Odd Azimuthal Anisotropy from CGC

A somewhat technical account

Vladimir Skokov (RBRC BNL)

June 20, 2017

Main Focus: L. McLerran and V. S., arXiv:1611.09870
Extended Analytical and Numerical results: future work
Outlook: A. Kovner, M. Lublinsky and V. S., arXiv:1612.07790
γ_{SS} in CGC: A. Kovner, M. Lublinsky and V. S., arXiv:1706.02330

Outline

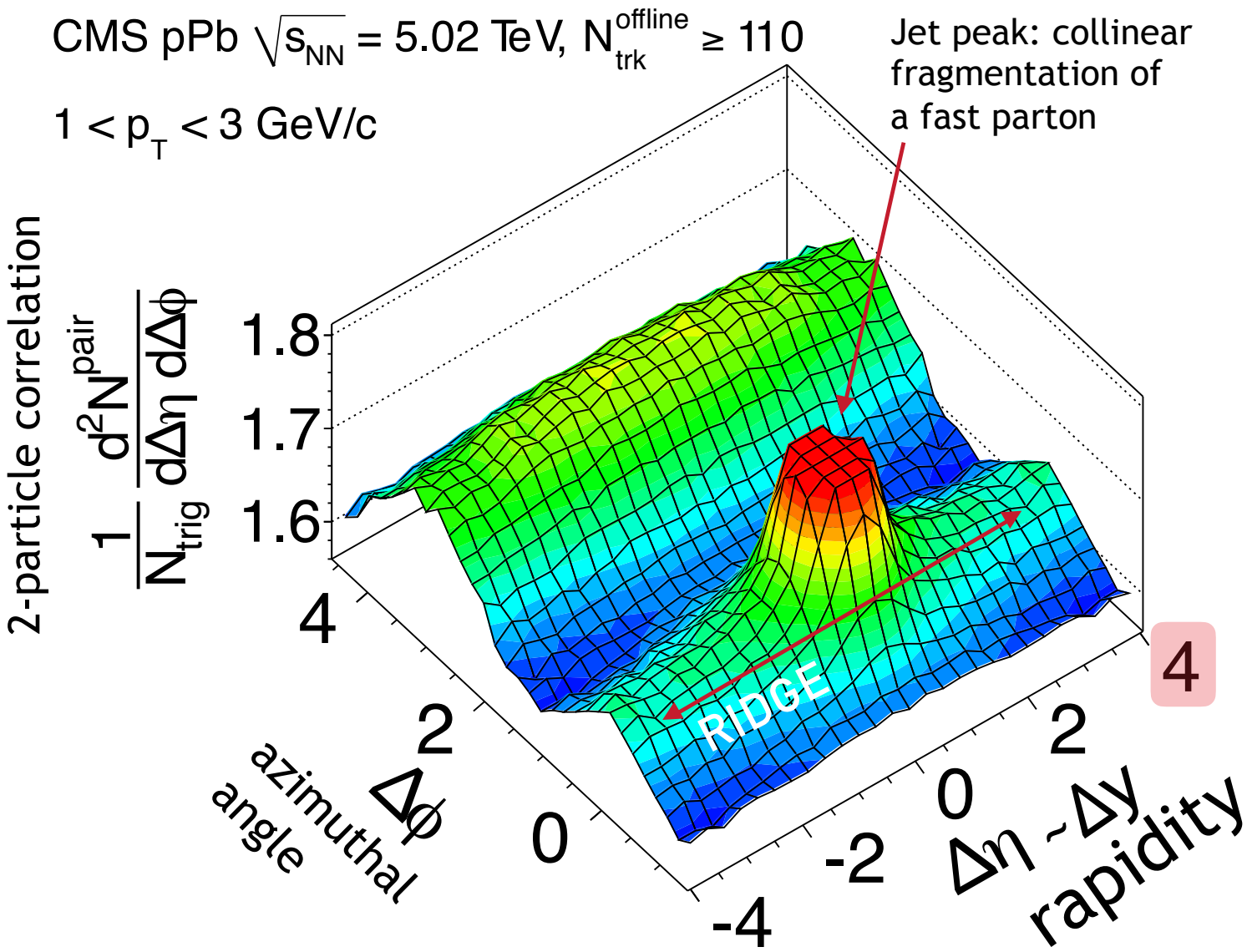
- Introduction:
  Quick recap on what we know from CGC
  & elusive odd azimuthal anisotropy ($v_3\{2\}$)

For a broader overview, Mark Mace, Wed 11 am

- Classical Yang-Mills: strategy of solving at LO and NLO in projectile density
- Back to basics: particle production and LSZ† reduction formula
- Dilute-dense limit of CGC and absence of odd anisotropy
- Beyond dilute-dense limit
- Single inclusive gluon production on configuration-by-configuration basis
- Double inclusive gluon production & odd azimuthal anisotropy
- Summary
- Outlook: beyond classical approximation and CME background from CGC

† Lehmann – Symanzik – Zimmermann

Introduction: the Ridge



CMS, Phys. Lett. B 718 (2013) 795

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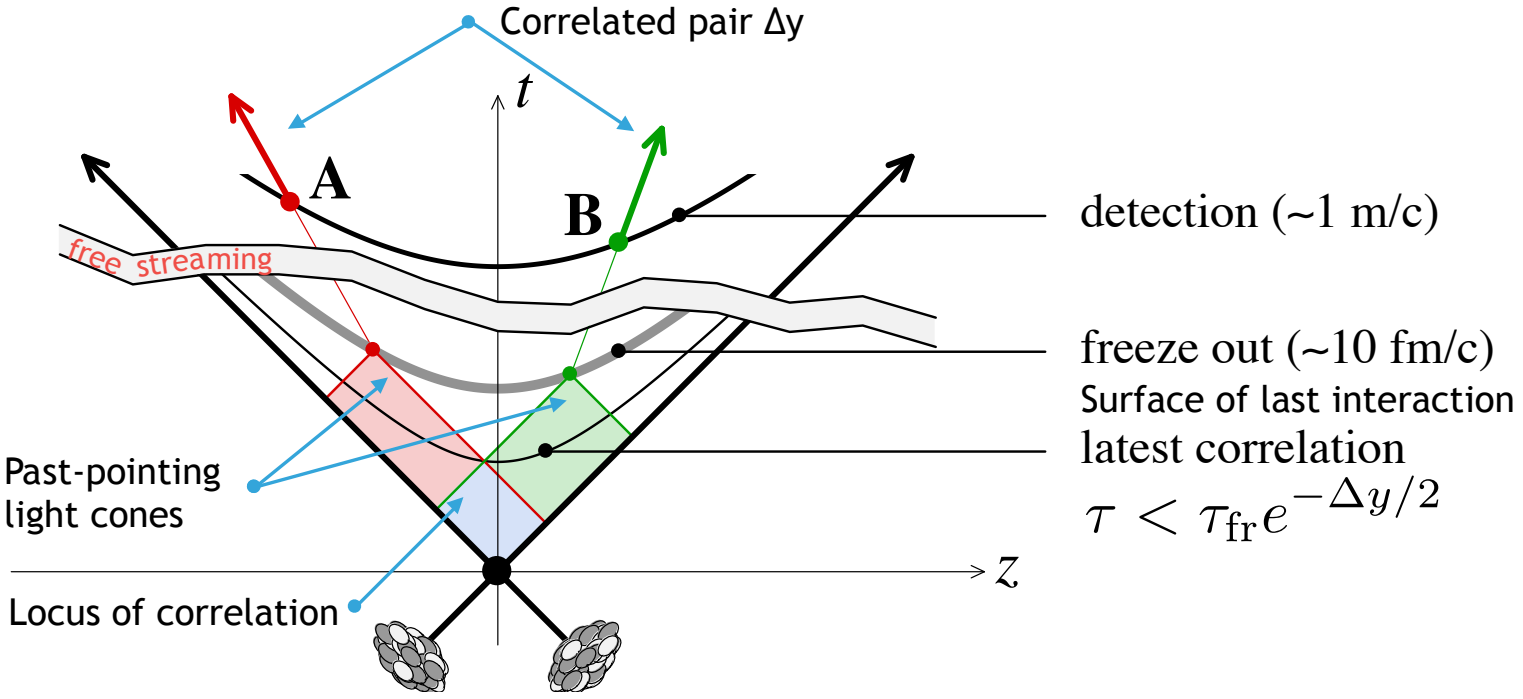
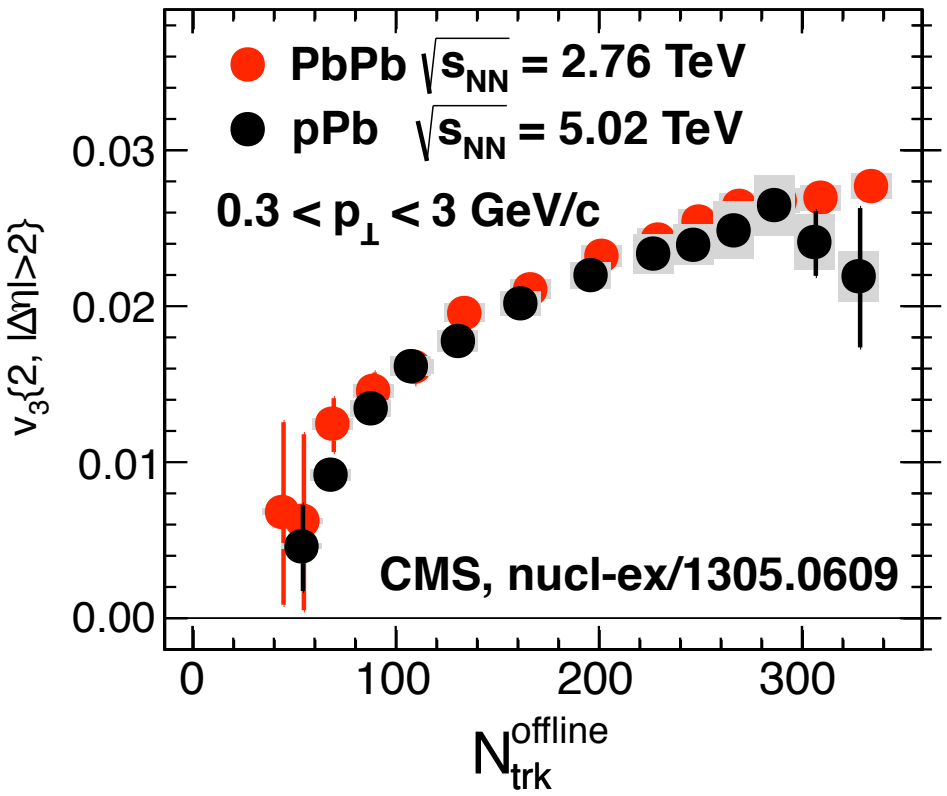


