The real evolution of non-Gaussian cumulants in QCD critical regime

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Based on: S. Mukherjee, R. Venugopalan and YY, 1506.00645.

Beam Energy Scan and Associated Theory Workshop,

BNL, Jun. 10th

Outline

- Part I: Memory effects and critical fluctuations.
- Part II: The evolution equations for cumulants.
- Part III: Evolution of cumulants in QCD critical regime.
- Part IV: Implications on beam energy scan and search for QCD critical point.

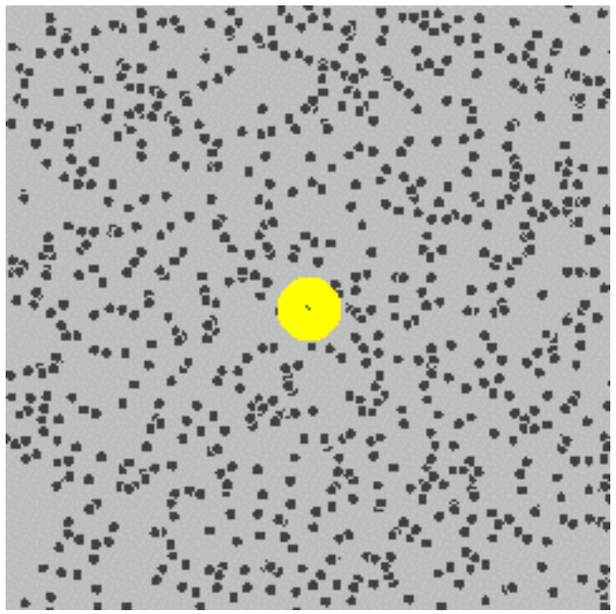
Memory effects and critical fluctuations

Fluctuations and memory effects: Brownian motion as an example

- Consider the random walk of a Brownian particle. For sufficiently long time $\tau \gg \tau_{\rm eff}$ ($\tau_{\rm eff}$ effective relaxation time), the fluctuations in position will approach the equilibrium values which are independent of the history of Brownian particle.

$$\langle (x(\tau) - x(0)) \rangle \to 0\,, \qquad \langle (x(\tau) - x(0))^2 \rangle \to D\tau\,,$$

- **Memory effects:** if τ is comparable to relaxation time $\tau_{\rm eff}$, the fluctuations can be substantially different from the equilibrium expectation and would depend on the history of Brownian particle.



Memory effects of critical fluctuations

- The relaxation time of critical mode also grows with correlation length:

$$\tau_{\text{eff}} \sim \xi^z \,,$$

  with z defines the dynamical scaling exponent and $z \sim 3$ for QCD critical point.

- τ_{eff} can be comparable or even larger than the time fireball spent in critical regime.

- Knowledge of equilibrium fluctuations might not be sufficient. Memory effects are important for understanding critical fluctuation in experiment.
  NB: as the sign of non-Gaussian cumulants can be either positive or negative, even a *qualitative* understanding of their behaviors in experiments requires taking memory effects into account!

This talk

- Understand how memory effects would affect the evolution of non-Gaussian cumulants.
- Understand the implication of such memory effects for detecting QCD critical point.

The evolution equations for cumulants

Moments (cumulants) of critical field

- We consider spatial average of critical field: $\sigma \equiv \left[\int d^3x\, \sigma(\mathbf{x})\right]/V$ (crtical field for QCD critical point is a combination of Baryon density and chiral condensate).

- The information of fluctuations are encoded in probability distribution $P(\sigma;\tau)$:

$$M(\tau) \equiv \langle \sigma \rangle\,, \qquad \kappa_2(\tau) \equiv \langle (\delta\sigma(\tau))^2 \rangle\,, \qquad \kappa_3(\tau) \equiv \langle (\delta\sigma(\tau))^3 \rangle\,,$$

$$\kappa_4(\tau) \equiv \langle (\delta\sigma(\tau))^4 \rangle - 3\kappa_2^2(\tau) \qquad \delta\sigma \equiv \sigma - M(\tau)\,.$$

- One could determine equilibrium distribution $P_0(\sigma) \sim \exp\left(-V_4\Omega_0(\sigma)\right)$, $V_4 \equiv V/T$ out of equilibrium $M^{\rm eq}, k_2^{\rm eq}, k_3^{\rm eq}, k_4^{\rm eq}$:

$$\Omega_0(\sigma) = \frac{1}{2}m_\sigma^2\left(\sigma-\sigma_0\right)^2 + \frac{\lambda_3}{3}\left(\sigma-\sigma_0\right)^3 + \frac{\lambda_4}{4}\left(\sigma-\sigma_0\right)^4\,,$$

where $\sigma_0, m_\sigma, \lambda_3, \lambda_4$ are determined by($m_\sigma = 1/\xi^{\rm eq}$):

$$\sigma_0 = M^{\rm eq}\,, \qquad \kappa_2^{\rm eq} = \frac{\xi_{\rm eq}^2}{V_4}\,, \qquad \kappa_3^{\rm eq} = -\frac{2\xi_{\rm eq}^6}{V_4^2}\lambda_3\,, \qquad \kappa_4^{\rm eq} = \frac{6\xi_{\rm eq}^8}{V_4^3}\left[2(\lambda_3\xi)^2 - \lambda_4\right]$$

The evolution of non-equilibrium cumulants

- Similarly to the case of Brownian motion, Fokker-Planck equation describes the relaxation of non-equilibrium distribution $P(\sigma, \tau)$ towards the equilibrium distribution (Hohenberg-Halperin, 1977),

$$\partial_\tau P(\sigma;\tau) = \frac{1}{m_\sigma^2 \tau_{\text{eff}}} \{\partial_\sigma \left[\partial_\sigma \Omega_0(\sigma) + V_4^{-1} \partial_\sigma\right] P(\sigma;\tau)\}\,, \qquad \tau_{\text{eff}} \sim \xi^z\,.$$

- Instead of directly solving Fokker-Planck equation, we derive a set of evolution equation for first four cumulants $M, \kappa_2, \kappa_3, \kappa_4$ (S. Mukherjee, R. Venugopalan and YY, 1506.00645). Similarly techniques have been used in studying turbulence.

- The results are valid in the regime that ξ is larger than microscopic scale but smaller than the size of the system. In this case,

$$\epsilon = \sqrt{\frac{\xi^3}{V}} \ll 1\,.$$

- For small ϵ, the evolution of κ_n is decoupled from higher cumulants $\kappa_{n+1}, \ldots$.

A set of equation for cumulants evolution:

- Structure of the evolution equation:
  - L.H.S. evolves the first time derivative of cumulants.
  - R.H.S evolves polynomials of those cumulants.

$$\begin{aligned} \partial_\tau M(\tau) &= -\tau_{\text{eff}}^{-1} L_1 \left[M(\tau)\right] \left[1 + \mathcal{O}(\epsilon^2)\right] , \\ \partial_\tau \kappa_n(\tau) &= -n\, \tau_{\text{eff}}^{-1} L_n \left[M(\tau), \kappa_2, \ldots, \kappa_n\right] \left[1 + \mathcal{O}(\epsilon^2)\right] , \end{aligned}$$

$n = 2, 3, 4\,.$

- $L_n, n = 1, 2, 3, 4$ are polynomials of their arguments and their functional form only depend on the equilibrium properties of the system.
- We will present two limits.

The Gaussian limit

- If the equilibrium distribution is Gaussian: $\Omega_0(\sigma) = \frac{1}{2} m_\sigma^2 (\sigma - \sigma_0)^2$, $\kappa_3^{\text{eq}} = \kappa_4^{\text{eq}} = 0$), the evolution among cumulants decouple:

$$\partial_\tau M = -\tau_{\text{eff}}^{-1} \left[M(\tau) - \sigma_0(\tau) \right] , \qquad \partial_\tau \kappa_2(\tau) = -2\tau_{\text{eff}}^{-1} \left[\kappa_2(\tau) - k_2^0(\tau) \right] ,$$

$$\partial_\tau \kappa_3(\tau) = -3\tau_{\text{eff}}^{-1} \kappa_3(\tau) , \qquad \partial_\tau \kappa_4(\tau) = -4\tau_{\text{eff}}^{-1} \kappa_4(\tau) .$$

- Simple relaxation equation, any non-Gaussian cumulants will be damped and the damping rate is faster for higher cumulants.

