermal noise.

Analytical calculations possible for the Bjorken expansion of an ultrarelativistic gas.

Thermal noise in ultrarelativistic gases



The relations between Green functions

The effect of perturbations on every physical quantity is determined by a *specific* Green function:

- Given $\delta\hat{H}(t) = \int d^3x j(\mathbf{x},t)\hat{\phi}(\mathbf{x},t)$, $\left\langle \delta\hat{\phi} \right\rangle = \int d^4x' G_R(x-x')j(x')$, where $G_R(x) \equiv -i\theta(t)\left\langle [\hat{\phi}(x), \hat{\phi}(0)] \right\rangle$.

- For the same $\delta\hat{H}(t)$, transition rates determined with $G_>(t) \equiv -i\int dt\ \left\langle \hat{\phi}(t)\hat{\phi(0)} \right\rangle \equiv G_<(-t)$.

- Variances determined with $G_S(t) \equiv \frac{1}{2}\left\langle \{\hat{\phi}(t), \hat{\phi}(0)\} \right\rangle$.

In $\mathbf{x}$ and t, several Green functions exist, each with their own domain of applicability.

The relations between Green functions

Relationships become clearer in Fourier space:

$$\mathrm{Im}\, G_R(\omega) = -\frac{i}{2}\Big[-i\int dt\, \theta(t) e^{i\omega t}\left\langle[\hat{\phi}(t),\hat{\phi}(0)]\right\rangle$$

$$-i\int dt\, \theta(t) e^{-i\omega t}\left\langle[\hat{\phi}(t),\hat{\phi}(0)]\right\rangle^*\Big] = -\frac{1}{2}\Big[\int dt\, \theta(t) e^{i\omega t}\left\langle[\hat{\phi}(t),\hat{\phi}(0)]\right\rangle$$

$$+\int dt\, \theta(t) e^{-i\omega t}\left\langle[\hat{\phi}(0),\hat{\phi}(t)]\right\rangle\Big] = -\frac{1}{2}\int dt\, e^{i\omega t}\left\langle[\hat{\phi}(t),\hat{\phi}(0)]\right\rangle$$

$$= -\frac{1}{2}(1-e^{-\omega/T})\int dt\, e^{i\omega t}\left\langle\hat{\phi}(t)\hat{\phi}(0)\right\rangle$$

$$= -\frac{i}{2}(1-e^{-\omega/T})G_>(\omega).$$

Hermiticity and the KMS relation lead to algebraic relations between Green functions in Fourier space.

The relations between Green functions

Going between coordinate and momentum space relates physics to facts from complex analysis:

Coordinate space	Momentum space
Causality	Analyticity of $G_R(\omega)$ in the upper-half plane
KMS relation	Detailed balance
Fluctuation-dissipation	$G_S(\omega) = -(1 + 2n_B(\omega))\mathrm{Im}\, G_R(\omega)$

In particular the correlation function $G_S \equiv \frac{1}{2}\left\langle \{\hat{\phi}(t), \hat{\phi}(0)\} \right\rangle$ (fluctuations) have an easy relationship to $\mathrm{Im}\, G_R(\omega)$ (the dissipation) in momentum space.

Viscosity and thermal noise

"Why didn't I notice thermal noise in liquids before?"

$$\frac{1}{2}\left\langle\{\delta u_T^i(x), \delta u_T^j(0)\}\right\rangle \approx \frac{T}{e+p}\left(\frac{\pi(e+p)}{\eta|t|}\right)^{3/2} \exp\left(-\frac{(e+p)|\mathbf{x}|^2}{4\eta|t|}\right)\frac{2}{3}\delta^{ij}:$$

For non-relativistic gases, $e+p \approx \rho$, making T/ρ *tiny*.

For ultrarelativistic gases, $\frac{4\eta|t|}{e+p}$ often small compared to length scales of interest.

In heavy-ion collisions, length and time scales ~ 1 fm $\sim \frac{1}{T}$, $e \sim p \sim T^4$, $s \sim T^3$: *no mass scale exists to suppress the importance of fluctuations*, only the smallness of η/s can.

Noise and observables

Now $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{av.}} + \delta T^{\mu\nu}$:
Each solution for a given $\delta T^{\mu\nu}$ corresponds to *one event* at a heavy-ion collider. The ensemble of $\delta T^{\mu\nu}$ is approximated by a data set. Most observables are averages of these sets.

$\langle \delta T^{\mu\nu} \rangle = 0 \rightarrow$ hydrodynamical noise has *no effect* on one-particle observables dN/dp_T, dN/dy, averaged over many events.