Suppress final state effects in order to probe initial state physics

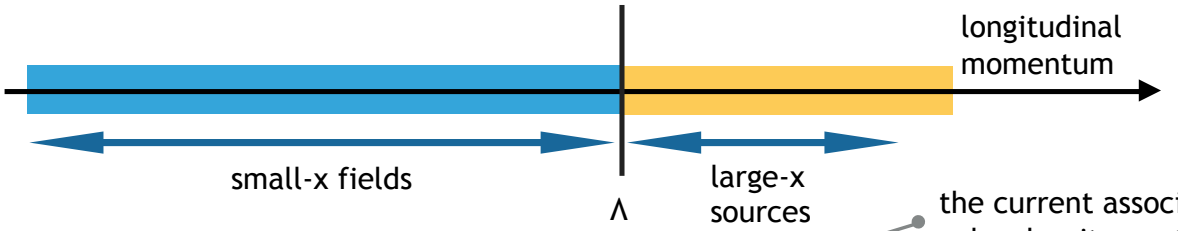
- Regardless of the nature of the ridge, we all agree that
  - long-range rapidity correlations either pre-exist in the initial wave function or develop very early after the collision
  - understanding the initial stage – the creation of the quarks and gluons before possible thermalization and formation of QGP – is of paramount importance for understanding A-A, p-A and p-p.

Odd harmonics



- Large v_3; magnitude is the same as in Pb-Pb collisions at the same centrality
- Suggests hydrodynamic interpretation
- At least consideration of studying observed experimental data with first-principle approach should be given
- QCD at high energy: CGC effective theory

Color Glass Condensate



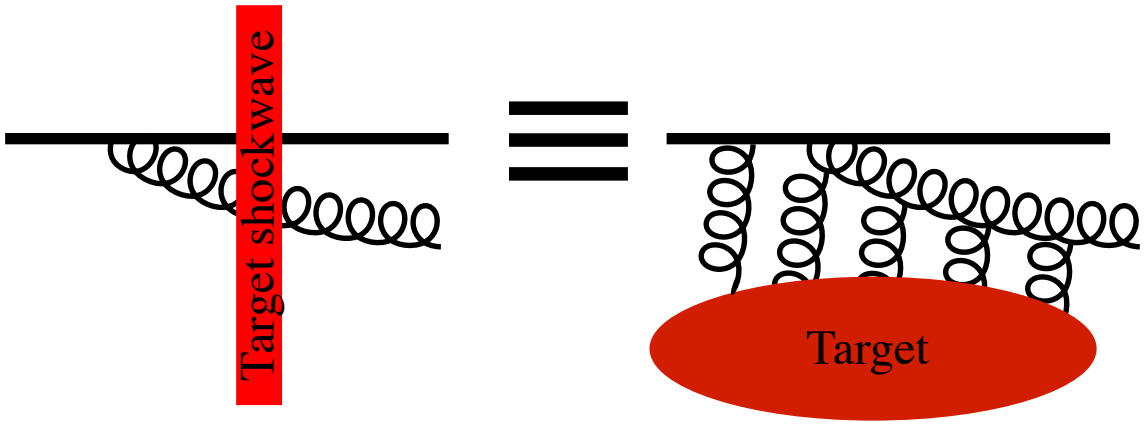
the current associated with
color density ρ; statistical ensemble $W[\rho]$

$$S = -\frac{1}{4}\int \underbrace{F_{\mu\nu}F_{\mu\nu}}_{\text{Yang-Mills}} + \int \underbrace{J^{\mu}A_{\mu}}_{\text{Sources — fast partons; } J^{\mu} = \delta^{\mu +}\rho(x^{-},x_{\perp})}$$

internal dynamics is frozen:
no x^{+} dependence

- Large x: color sources from different overlapping in longitudinal direction nucleons
  The sources are approximately Gaussian. Correlations are local in the transverse plane because they originate from uncorrelated nucleons $\rightsquigarrow$ McLerran-Venugopalan model for color charge distribution $W[\rho]$
- Small x: high gluon densities at leading order, classical field description
- To organize calculations use an EFT of QCD: the Color Glass Condensate

Multiple rescattering



- Multiple rescattering in higher energy limit is described by the eikonal phase

$$U(x_\perp) = \mathcal{P}\exp\left(ig\int dx^- A_+(x^-, x_\perp)\right)$$

- Only important property I need for further narrative: the classical gluon field is real
  (for rigorous discussion see L. McLerran and V. S., arXiv:1611.09870)
- $A(x)$ is real function; therefore $A^*(k) = A(-k)$
- $s(k) \propto \underbrace{\left(\int d^2x A(x)\exp(-ikx)\right)}_{\text{amplitude}}\underbrace{\left(\int d^2y A(y)\exp(-iky)\right)^*}_{\text{c.c.amplitude}} = A(k)A(-k)$
- Thus $\sigma(k) = \sigma(-k)$

Single-inclusive gluon production

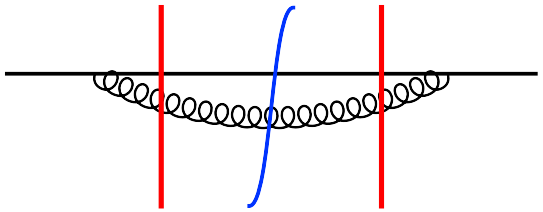
- In general, single-inclusive gluon production

$$\left\langle \frac{d^2N}{dk_\perp^2} \right\rangle_{\rho_1,\,\rho_2} \propto \frac{1}{\alpha_s} \sum_{n=1,\,m=1}^{n\le N,\,m\le M} c_{n,m} \left(\frac{Q_{s1}^2}{k_\perp^2}\right)^n \left(\frac{Q_{s2}^2}{k_\perp^2}\right)^m$$