Near equilibrium limit

- If the deviation from equilibrium of cumulants is small $\delta\kappa_n \equiv \kappa_n - \kappa_n^{\rm eq}$, evolution equation is linearized:

$$
\begin{aligned}
\partial_\tau M(\tau) &= -\tau_{\rm eff}^{-1}\delta M\,, \qquad \partial_\tau \kappa_2(\tau) = -2\,\tau_{\rm eff}^{-1}\delta\tilde{\kappa}_2\,,\\
\partial_\tau \kappa_3(\tau) &= -3\,\tau_{\rm eff}^{-1}(\epsilon b^3)\left[\left(\frac{\delta\kappa_3}{\epsilon b^3}\right) + 4\tilde{\lambda}_3\left(\frac{\delta\kappa_2}{b^2}\right)\right]\,,\\
\partial_\tau\left[\kappa_4(\tau)\right] &= -4\,\tau_{\rm eff}^{-1}(\epsilon^2 b^4)\left\{\left(\frac{\delta\kappa_4}{\epsilon^2 b^4}\right) + 6\tilde{\lambda}_3\left(\frac{\delta\kappa_3}{\epsilon b^3}\right)\right.\\
&- \left. 6\left(2\tilde{\lambda}_3^2 - 3\tilde{\lambda}_4\right)\left(\frac{\delta\kappa_2}{b^2}\right)\right\}.
\end{aligned}
$$

- The evolution of higher cumulants are coupled to lower ones. Lower moments will be relaxed back to the equilibrium first.

Summary of Part I:

- We have derived a set of equations for the evolution of cumulants.

- We now apply it to the QCD critical regime. (We will restrict ourselves to the cross-over side of critical regime as effects such as bubble nucleation are not incorporated in the present approach.)

Evolution of cumulants in QCD critical regime.

Universal inputs

- We will apply our equations to study the evolution of cumulants in QCD critical regime. Both universal and non-universal (but phenomenologically motivated) inputs are needed.

- The QCD critical point is in the same universality class as that of 3-d Ising Model. The equilibrium distribution $P^{\mathrm{eq}}(\sigma; r, h)$ is known in terms of Ising variables r (Ising temperature), h (magnetic fields) and is universal.

- Parametrization of τ_{eff} on ξ_{eq} is universal:

$$\tau_{\mathrm{eff}} = \tau_{\mathrm{rel}} \left(\frac{\xi}{\xi_{\mathrm{min}}} \right)^{z} .$$

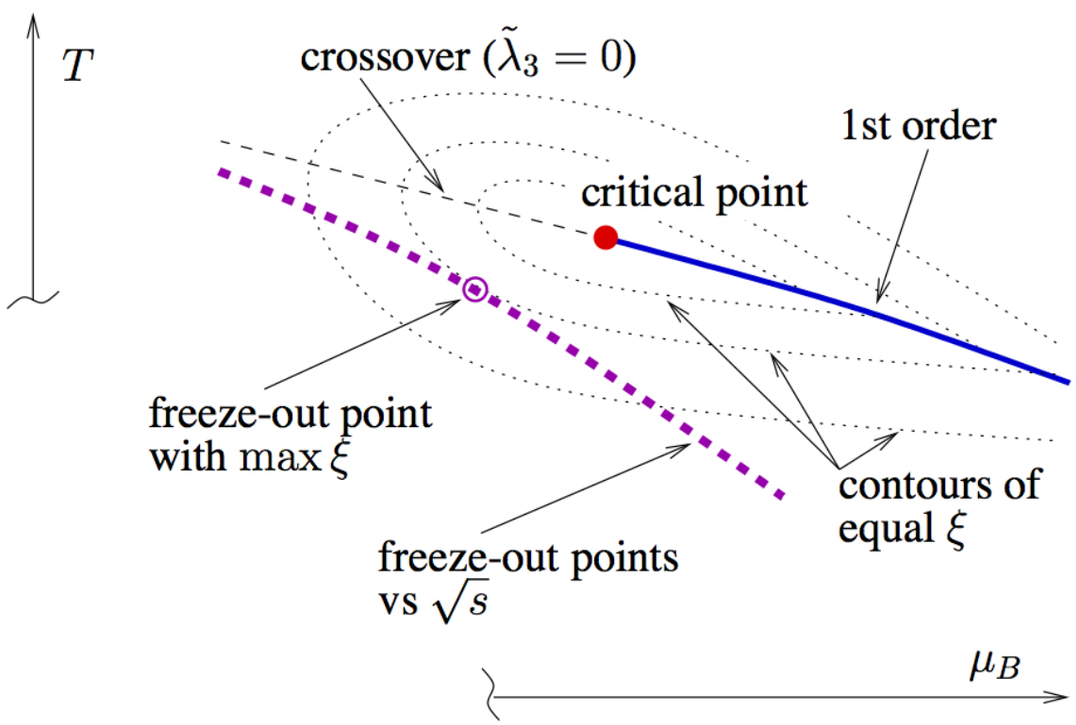
  and we use results from Model H, i.e. , $z = 3$. τ_{rel} , the relaxation time at $\xi = \xi_{\mathrm{min}}$ are model-dependent.

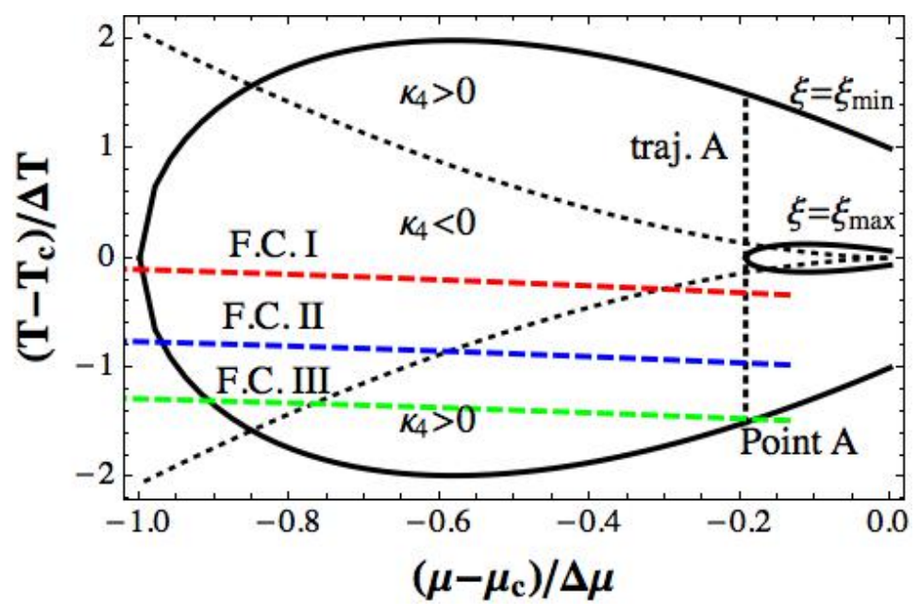
Mapping to QCD phase diagram

- We need to map Ising variables(r, h) to QCD variables T, μ_B (Non-universal, major uncertainty). We use linear mapping with $\Delta T, \Delta\mu$ the width of critical regime in QCD phase diagram.

$$\frac{T - T_c}{\Delta T} = \frac{h}{\Delta h}, \qquad \frac{\mu - \mu_c}{\Delta \mu} = -\frac{r}{\Delta r}.$$

- We define the scaling regime with the criterion: $\xi_{min} < \xi_{\rm eq} < \xi_{\rm max}$ and to be specific, we will take $\xi_{\rm max}/\xi_{\rm min} = 3$ below.





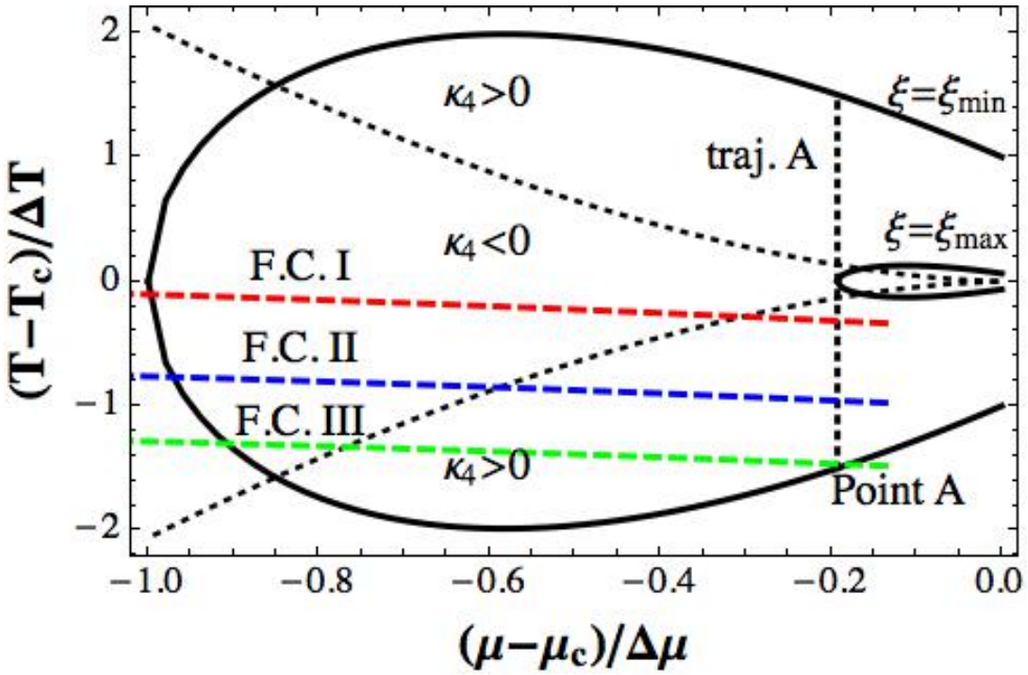
Trajectory

- We will assume for each trajectory, the μ_B of the fireball is constant. It would then be corresponding to a vertical line in the critical regime due to our mapping relation.
- Along each trajectory, we parametrize the evolution of volume and temperature by expansion rate $n_V = 3$ and speed of sound c_s^2:

$$\frac{V(\tau)}{V_I} = \left(\frac{\tau}{\tau_I}\right)^{n_V}, \qquad \frac{T(\tau)}{T_I} = (\frac{\tau}{\tau_I})^{-n_V c_s^2},$$

where V_I, T_I are volume and temperature of the system at τ_I, the time when the trajectory hits the boundary of critical regime.

- Initial condition: we will assume $M, \kappa_2, \kappa_3, \kappa_4$ equal to their equilibrium value at $\tau = \tau_I$ (τ_{eff} is small).



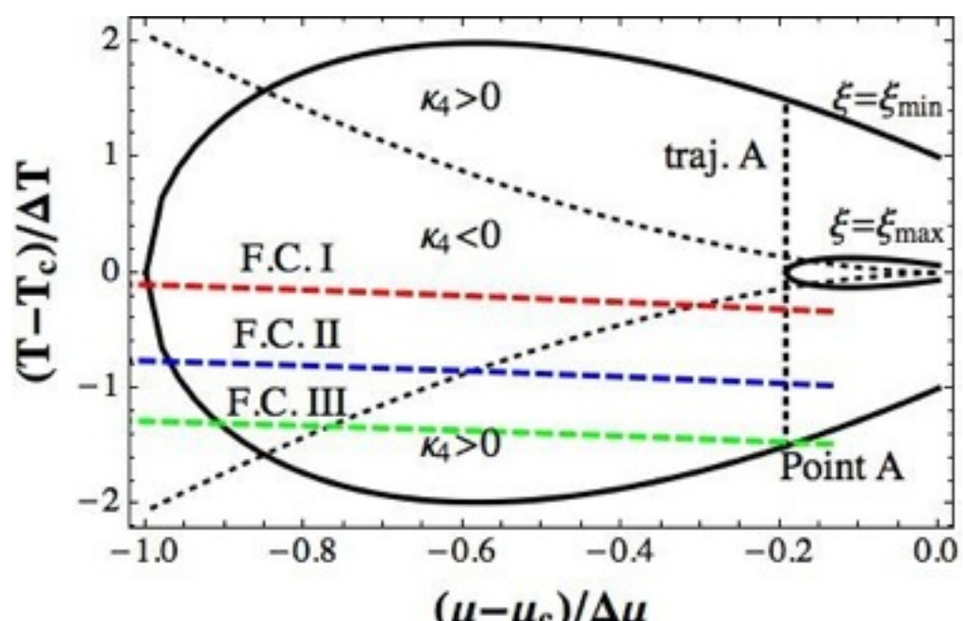
The evolutions

- Only one free parameter: $\tau_{\rm rel}/\tau_I$

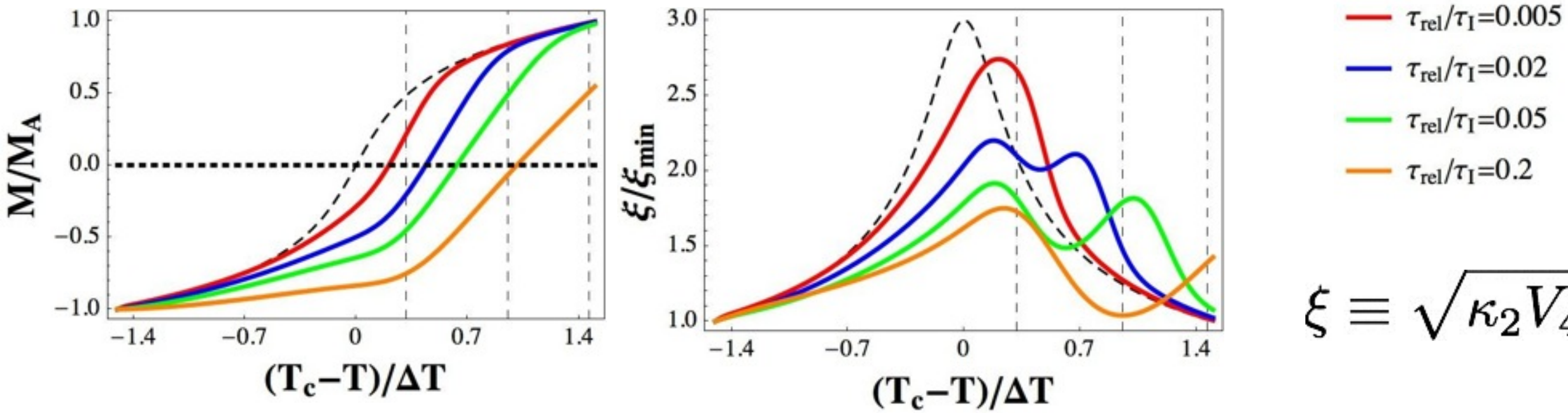
- We have solved evolution equations along trajectories passing through the critical regime.

- We will present the non-equilibrium values as a function of the corresponding temperature on the trajectory.

- We rescale our results by the corresponding equilibrium value at point A.



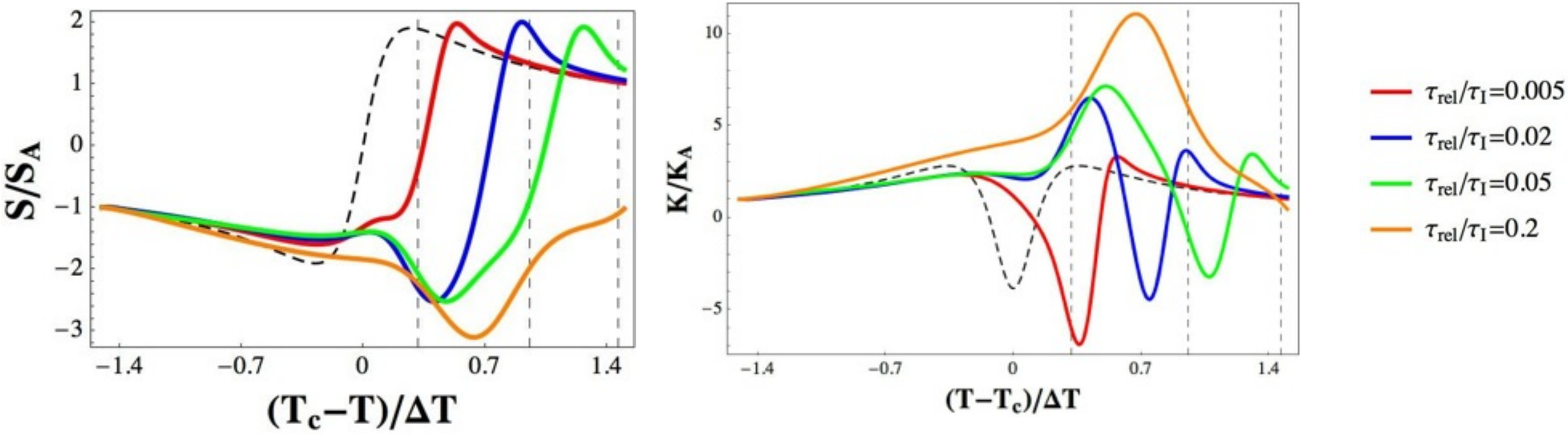
Evolution of magnetization and effective correlation length



$$\xi \equiv \sqrt{\kappa_2 V_4}$$

- M tend to approach its equilibrium value but still fall behind.
- The effects of critical slowing down would delay the growth of non-equilibrium length.
- Memory effect also protects the memory of the system in critical regime from being completely washed out (Similar to previous results by Berdnikov-Rajagopal).
- As expected, the slowing down is most visible around Tc where the equilibrium correlation becomes large.