$\left\langle \delta T^{\mu\nu} \delta T^{\alpha\beta} \right\rangle \neq 0 \rightarrow$ noise affects two-particle correlations even after averaging over events.

$\delta T^{\mu\nu} \neq 0 \rightarrow$ noise drives non-trivial variance of *all* observables, within a centrality class.

The shape of second-order noise

The Israel-Stewart formalism of the hydrodynamical equations

$$\begin{aligned}\partial_\mu(T^{\mu\nu}_{\text{ideal}} + \partial_\mu W^{\mu\nu}) &= 0 \\ \Delta^\mu_\alpha \Delta^\nu_\beta (u\cdot\partial) W^{\alpha\beta} &= -\frac{1}{\tau_\pi}(W^{\mu\nu} - \Pi^{\mu\nu}),\end{aligned}$$

where $\Pi^{\mu\nu} = \eta \Delta^\mu u^\nu + \Delta^\nu u^\mu - \frac{2}{3}(\partial\cdot u)\Delta^{\mu\nu}$ is the first-order viscous energy-momentum and τ_π is the relaxation time. What is the correlation function for thermal noise here?

Noise in energy-momentum: $T^{\mu\nu} + \Xi^{\mu\nu}$:
Current conservation ($\partial_\mu(T^{\mu\nu} + \Xi^{\mu\nu}) = 0$) and the fluctuation-dissipation relation give the autocorrelation of thermal noise (for a fluid at rest):

$$\begin{aligned}&\left\langle ((\tau_\pi \partial_t \Xi^{ij}(x) + \Xi^{ij}(x))(\tau_\pi \partial_t \Xi^{kl}(x') + \Xi^{kl}(x')) \right\rangle \\ &= \left[2\eta T(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) + 2(\zeta - 2\eta/3) T \delta^{ij}\delta^{kl}\right]\delta^4(x-x').\end{aligned}$$

Colored noise in numerical simulations

This makes the noise autocorrelation *colored*:

$$\left\langle \Xi^{ij}(x)\Xi^{kl}(x') \right\rangle = \left[2\eta T(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) + 2(\zeta - 2\eta/3)T\delta^{ij}\delta^{kl}\right] \times \delta^3(x - x')\frac{\exp(-|t - t'|/\tau_\pi)}{2\tau_\pi};$$

the noise decorrelates slowly in time. Numerically, it is easier to use the first equation to define a differential equation for $\Xi^{\mu\nu}$:

$$\tau_\pi \dot{\Xi}^{\mu\nu} = -(\Xi^{\mu\nu} - \xi^{\mu\nu}),$$

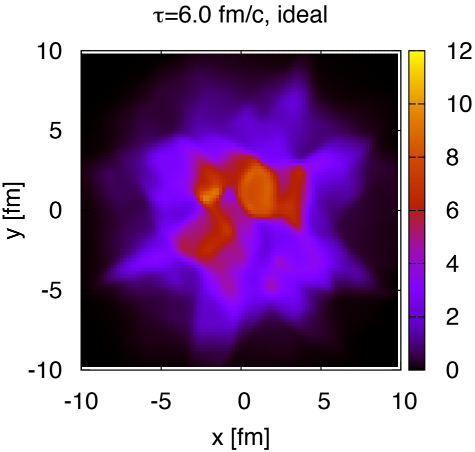
where $\xi^{\mu\nu}$ is now white noise.

MUSIC for 3+1-dimensional viscous hydrodynamics

Noise breaks boost invariance
$\rightarrow$ (3+1)-dimensional simulation necessary for $K(\Delta\eta, \Delta\phi)$.

Fluctuations related to dissipation
$\rightarrow$ viscous hydrodynamics necessary.

MUSIC (Schenke, Jeon, and Gale) solves the Israel-Stewart model of viscous hydrodynamics in (τ, x, y, η), uses lattice EOS, determines 3-dimensional freeze-out surface for hadron production.