  1 (2) stands for Projectile (Target)

- Lowest order $c_{1,1}$ was computed by A. Kovner, L. McLerran, H. Weigert, arXiv:9506320
  reproducing results of E. Kuraev, L. Lipatov, V. Fadin, '77

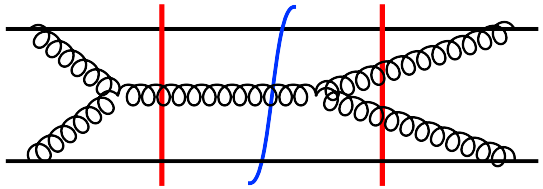
- Analytic results for dilute-dense limit $N = 1$, $M = \infty$
  Yu. Kovchegov, A. Mueller, arXiv:9802440; A. Dumitru, L. McLerran, arXiv:0105268;
  J.-P. Blaizot, F. Gelis, R. Venugopalan, arXiv:0402256



- Numerical results for dense-dense limit $N = \infty$, $M = \infty$
  Pioneering work by A. Krasnitz, R. Venugopalan, arXiv:9809433

- First saturation correction in the projectile $N = 2$, $M = \infty$ – Incomplete

  (diagrams where one of the nucleons is
  a spectator in the amplitude of order g are not included)
  G. Chirilli, Yu. Kovchegov, D. Wertepny, arXiv:1501.03106



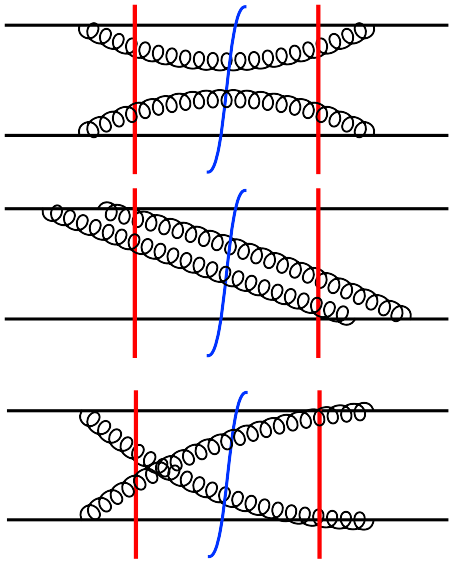
- To describe physics of the ridge: double-inclusive gluon production...

Double-inclusive gluon production

- "Glasma" graph – the correlated leading order contribution

  A. Dumitru, F. Gelis, L. McLerran, R. Venugopalan, arXiv:0804.3858

$$\left\langle \frac{d^2N(\vec{k}_\perp,\vec{p}_\perp)}{dk_\perp^2\,dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2} \propto \frac{1}{\alpha_s^2}\left(\frac{Q_s^2}{k_\perp^2}\right)^2\left(\frac{Q_s^2}{p_\perp^2}\right)^2$$



- Analytical calculations to any order in the target density

  Yu. Kovchegov, D. Wertepny, arXiv:1212.1195

- The nature of the "glasma" graph correlations: pre-existing gluon correlation in projectile

  T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and M. Lublinsky, arXiv:1503.07126
  "Bose enhancement of gluons in the projectile leads to azimuthal collimation of . . . ";
  Conventionally: average over soft degrees of freedom at fixed ρ; then average over ρ.
  1503.07126: reverse the conventional order to get a density matrix for soft degrees of freedom.

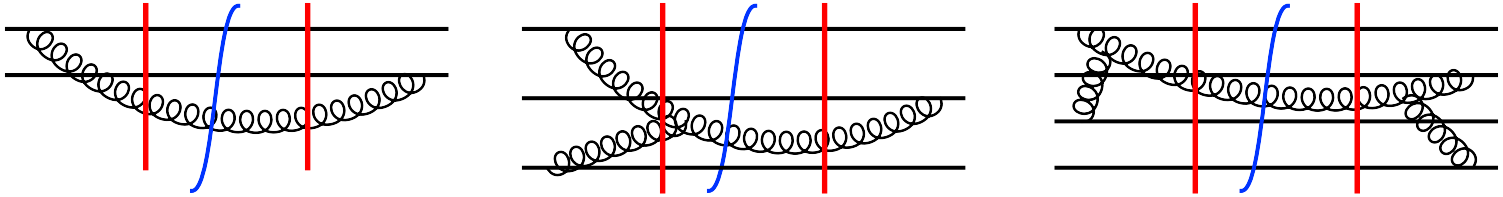
- Dense-dense limit: numerical simulations of Classical Yang-Mills

  T. Lappi, S. Srednyak and R. Venugopalan, arXiv:0911.2068; B. Schenke, S. Schlichting and R. Venugopalan, arXiv:1502.01331

Key properties of single- & double-inclusive production I

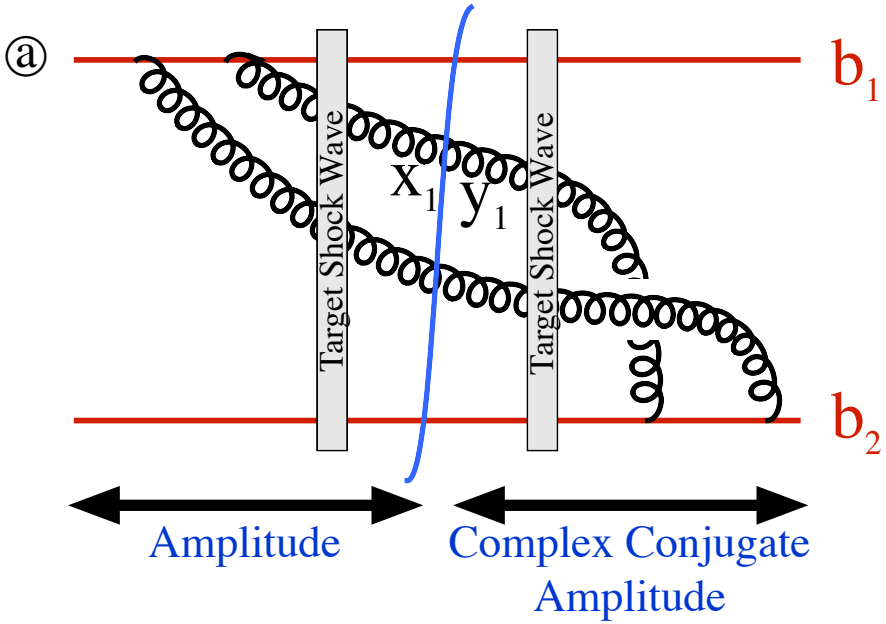
- Dilute-dense power-counting on configuration-by-configuration basis (for a given ρ_1):

$$\frac{dN}{dk_\perp^2} \propto g^2\rho_1^2 + g^4\rho_1^3 + g^6\rho_1^4 + \cdots$$



- The leading contribution $g^2\rho_1^2$ (dilute-dense limit) is symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$!
  Corollary: the leading log double-inclusive particle production
  $\left\langle \frac{d^2N(\vec{k}_\perp,\vec{p}_\perp)}{dk_\perp^2\, dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2} = \left\langle \frac{dN}{dk_\perp^2}\frac{dN}{dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2}$ shares this symmetry at the leading order. No $v_3\{2\}$!
  Yu. Kovchegov, D. Wertepny, arXiv:1212.1195
- The first saturation correction contribution $g^6\rho_1^4$ is also symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$
  see the argument in L. McLerran, V. S. arXiv:1611.09870;
- The contribution $g^4\rho_1^3$ averages out (i.e. it is zero) in a Gaussian ensemble.
  My goal is to derive this contribution and show that it is not symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$.
  I will argue that it gives a non-vanishing $g^8\rho_1^6$ contribution to double inclusive production.