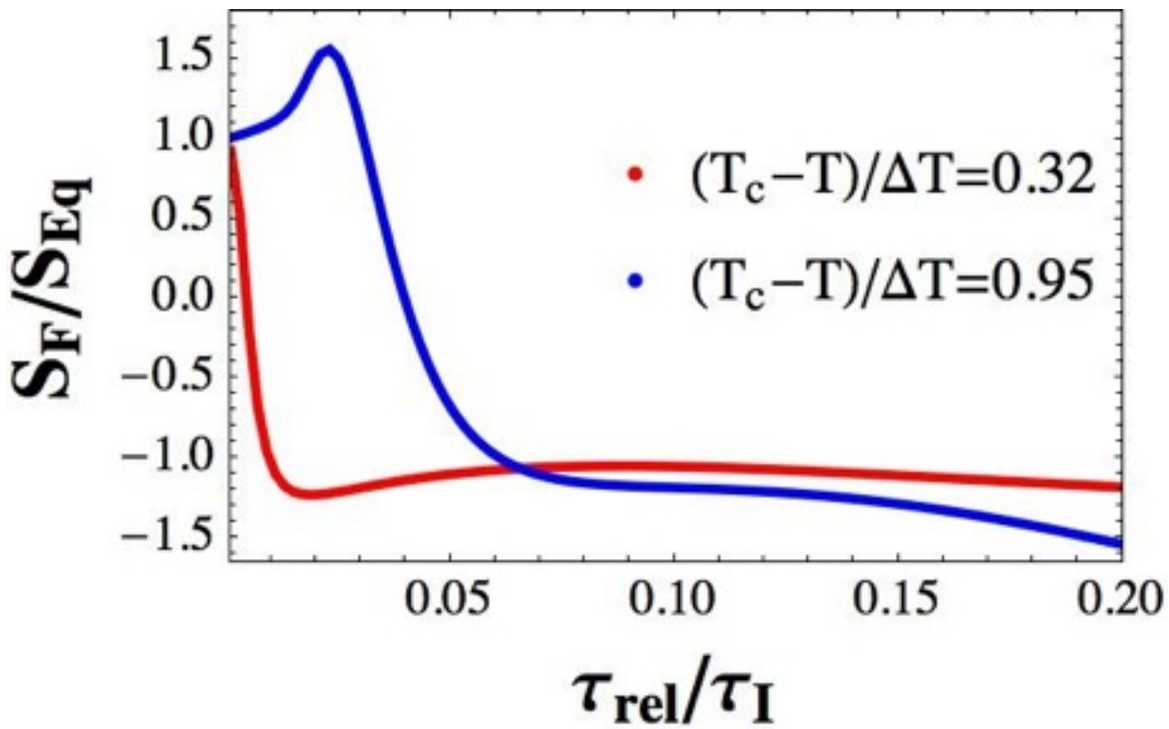
Skewness and Kurtosis

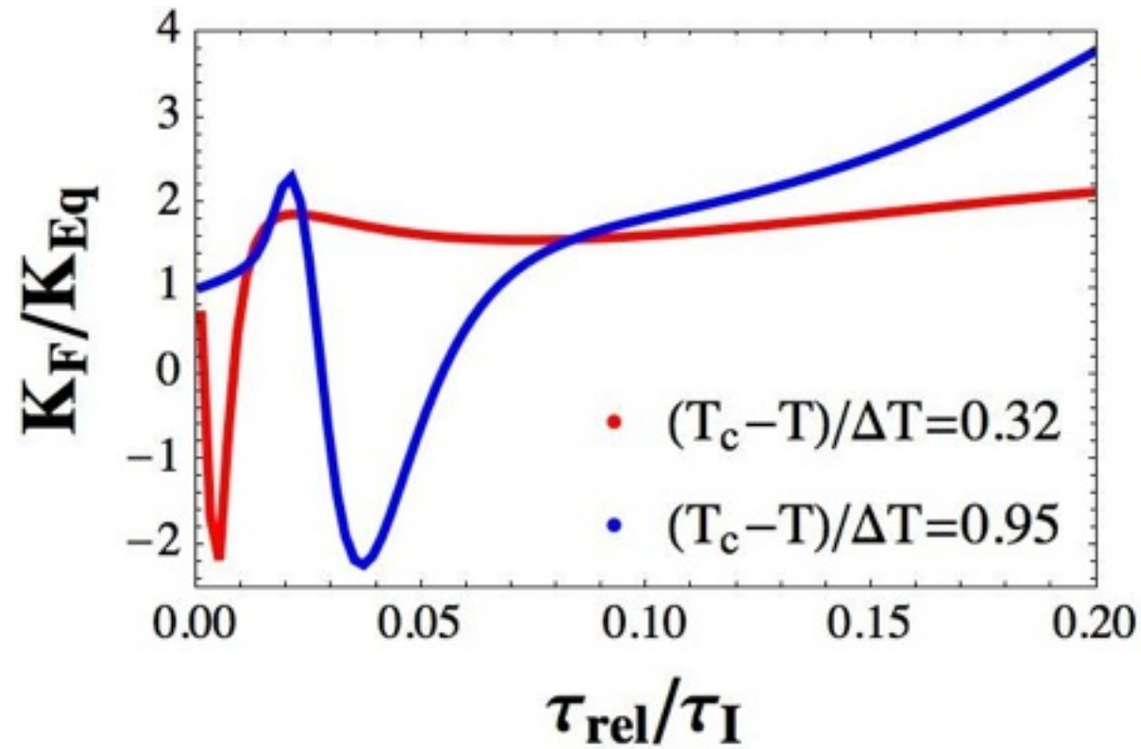


$$S(\text{Skewness}) \equiv \frac{V_4^{1/2}\kappa_3}{\kappa_2^{3/2}}, \qquad K(\text{Kurtosis}) \equiv \frac{V_4\kappa_3}{\kappa_2^2}$$

- The evolution of higher cumulants might not follow the equilibrium moments (lower cumulants will affect the evolution of the higher one).
- The non-equilibrium value can be substantially different (including sign) from the equilibrium one.

Oscillation of non-Gaussian cumulants





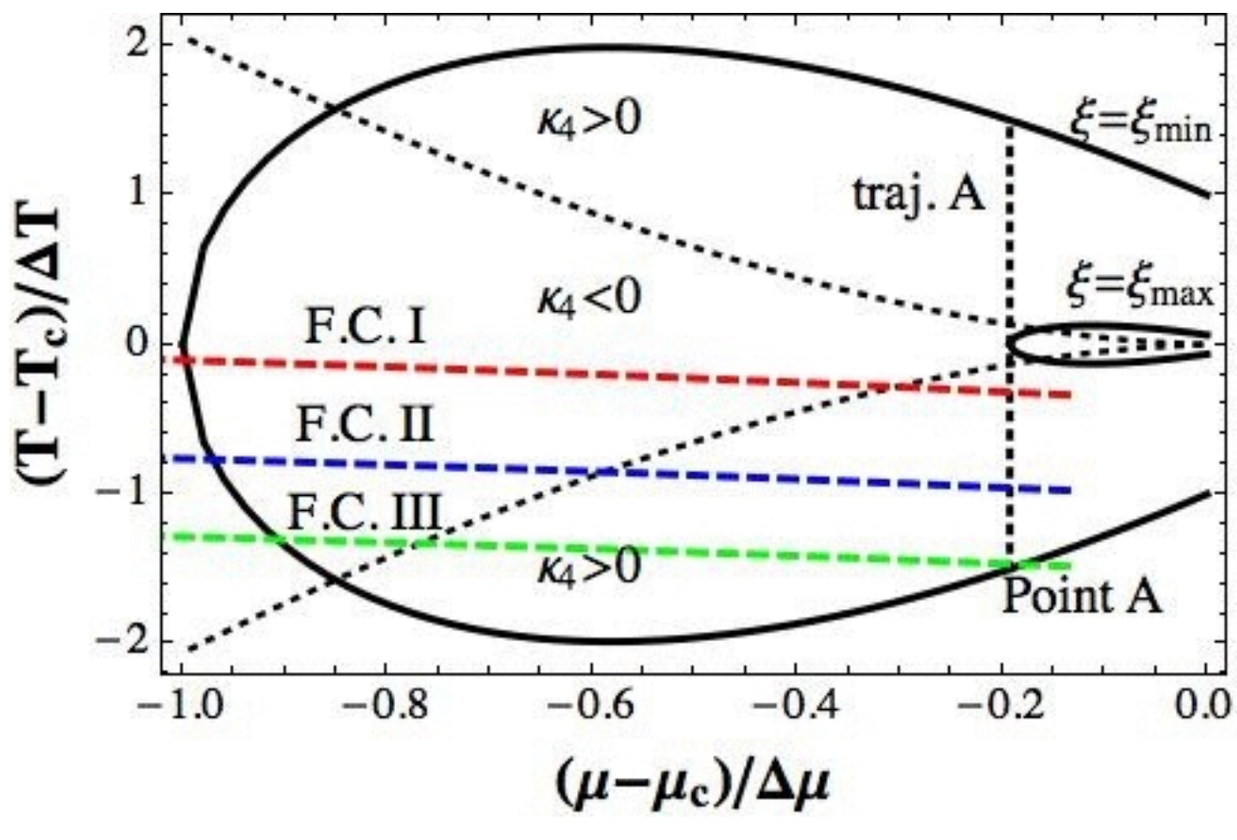
- To further illustrate dependence on $\tau_{\rm rel}/\tau_I$, we take non-Gaussian cumulants at fixed T and plot them as a function of $\tau_{\rm rel}/\tau_I$.
- The non-equilibrium values would approach the equilibrium ones only for small $\tau_{\rm rel}/\tau_I$.
- Skewness and in particular kurtosis would oscillate $\tau_{\rm rel}/\tau_I$ when changes.
- In experiment, $\tau_{\rm I}$ can be different for different centrality bins. Centrality dependence of cumulants at fixed energy would potentially probe memory effects.