Calculating linearized fluctuations numerically

The equations for $\delta T^{\mu\nu}_{\rm id.}$ and $\delta W'^{\mu\nu} = W^{\mu\nu} + \Xi^{\mu\nu}$ can be linearized:

$$\partial_t \delta T^{t\nu}_{\rm id.} = -\partial_i \delta T^{i\nu}_{\rm id.} - \partial_\mu \delta W'^{\mu\nu},$$

$$(u\cdot\partial)\delta W'^{\mu\nu} = -\frac{1}{\tau_\pi}(\delta W'^{\mu\nu} - \delta S^{\mu\nu} - \Xi^{\mu\nu}) - \frac{4}{3}(\partial\cdot\delta u)W^{\mu\nu} - \frac{4}{3}(\partial\cdot u)\delta W'^{\mu\nu} - (\delta u\cdot\partial)W^{\mu\nu}$$

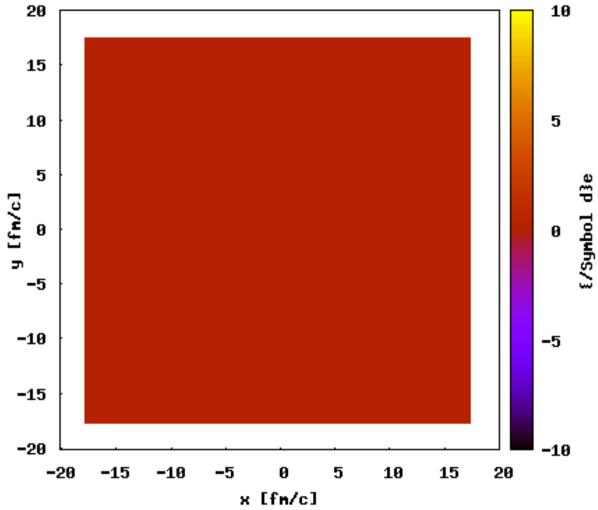
$$-\delta u^\mu((u\cdot\partial)u_\alpha)W^{\alpha\nu} - u^\mu((\delta u\cdot\partial)u_\alpha)W^{\alpha\nu} + (u\cdot\partial)\delta u_\alpha)W^{\alpha\nu} + (u\cdot\partial)u_\alpha)\delta W'^{\alpha\nu})$$
$$-\delta u^\nu((u\cdot\partial)u_\alpha)W^{\alpha\mu} - u^\nu((\delta u\cdot\partial)u_\alpha)W^{\alpha\mu} + (u\cdot\partial)\delta u_\alpha)W^{\alpha\mu} + (u\cdot\partial)u_\alpha)\delta W'^{\alpha\mu}).$$

The equations are discretized: $\xi(x) \to \xi_i = \frac{1}{\Delta V \Delta t}\int d^4x \xi(x)$.

$\langle \xi(x)\xi(x')\rangle \propto \delta^4(x-x') \to \left\langle \xi^i \xi^j \right\rangle \propto \frac{1}{\Delta V \Delta t}$:

hyperbolic equations with large gradients and sources.

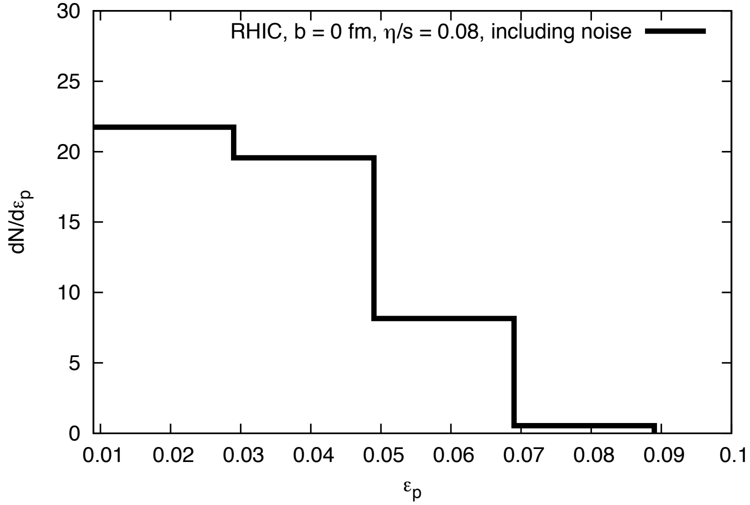
MUSIC with noise



$\delta e(x, y, \tau)$, in a $\sqrt{s_{NN}} = 19.6$ GeV Au+Au collision, $b = 0$.

The variance of ϵ_p in very central collisions

$\epsilon_p = \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle}$ and $\frac{2\langle T^{xy} \rangle}{\langle T^{xx} + T^{yy} \rangle}$ added in quadrature:



Conclusions

- Hydrodynamical noise in heavy-ion collisions produces a quiet but important correlation in heavy-ion collisions, capable of providing independent measurements of viscosity.
- Thermal noise contributes to event-by-event variance of v_2.
- Simulations with initial and freeze-out fluctuations necessary to find signal of hydrodynamical noise in experimental observables.

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