Why are there no odd harmonics in Glasma graph?



$$\sigma_{ⓐ}(k,p) = \int_{x_1,y_1,x_2,y_2} e^{-ik\cdot(x_1-y_1)} e^{-ip\cdot(x_2-y_2)} \underbrace{A(x_1,b_1)A(x_2,b_1)}_{\text{ampl.}} \underbrace{A^*(y_1,b_2)A^*(y_2,b_2)}_{\text{c.c.ampl.}}$$

A. Kovner and M. Lublinsky, arXiv:1012.3398,
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Y. Kovchegov, V.S., work in progress

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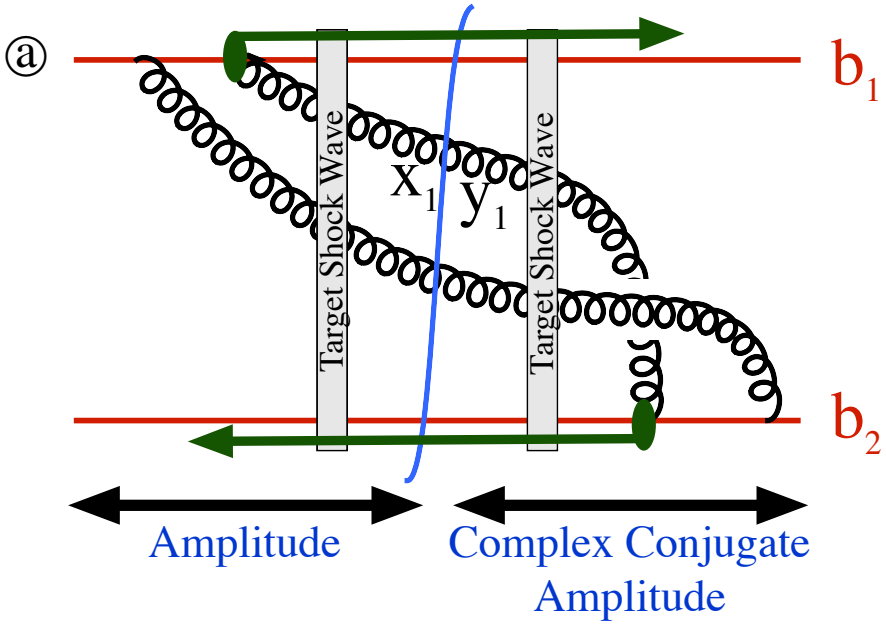

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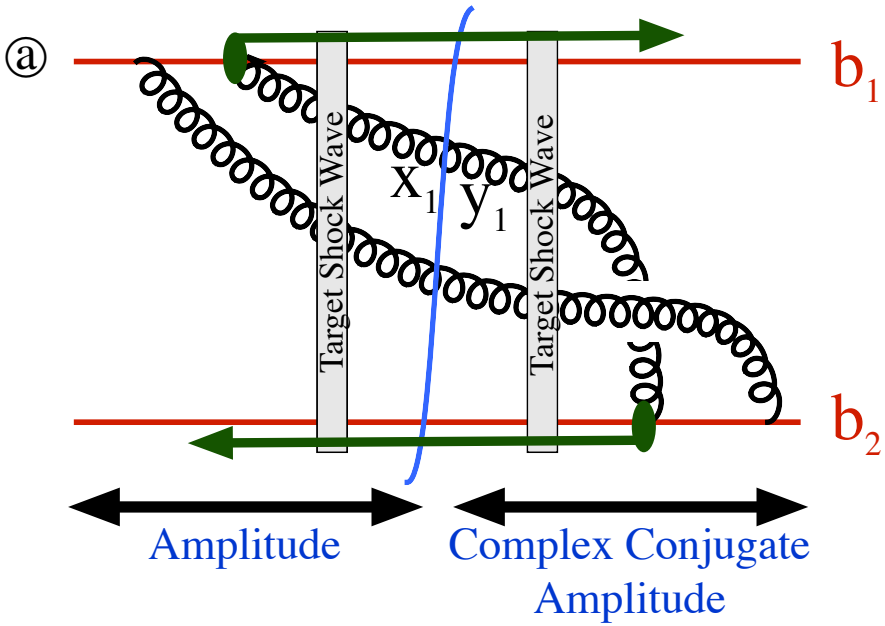

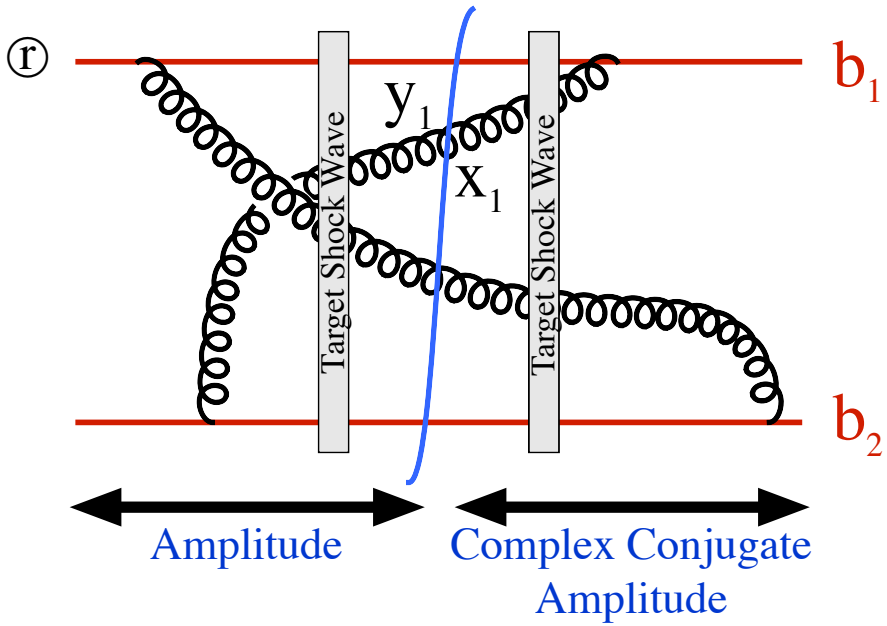

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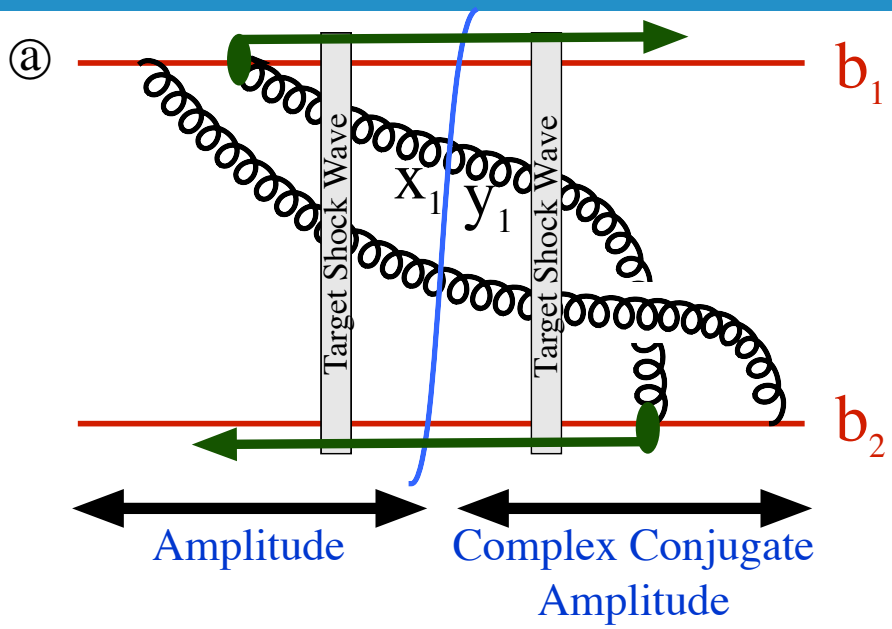