Part III: Implications on BES and the search for QCD critical point

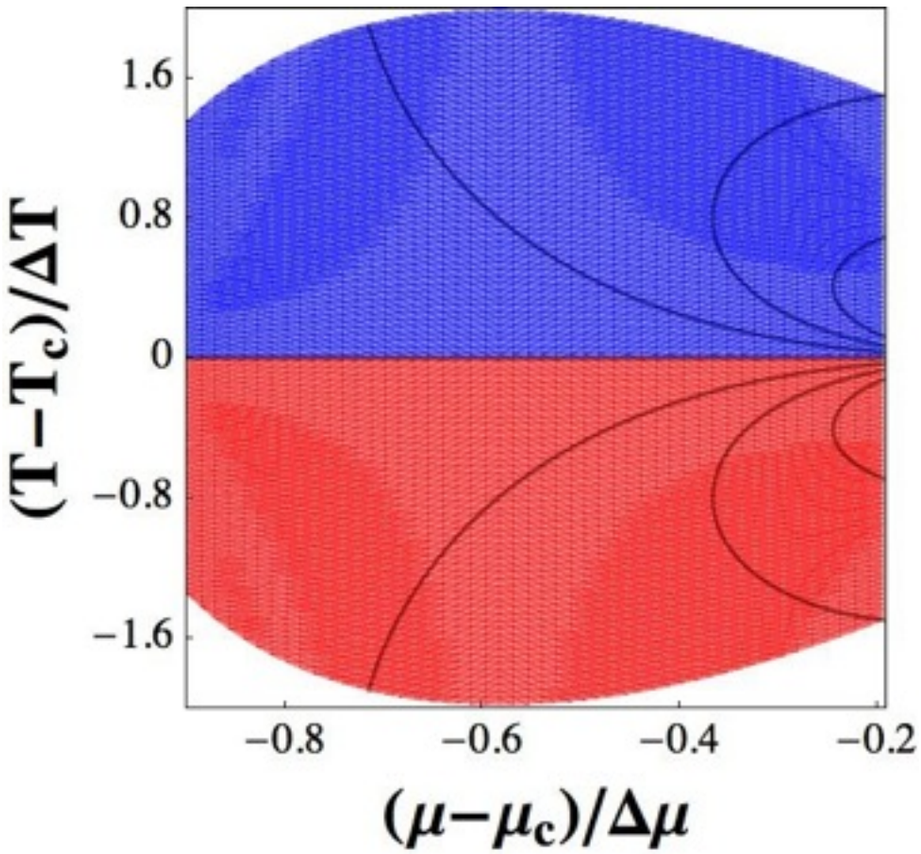
Mimicking Beam Energy Scan

- To mimic the beam energy scan, we also solved the evolution equations for all constant μ trajectories. We therefore obtain non-equilibrium at each point in the critical regime.

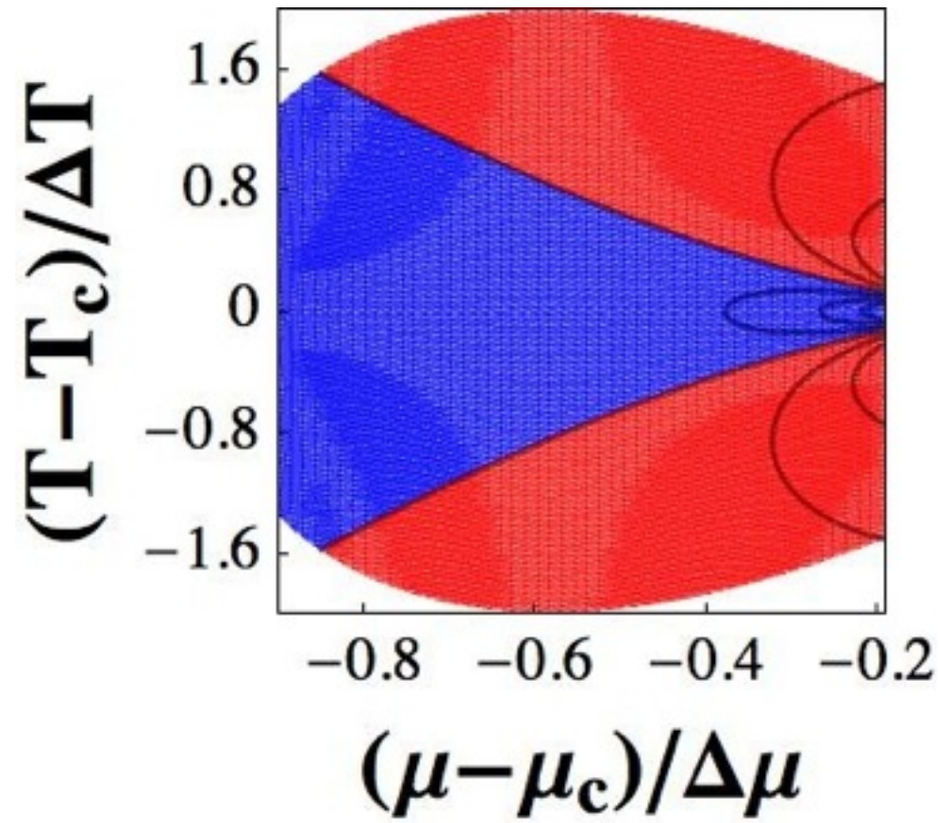
- We will concentrate on the Skewness and Kurtosis and will start with their most prominent feature: sign.



(Sign of) Equilibrium Skewness and Kurtosis



Skewness



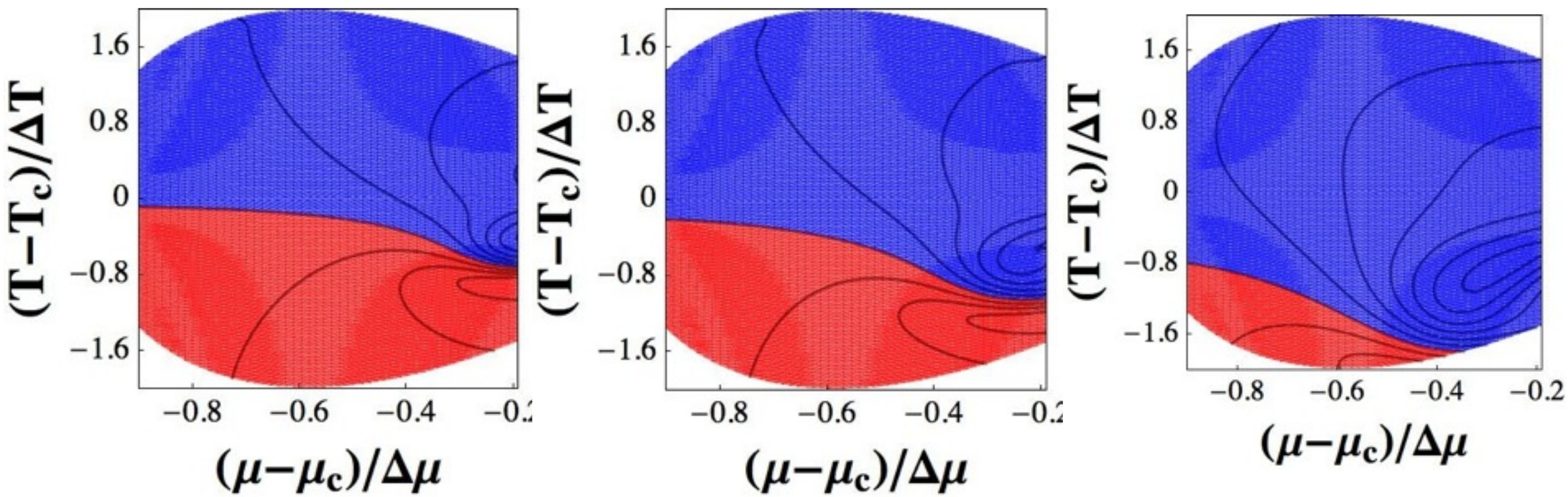
Kurtosis

Red >0

Blue<0

- Following the argument by Stephanov(Phys.Rev.Lett. 102 (2009) 032301), we fix the sign of skewness is positive below cross-over line.
- How would non-equilibrium effects change the above picture?