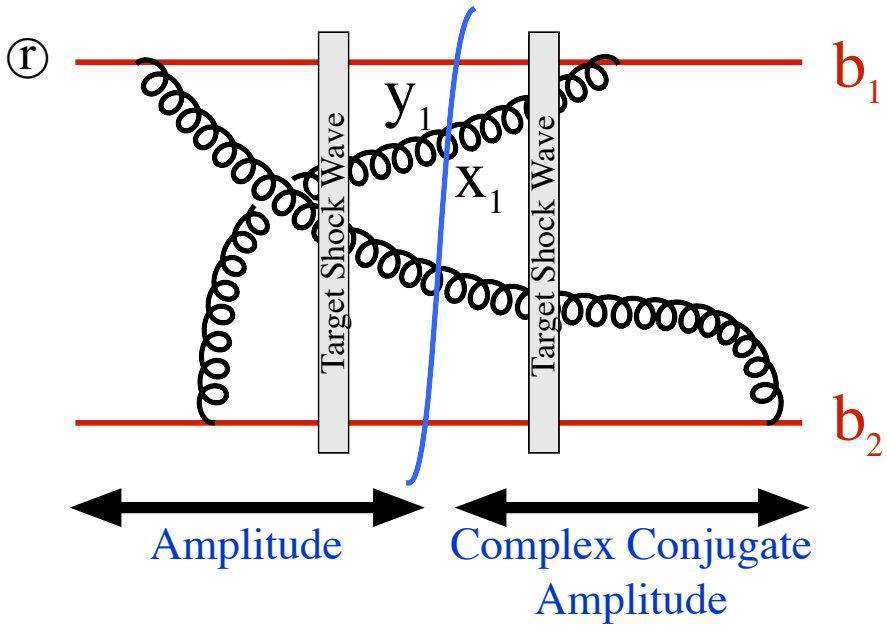

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But $A(x,b)$ is real: $A(x,b) = A^*(x,b) \rightsquigarrow$

$$\sigma_{ⓐ}(k,p) + \sigma_{ⓡ}(k,p) = \sigma_{ⓐ}(k,p) + \sigma_{ⓐ}(-k,p)$$

$$\rightsquigarrow v_3 = 0$$

A. Kovner and M. Lublinsky, arXiv:1012.3398,
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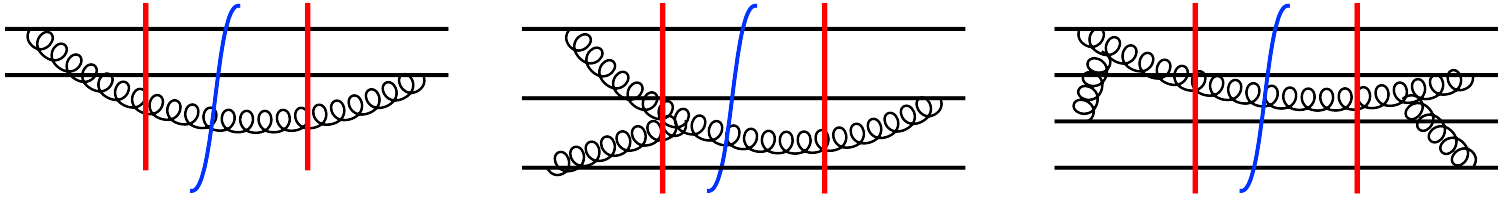

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Key properties of single- & double-inclusive production II

- Dense-dense limit: analysis of 0911.2068 (section "3.1 Numerical approach") showed that

$$\frac{dN}{dk_\perp^2} \propto f_\gamma(\vec{k}_\perp) f_\gamma(-\vec{k}_\perp) + i\left[E_\gamma(\vec{k}_\perp)A_\gamma(-\vec{k}_\perp) - A_\gamma(\vec{k}_\perp)E_\gamma(-\vec{k}_\perp)\right]$$

T. Lappi, S. Srednyak and R. Venugopalan, arXiv:0911.2068

E_γ and A_γ are solutions of CYM at the asymptotic infinity, $\tau \to \infty$.

- $\rightsquigarrow$ non-vanishing odd azimuthal anisotropy in double-inclusive production!

T. Lappi, S. Srednyak and R. Venugopalan, arXiv:0911.2068; B. Schenke, S. Schlichting and R. Venugopalan, arXiv:1502.01331

- This apparent contradiction between the dilute-dense limit and dense-dense argument was an outstanding problem for the community.

a quote from a well-cited review

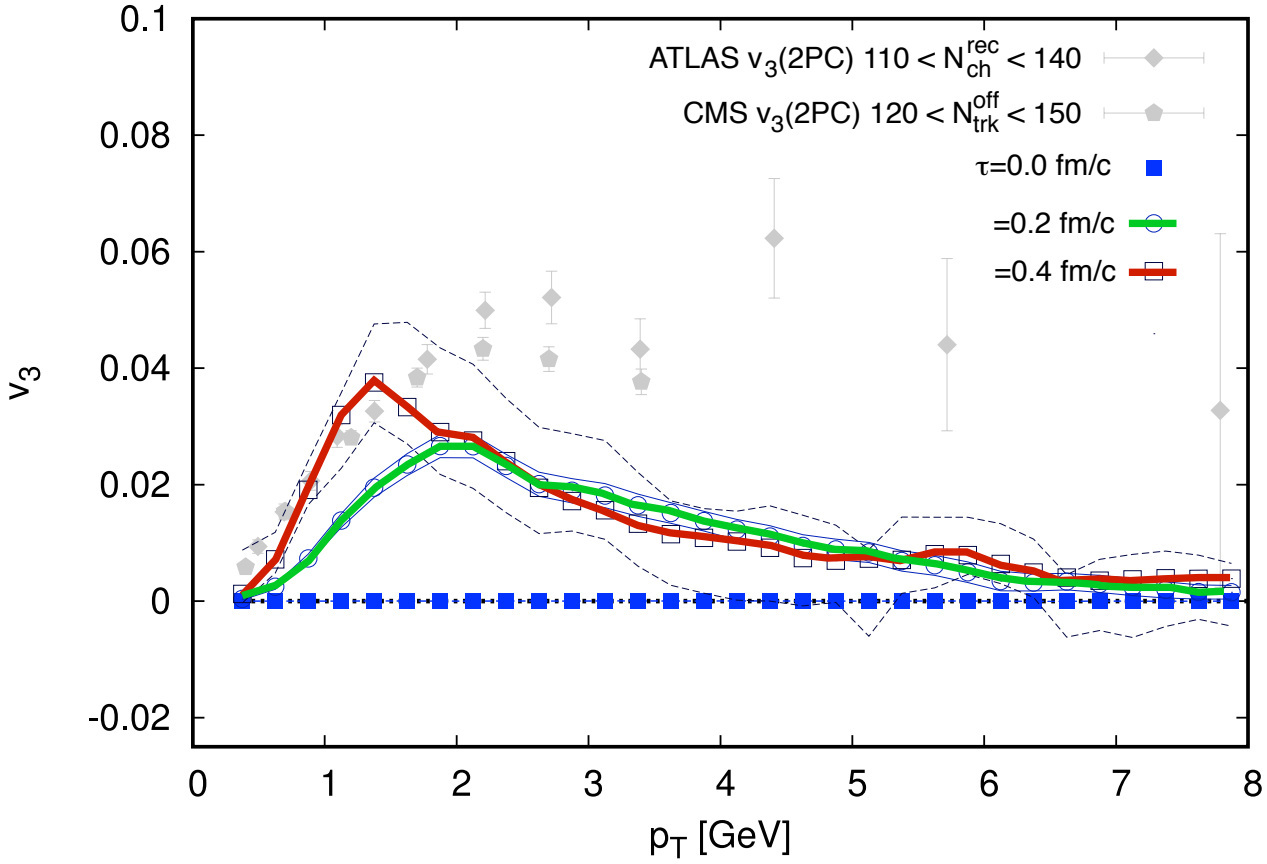
> ~~*"We note that numerical results do not seem to display the exact symmetry $\vec{k}_\perp \to -\vec{k}_\perp$, which may be an indication of some subtlety of the numerical procedure."*~~

- This program is a) to build a bridge between dilute-dense and dense-dense regimes; b) to provide analytical explicit expression for the odd asymmetry

L. McLerran, V. S. arXiv:1611.09870;

- Why do we care about an analytical result? a) no need to solve time evolution numerically; b) low-x evolution; c) running-coupling corrections