Deformation effects: Skewness

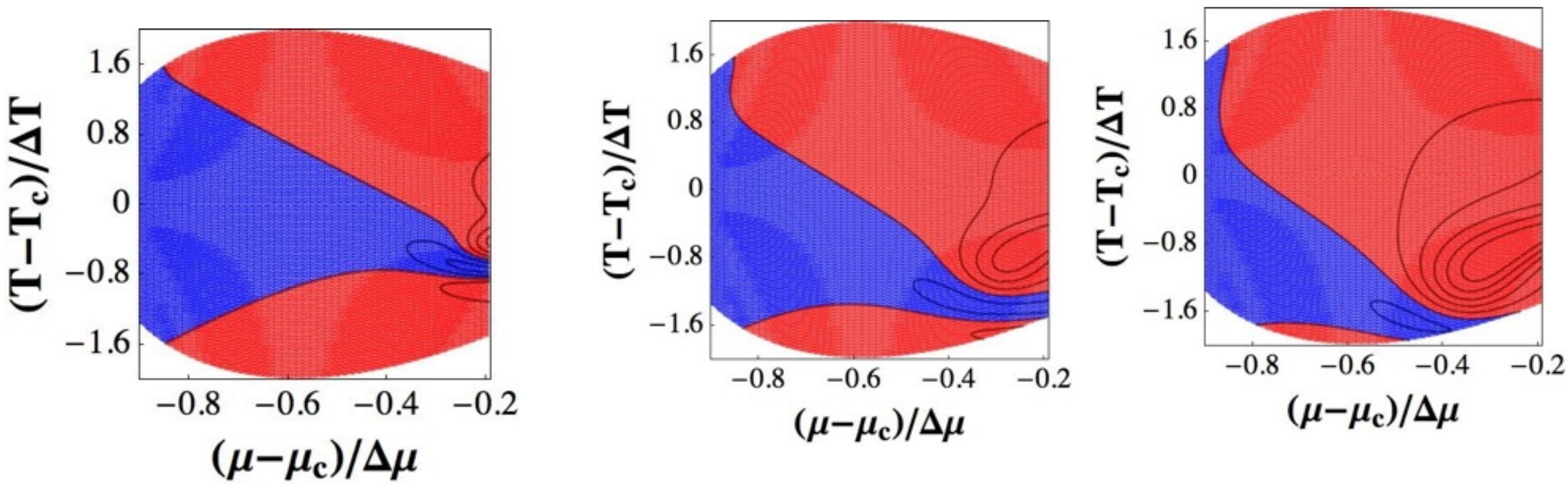


$\tau_{\rm rel}/\tau_I = 0.02$ $\tau_{\rm rel}/\tau_I = 0.2$ $\tau_{\rm rel}/\tau_I = 0.5$

Non-equilibrium skewness in critical regime

- Non-equilibrium effects deforms the regime that skewness is positive(negative).
- Non-equilibrium skewness carries the memory from deconfined phase(negative sign).

Deformation effects: Kurtosis



$\tau_{\rm rel}/\tau_I = 0.02$ $\tau_{\rm rel}/\tau_I = 0.2$ $\tau_{\rm rel}/\tau_I = 0.5$

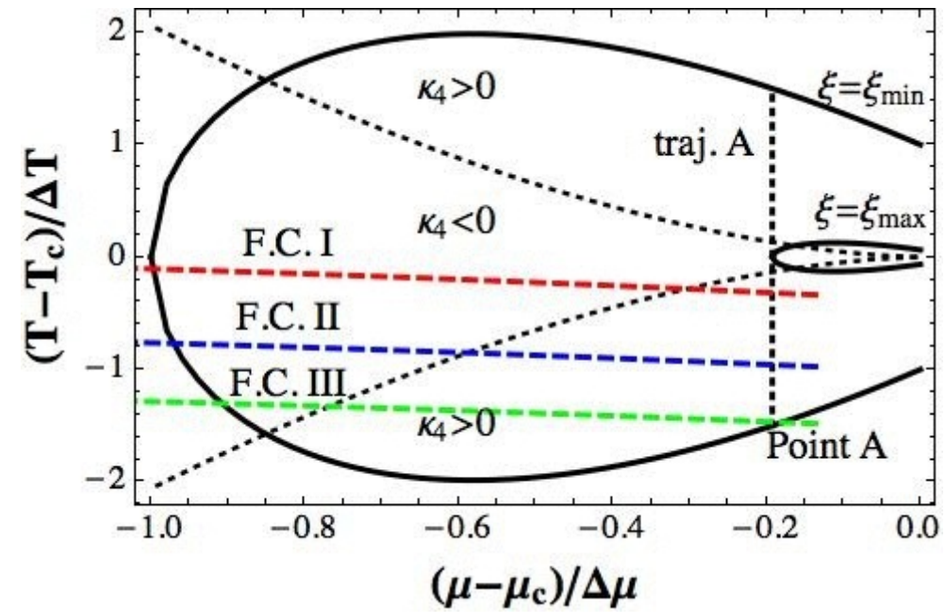
Non-equilibrium kurtosis in critical regime

- Similar for kurtosis. The boundary that kurtosis will change sign also deform.

- Flipping of sign of kurtosis is still generic!

Skewness and Kurtosis on freeze-out curves

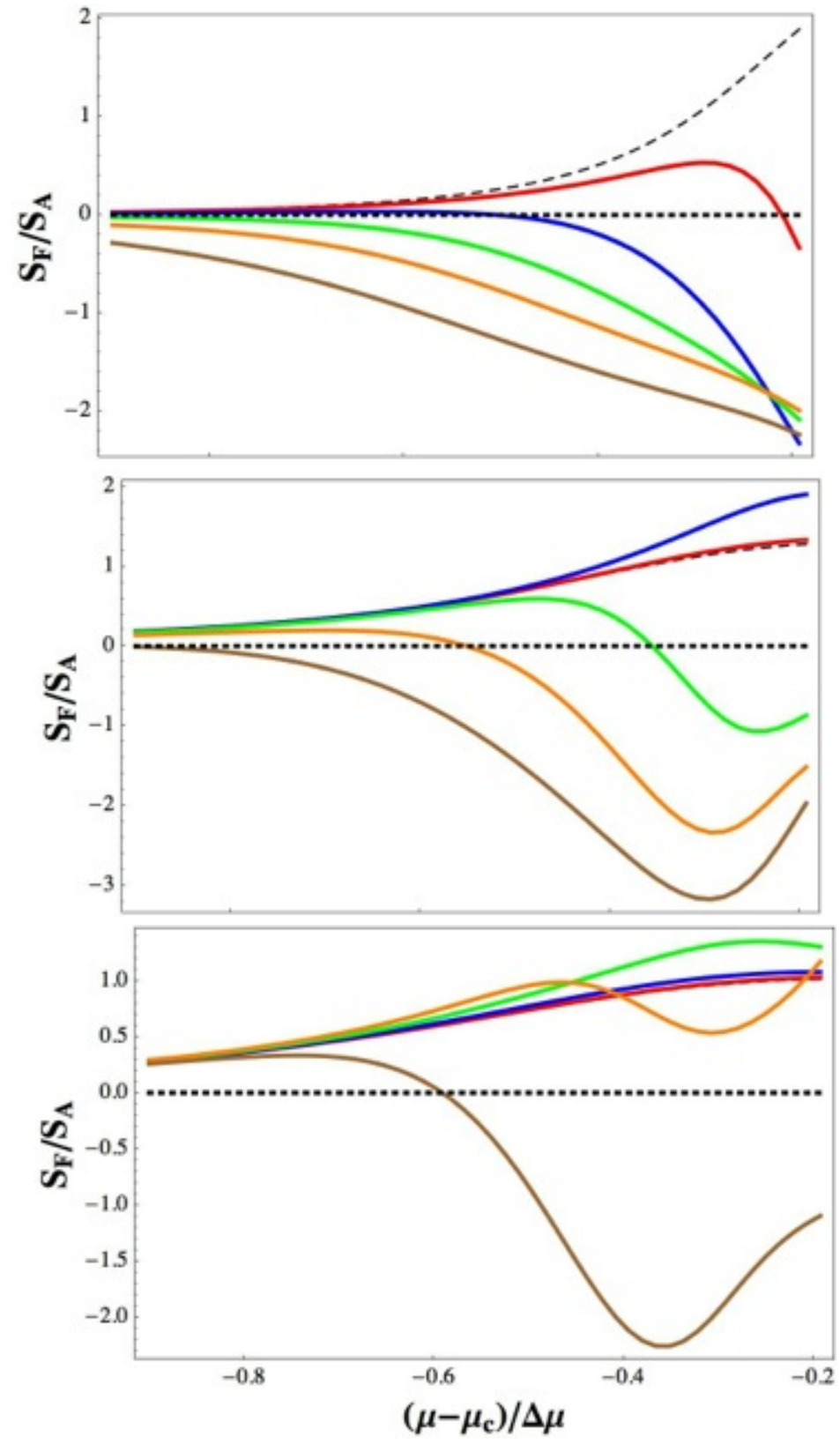
- We plot non-equilibrium skewness and kurtosis on the freeze-out curves as functions of (freeze-out) μ_B

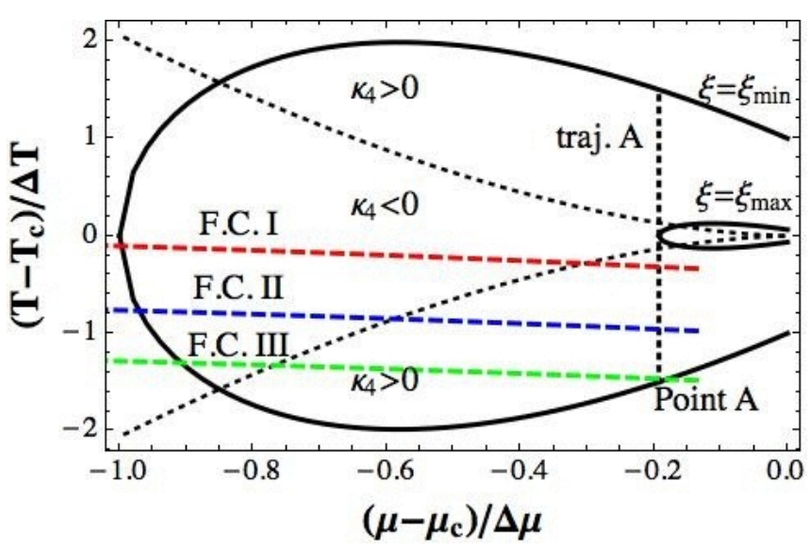


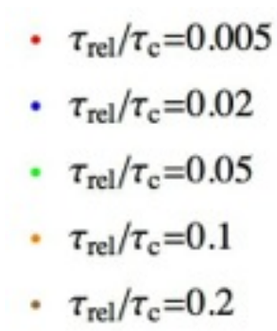
- Results depends on the relative location of freeze-curves and critical points. We have taken three generic freeze-out curves.

Skewness on freeze-out curves

- The behavior of non-equilibrium skewness can be non-monotonous even if the equilibrium skewness is monotonous.
- The sign of non-equilibrium skewness can be opposite to the equilibrium skewness.