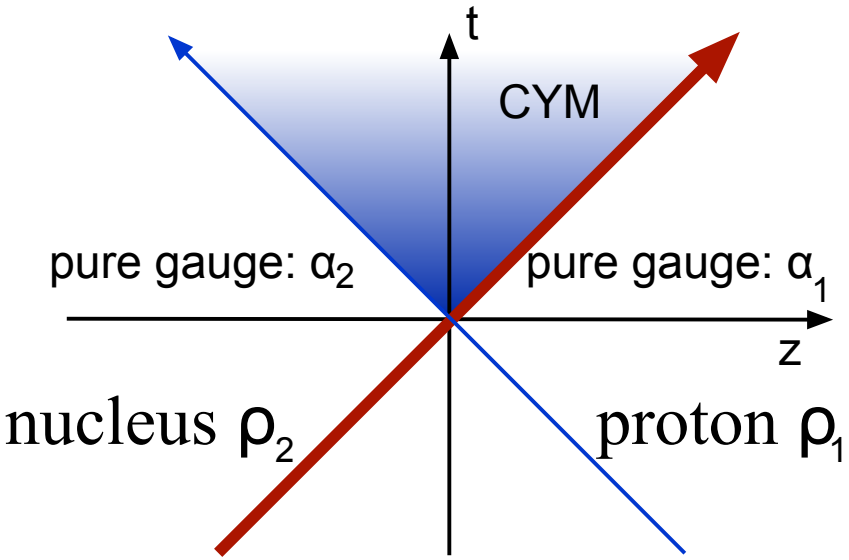
Odd harmonics in dense-dense scattering



B. Schenke, S. Schlichting, R. Venugopalan, 1502.01331

- Zero v_3 from the initial state; expected in classical limit, as initial conditions for classical field are real functions.
- v_3 develops quickly with time

Classical Yang-Mills



- Before collision, pure gauge soft fields created by "valence" currents

$$\partial_i \alpha^i_{1,2}(\vec{x}_\perp) = g\,\rho_{1,2}(\vec{x}_\perp)$$
$$\alpha^i_{1,2}(\vec{x}_\perp) = -\tfrac{1}{ig} U_{1,2}(\vec{x}_\perp)\partial^i U^\dagger_{1,2}(\vec{x}_\perp)$$

- Just after the collision, $\tau \to 0+$, (Fock-Schwinger gauge $A_\tau = 0$)
  A. Kovner, L. McLerran, H. Weigert, arXiv:9506320

$$\alpha^i(\tau \to 0, \vec{x}_\perp) = \alpha^i_1(\vec{x}_\perp) + \alpha^i_2(\vec{x}_\perp)$$

$$A_\eta(\tau \to 0, \vec{x}_\perp) = \tau^2 \alpha(\tau \to 0, \vec{x}_\perp); \quad \alpha(\tau \to 0, \vec{x}_\perp) = \frac{ig}{2}[\alpha^i_1(\vec{x}_\perp), \alpha^i_2(\vec{x}_\perp)]$$

- In the forward light-cone $\quad [D_\mu, F^{\mu\nu}] = 0$

Classical Yang-Mills: expansion in $g\rho_1$

- Initial conditions

$$\alpha^i(\tau \to 0, \vec{x}_\perp) = \alpha_1^i(\vec{x}_\perp) + \alpha_2^i(\vec{x}_\perp)$$

$$A_\eta(\tau \to 0, \vec{x}_\perp) = \tau^2 \alpha(\tau \to 0, \vec{x}_\perp); \quad \alpha(\tau \to 0, \vec{x}_\perp) = \frac{ig}{2}[\alpha_1^i(\vec{x}_\perp), \alpha_2^i(\vec{x}_\perp)]$$

complemented with the expansion in $g\rho_1$ ($\Phi_1 = \frac{g}{\partial_\perp^2}\rho_1$):

$$\alpha_1^i = \underbrace{\partial^i \Phi_1}_{(1)} - \underbrace{\frac{ig}{2}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right)\left[\partial^j \Phi_1, \Phi_1\right]}_{(2)} + O(\Phi_1^3)$$

- YM equations describing the dynamics of the fields $[D_\mu, F^{\mu\nu}] = 0$
- In order to perform the expansion, it is convenient to rotate out the **nucleus field** from the initial conditions:

$$\alpha(\tau, \vec{x}_\perp) = U_2(\vec{x}_\perp)\beta(\tau, \vec{x}_\perp)U_2^\dagger(\vec{x}_\perp)$$

$$\alpha^i(\tau, \vec{x}_\perp) = U_2(\vec{x}_\perp)\left(\beta^i(\tau, \vec{x}_\perp) - \frac{1}{ig}\partial_i\right)U_2^\dagger(\vec{x}_\perp)$$

$$\beta(\tau \to 0, \vec{x}_\perp) = U_2^\dagger(\vec{x}_\perp)\alpha(\tau \to 0, \vec{x}_\perp)U_2^\dagger(\vec{x}_\perp)$$

$$\beta^i(\tau \to 0, \vec{x}_\perp) = U_2^\dagger(\vec{x}_\perp)\alpha_1^i(\vec{x}_\perp)U_2^\dagger(\vec{x}_\perp)$$

- Perform expansion in powers of ρ_1: $\beta_\gamma = \beta_\gamma^{(1)} + \beta_\gamma^{(2)} + \cdots$

First order, $\beta^{(1)}$

- At the leading order, the CYM equations are (in Milne coordinates)

$$\left[\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_\perp^2\right]\beta^{(1)}(\tau,\vec{x}_\perp) = 0,$$
$$\partial_\tau\partial_i\beta_i^{(1)}(\tau,\vec{x}_\perp) = 0,$$
$$\left[\delta^{ij}\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_\perp^2\right) + \partial_i\partial_j\right]\beta_j^{(1)}(\tau,\vec{x}_\perp) = 0$$

No non-linear terms $\rightsquigarrow$ the solution can be found trivially.

- The solutions is

in momentum space:

$$\beta^{(1)}(\tau,\vec{k}_\perp) = b_1(\vec{k}_\perp)\frac{J_1(k_\perp\tau)}{k_\perp\tau},$$
$$\beta_i^{(1)}(\tau,\vec{k}_\perp) = i\frac{\varepsilon^{ij}k_j}{k_\perp^2}b_2(\vec{k}_\perp)J_0(k_\perp\tau) + ik_i\Lambda(\vec{k}_\perp)$$

$[U(\vec{x}_\perp) \equiv U_2(\vec{x}_\perp)]$

$$b_1(\vec{x}_\perp) = \delta^{ij}\Omega_{ij}(\vec{x}_\perp),$$
$$b_2(\vec{x}_\perp) = \epsilon^{ij}\Omega_{ij}(\vec{x}_\perp),$$
$$\Omega_{ij}(\vec{x}_\perp) = g\left[\frac{\partial_i}{\partial_\perp^2}\rho_1^a(\vec{x}_\perp)\right]\partial^j\left(U^\dagger(\vec{x}_\perp)t_aU(\vec{x}_\perp)\right),$$
$$\Lambda(\vec{x}_\perp) = g\frac{\partial^i}{\partial_\perp^2}\left(U^\dagger(\vec{x}_\perp)\left[\frac{\partial_i}{\partial_\perp^2}\rho_1^a(\vec{x}_\perp)\right]U(\vec{x}_\perp)\right)$$

Second order, $\beta^{(2)}$

- At the next-to-leading order:

$$\left[\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_\perp^2\right]\beta^{(2)}(\tau,\vec{x}_\perp) = -ig\left(\partial_i[\beta^{(1)}{}_i(\tau,\vec{x}_\perp),\beta^{(1)}(\tau,\vec{x}_\perp)] + [\beta^{(1)}{}_i(\tau,\vec{x}_\perp),\partial_i\beta^{(1)}(\tau,\vec{x}_\perp)]\right)$$

$$\left[\delta^{ij}\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_\perp^2\right) + \partial_i\partial_j\right]\beta^{(2)}{}_j(\tau,\vec{x}_\perp) = -ig\left(\partial_j[\beta^{(1)}{}_j(\tau,\vec{x}_\perp),\beta^{(1)}{}_i(\tau,\vec{x}_\perp)]\right.$$
$$\left.+[\beta^{(1)}{}_j(\tau,\vec{x}_\perp),\partial_j\beta^{(1)}{}_i(\tau,\vec{x}_\perp) - \partial_i\beta^{(1)}{}_j(\tau,\vec{x}_\perp)] - \tau^2[\beta^{(1)}(\tau,\vec{x}_\perp),\partial_i\beta^{(1)}(\tau,\vec{x}_\perp)]\right)$$

First non-linear corrections!

- This looks very discouraging.
- The goal is to compute $g^6\rho_1^3$ correction to particle production.
  Do we have to solve these equations?

Particle production: Lehmann-Symanzik-Zimmermann I

- For simplicity – Minkowski-space and semi-classical scalar field $\phi(x)$.
  The creation operator

$$a^+(\vec{k},t) = \frac{1}{i}\int d^3x\ \exp(-ik\cdot x)\ \overleftrightarrow{\partial_0}\ \phi(x) \qquad k\cdot x = k_\mu x^\mu$$

- The difference

$$a^+(\vec{k},t\to\infty) - a^+(\vec{k},t\to t_0) = \frac{1}{i}\int_{t_0}^{\infty} dt\ \partial_0\left(\int d^3x\ \exp(-ik\cdot x)\ \overleftrightarrow{\partial_0}\ \phi(x)\right)$$
$$= \frac{1}{i}\int_{t_0}^{\infty} dt\ d^3x\ \exp(-ik\cdot x)\ \underbrace{\left(\Box + m^2\right)\ \phi\ (x)}_{\text{interaction}}$$

  Instead of a usual choice $t_0 \to -\infty$, in order to mimic initial conditions on the light cone, $t_0 \to 0$.

- Thus for the creation operator at the out-state, there are two contributions

$$a^+(\vec{k},\infty) = \underbrace{\frac{1}{i}\int_{t=0} d^3x \exp(-ik\cdot x)\ \overleftrightarrow{\partial_0}\ \phi(x)}_{\text{initial flux through t=0 hypersurface}} + \underbrace{\frac{1}{i}\int_0^{\infty} dt \int d^3x \exp(-ik\cdot x)\ \left(\Box + m^2\right)\ \phi(x)}_{\text{interaction; evolution in the forward light cone}}$$

- Single-inclusive gluon production $E_k \frac{dN}{d^3k} = \frac{1}{2(2\pi)^3} a^+(\vec{k},\infty)\ a(\vec{k},\infty)$

Particle production: Lehmann-Symanzik-Zimmermann II

- Single-inclusive gluon production

$$E_k \frac{dN}{d^3k} = \frac{1}{2(2\pi)^3} \left| \overbrace{\underbrace{\frac{1}{i}\int_{t=0} d^3x \exp(-ik\cdot x)\, \overleftrightarrow{\partial}_0\, \phi(x)}_{\text{initial flux}}}^{\text{"surface"}} + \overbrace{\underbrace{\frac{1}{i}\int_0^\infty dt \int d^3x \exp(-ik\cdot x)\, \left(\Box + m^2\right)\, \phi(x)}_{\text{interaction; evolution in the forward light cone}}}^{\text{"bulk"}} \right|^2$$

- Leading order: no "bulk" contribution
- "Surface": Initial conditions (to any order) are real functions of $\vec{x}_\perp \rightsquigarrow$ symmetric under $\vec{k}_\perp \to -\vec{k}_\perp \rightsquigarrow$ absence of odd azimuthal anisotropy
- Both contributions schematically:

$$E_k \frac{dN}{d^3k} = \left(\underbrace{a^{(1)}(\vec{k}_\perp)}_{\text{surface only}} + a^{(2)}(\vec{k}_\perp) + \ldots \right) \left(a^{(1)}(\vec{k}_\perp) + \underbrace{a^{(2)}(\vec{k}_\perp)}_{\text{surface and bulk}} + \ldots \right)^*$$

$$\approx \underbrace{a^{(1)}(\vec{k}_\perp) a^{(1)}(-\vec{k}_\perp)}_{\text{symmetric}} + \underbrace{a^{(1)}(\vec{k}_\perp) \left(a^{(2)}(\vec{k}_\perp)\right)^* + a^{(1)}(-\vec{k}_\perp) a^{(2)}(\vec{k}_\perp)}_{\text{odd asymmetry is possible}}$$

- Realistic calculations: Milne coordinates (Hankel functions instead of exp), gluon polarization, non-trivial time integrals involving up to three Bessel functions

LSZ: Milne coordinates

- Single-inclusive gluon production

$$E_k \frac{dN}{d^3k} = \frac{1}{16\pi}\left[\left|\underbrace{\mathfrak{S}_\perp(\vec{k}_\perp)}_{\text{surface}} + \underbrace{\mathfrak{B}_\perp(\vec{k}_\perp)}_{\text{bulk}}\right|^2 + \left|\mathfrak{S}_\eta(\vec{k}_\perp) + \mathfrak{B}_\eta(\vec{k}_\perp)\right|^2\right]$$

- The surface contributions:

$$\mathfrak{S}_\perp(\vec{k}_\perp) = \lim_{\tau\to 0+}\left(\tau\, \underbrace{H_0^{(1)}(k_\perp\tau)}_{H_n^{(1,2)}=J_n\pm iY_n}\, \overleftrightarrow{\partial}_0\, \beta_\perp(\tau,\vec{k}_\perp)\right)$$

$$\beta_\perp(\tau,\vec{k}_\perp) = i\frac{\epsilon^{ij}k_j}{k_\perp}\beta_i(\tau,\vec{k}_\perp)$$

$$\mathfrak{S}_\eta(\vec{k}_\perp) = \lim_{\tau\to 0+}\left(\tau^3\left\{\frac{H_1^{(1)}(k_\perp\tau)}{\tau}\, \overleftrightarrow{\partial}_0\, \beta(\tau,\vec{k}_\perp)\right\}\right)$$

- The bulk contributions:

$$\mathfrak{B}_\perp(\vec{k}_\perp) = \int_0^\infty d\tau\tau H_0^{(1)}(k_\perp\tau)\left\{\frac{1}{\tau}\partial_\tau\tau\partial_\tau\beta_\perp(\tau,\vec{k}_\perp) - \partial_\perp^2\,\beta_\perp(\tau,\vec{k}_\perp)\right\}$$

$$\mathfrak{B}_\eta(\vec{k}_\perp) = \int_0^\infty d\tau\tau^2 H_1^{(1)}(k_\perp\tau)\left\{\frac{1}{\tau^3}\partial_\tau\tau^3\partial_\tau\beta(\tau,\vec{k}_\perp) - \partial_\perp^2\,\beta(\tau,\vec{k}_\perp)\right\}$$

Single-inclusive production: leading order I

$$E_k \frac{dN^{\rm LO}}{d^3k} = \frac{1}{16\pi}\left[\left|\mathfrak{S}_\perp(\vec{k}_\perp)\right|^2 + \left|\mathfrak{S}_\eta(\vec{k}_\perp)\right|^2\right]$$

with

$$\mathfrak{S}_\perp^{(1)}(\vec{k}_\perp) = -\lim_{\tau\to 0+}\left(\tau\partial_\tau H_0^{(1)}(k_\perp\tau)\beta_\perp^{(1)}(\tau,\vec{k}_\perp)\right) = -\frac{2}{\pi}i\,\beta_\perp^{(1)}(\tau=0,\vec{k}_\perp) = \frac{2i}{\pi k_\perp}\underbrace{b_2(\vec{k}_\perp)}_{\text{init. cond.}}$$

$$\mathfrak{S}_\eta^{(1)}(\tau,\vec{k}_\perp) = -\frac{4}{\pi}i\,\frac{\beta(\tau=0,\vec{k}_\perp)}{k_\perp} = -\frac{2i}{\pi k_\perp}\underbrace{b_1(\vec{k}_\perp)}_{\text{init. cond.}}$$

- Substituting the initial condition, we get

  reproducing the result of J.-P. Blaizot, F. Gelis, R. Venugopalan, arXiv:0402256

$$\begin{aligned} E_k\frac{dN^{\rm LO}}{d^3k} &= \frac{1}{8\pi^3k_\perp^2}(\delta_{ij}\delta_{lm}+\epsilon_{ij}\epsilon_{lm})\,\Omega_{ij}^b(\vec{k}_\perp)\,\left[\Omega_{lm}^b(\vec{k}_\perp)\right]^* \\ &= \frac{g^2}{8\pi^3k_\perp^2}(\delta_{ij}\delta_{lm}+\epsilon_{ij}\epsilon_{lm})\int\frac{d^2p_\perp}{(2\pi)^2}\frac{d^2q_\perp}{(2\pi)^2}\frac{p_i(k-p)_j}{p_\perp^2}\frac{q_l(k-q)_m}{q_\perp^2} \\ &\qquad\times\rho_a^*(\vec{q}_\perp)\left[W^\dagger(\vec{k}_\perp-\vec{q}_\perp)\underbrace{W(\vec{k}_\perp-\vec{p}_\perp)}_{\text{adjoint Wilson line}}\right]_{ab}\rho_b(\vec{p}_\perp) \end{aligned}$$