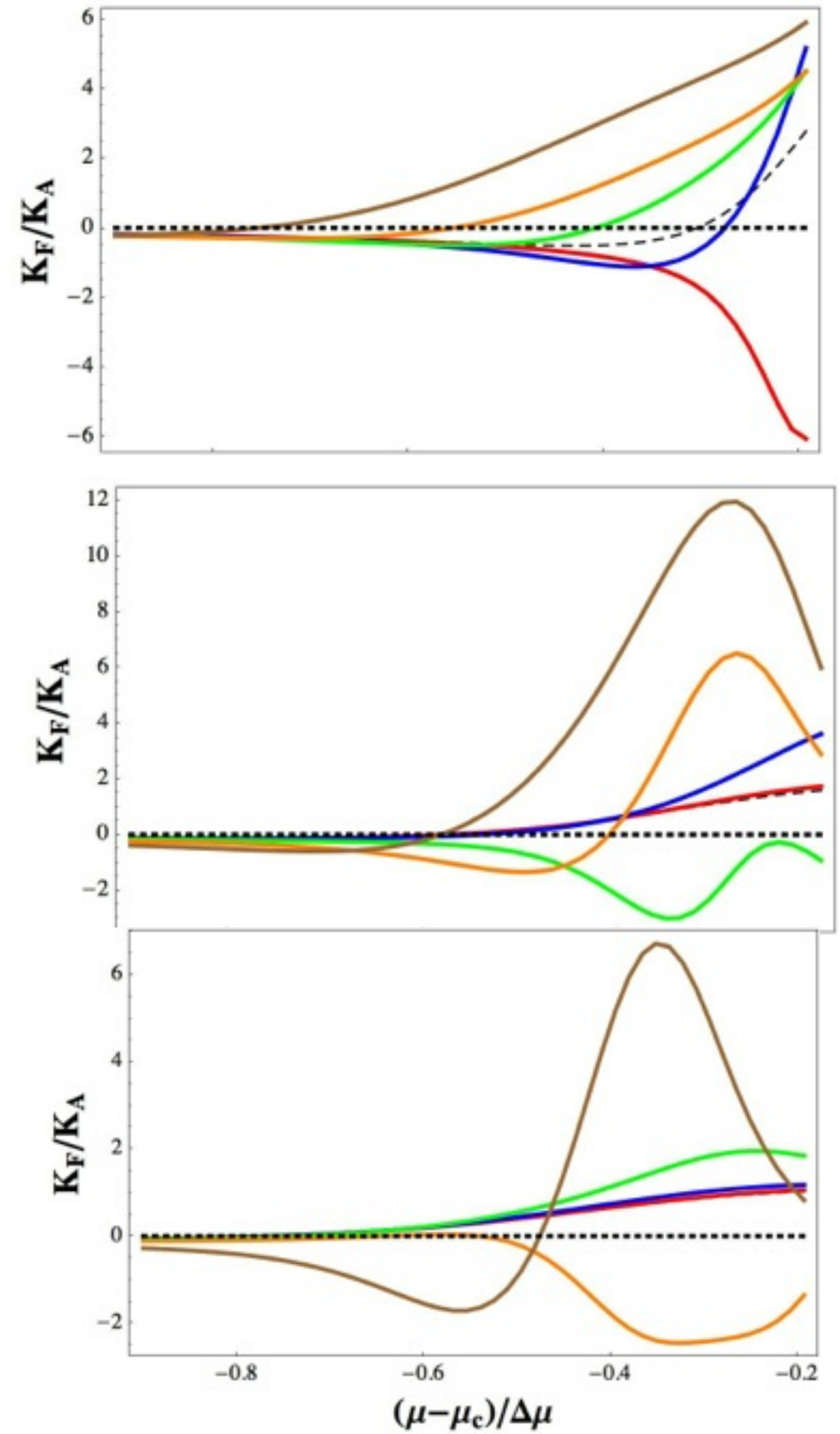


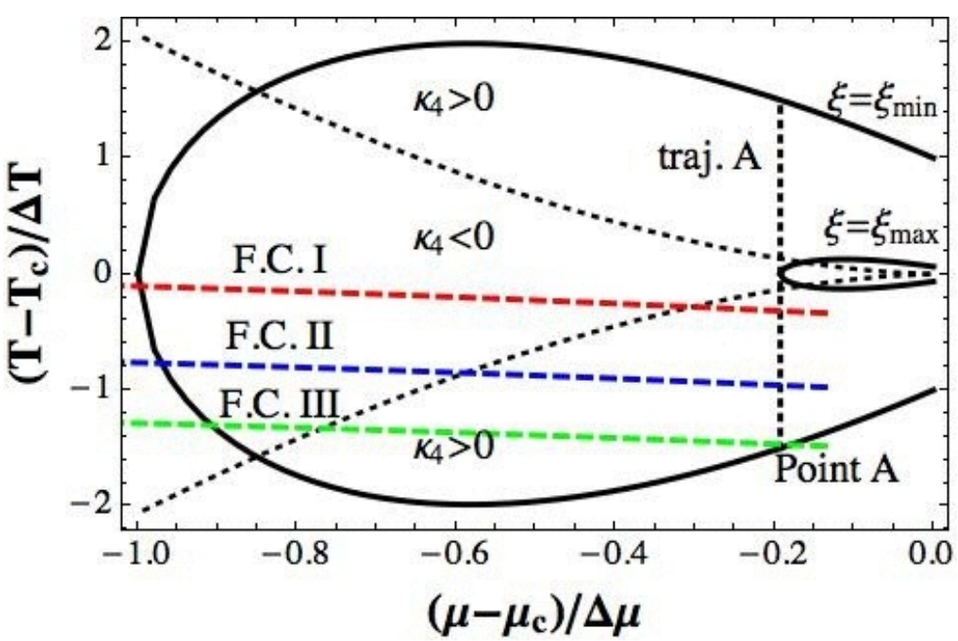




Non-equilibrium Kurtosis(of critical field) on freeze-out curves

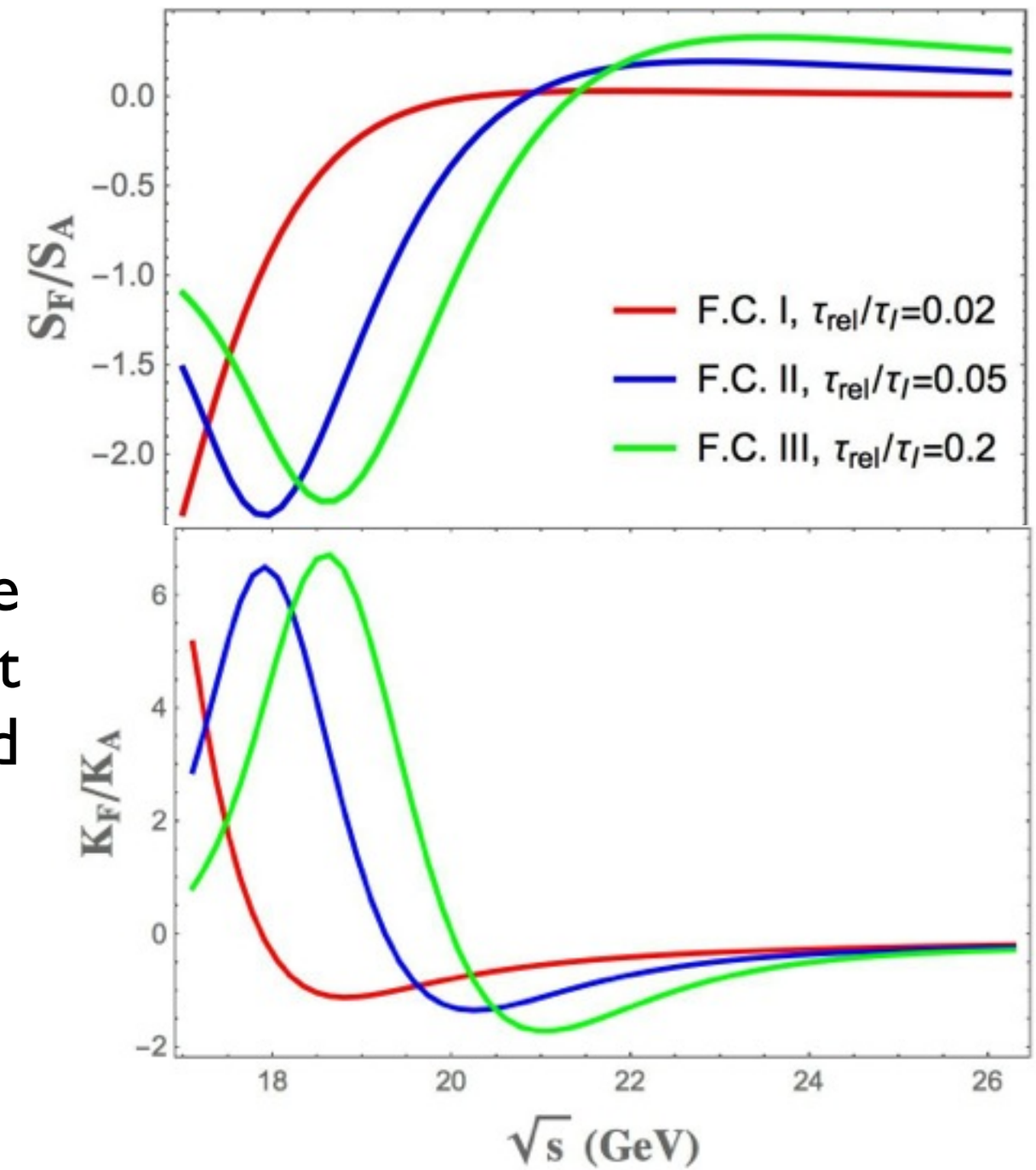
- The flipping of sign of kurtosis is still likely to happen!





Beam energy dependence

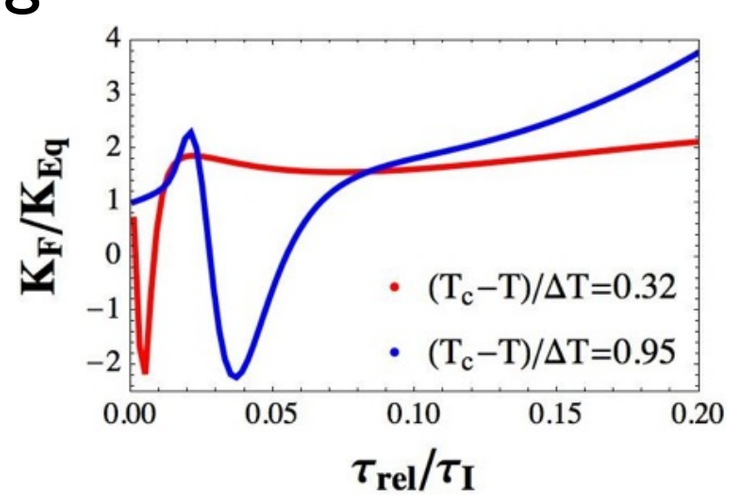
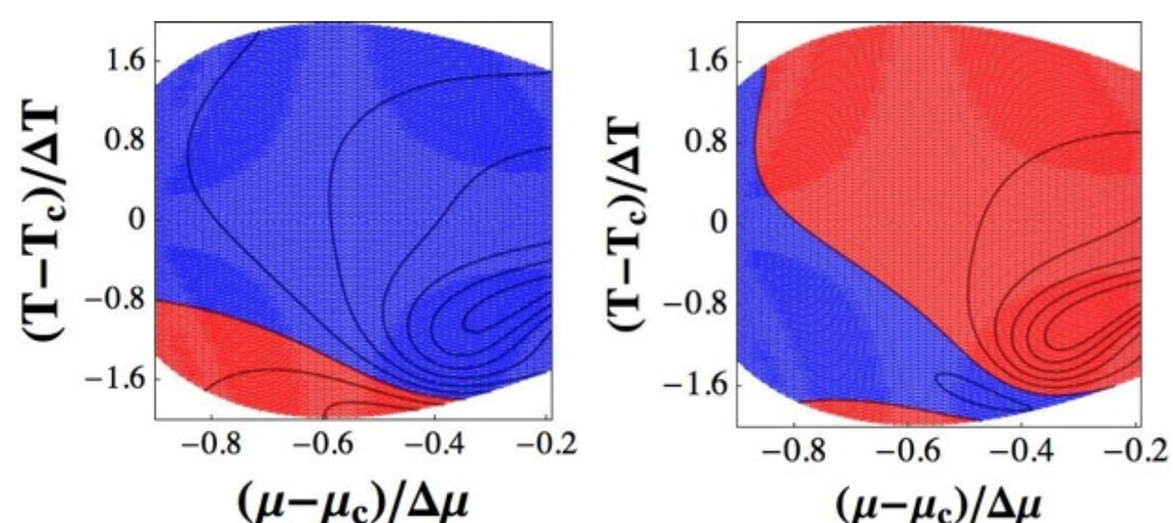
- We could to one step further and convert mu dependence into $\sqrt{s}$ dependence.



- Different combination of the relative locations of freeze-out curves and relaxation time would give similar trend.

- Locating QCD critical point requires extensive modeling.

Summary

Summary

- We have developed a set of equations to describe the evolution of cumulants in QCD critical regime.
- We have illustrated importance of memory effects and illustrate how would non-equilibrium cumulants behave in QCD critical regime.
- Keeping non-equilibrium effects in mind are important for understudying critical contributions to the data.
- For fixed trajectory and fixed T, non-Gaussian cumulants could oscillate with varying $\tau_{\rm rel}/\tau_I$.
- Deformation of the boundary that sign of higher cumulants will change.