- Manifestly symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$

Next-to-leading order I

$$E_k \frac{dN^{\rm NLO}}{d^3k} = \frac{1}{16\pi} \sum_{\gamma=\perp,\eta} \left| \underbrace{\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp) + \mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp)}_{\text{init. cond.}} + \underbrace{\mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp)}_{\text{evolution}} \right|^2$$

$$= \frac{1}{16\pi} \sum_{\gamma=\perp,\eta} \left[\left|\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right|^2 + \underbrace{\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\left(\mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp) + \mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp)\right)^*}_{!} + \underbrace{\left(\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right)^*\left(\mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp) + \mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp)\right)}_{!} \right]$$

- The surface contributions are defined by the initial condition on the light cone and can be readily found. At any order n, $\mathfrak{S}_\gamma^{(n)}(\vec{k}_\perp) = -(\mathfrak{S}_\gamma^{(n)}(-\vec{k}_\perp))^*$
- Anti-symmetric combination:

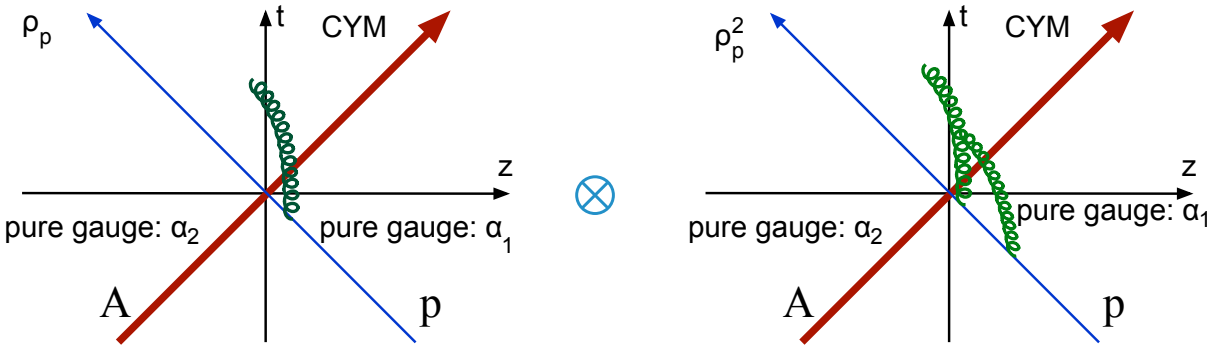
$$\frac{E_k}{2}\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right) = \frac{1}{16\pi}\mathrm{Re}\sum_{\gamma=\perp,\eta}\left(\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right)^* \underbrace{\left[\mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp) + \left(\mathfrak{B}_\gamma^{(2)}(-\vec{k}_\perp)\right)^*\right]}_{!}$$

- This combination is defined by the NLO solution.

Next-to-leading order II

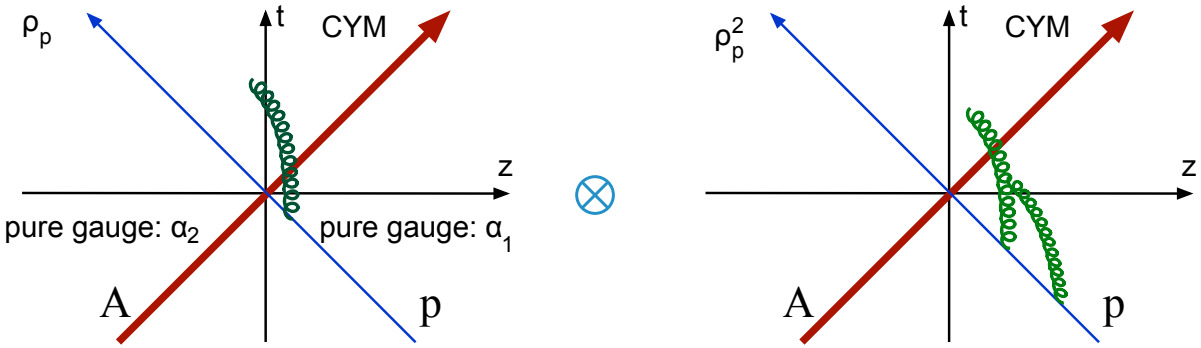
Graphically

$\left(\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right)^*$ $\mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp)$



This diagram does not contribute:

$\left(\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right)^*$ $\mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp)$



Next-to-leading order III

- A reminder of the definition the bulk contributions (η-component only):

$$\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) = \int_0^\infty d\tau\tau^2 H_1^{(1)}(k_\perp\tau)\underbrace{\left\{\frac{1}{\tau^3}\partial_\tau\tau^3\partial_\tau\beta^{(2)}(\tau,\vec{k}_\perp) - \partial_\perp^2\,\beta^{(2)}(\tau,\vec{k}_\perp)\right\}}_{\square}$$

- There is no need to solve for $\beta^{(2)}$!

$$\underbrace{\left[\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_\perp^2\right]}_{\square}\beta^{(2)}(\tau,\vec{x}_\perp) = -ig\left(\partial_i[\beta^{(1)}{}_i(\tau,\vec{x}_\perp),\beta^{(1)}(\tau,\vec{x}_\perp)] + [\beta^{(1)}{}_i(\tau,\vec{x}_\perp),\partial_i\beta^{(1)}(\tau,\vec{x}_\perp)]\right)$$

- $\rightsquigarrow$

$$\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) = \int_0^\infty d\tau\tau^2 H_1^{(1)}(k_\perp\tau)\left(g\int\frac{d^2q}{(2\pi)^2}[(2\vec{k}_\perp - \vec{q}_\perp)\cdot\underbrace{\vec{\beta}_\perp^{(1)}(\tau,\vec{q}_\perp)}_{\propto J_0(\tau q_\perp)},\underbrace{\beta^{(1)}(\tau,\vec{k}_\perp - \vec{q}_\perp)}_{\propto J_1(\tau|\vec{k}_\perp-\vec{q}_\perp|)/\tau}]\right)$$

τ-integrals involving three Bessel functions and one power of τ

Some of the integrals

$$\int d\tau\, \tau\, H_0^{(1)}(k_\perp\tau) J_0(|\vec{q}_\perp - \vec{k}_\perp|\tau) J_0(q_\perp\tau) = \frac{1}{\pi}\frac{1}{|\vec{k}_\perp \times \vec{q}_\perp|}$$

$$\int d\tau\, \tau\, J_1(q_\perp\tau) J_0(|\vec{q}_\perp - \vec{k}_\perp|\tau) Y_1(k_\perp\tau) = -\frac{1}{\pi}\frac{1}{q_\perp k_\perp}$$

$$\int d\tau\, \tau\, J_1(q_\perp\tau) J_1(|\vec{q}_\perp - \vec{k}_\perp|\tau) Y_0(k_\perp\tau) = \frac{1}{\pi}\frac{1}{q_\perp |\vec{k}_\perp - \vec{q}_\perp|}$$

$$\int d\tau\, \tau\, J_1(q_\perp\tau) J_1(k_\perp\tau) J_0(|\vec{k}_\perp - \vec{q}_\perp|\tau) = \frac{1}{\pi q_\perp k_\perp}\frac{\vec{q}_\perp \cdot \vec{k}_\perp}{|\vec{q}_\perp \times \vec{k}_\perp|}$$

$$\int d\tau\, \tau\, J_1(q_\perp\tau) J_1(|\vec{k}_\perp - \vec{q}_\perp|\tau) J_0(k_\perp\tau) = \frac{1}{\pi q_\perp |\vec{k}_\perp - \vec{q}_\perp|}\frac{\vec{q}_\perp \cdot (\vec{q}_\perp - \vec{k}_\perp)}{|\vec{q}_\perp \times \vec{k}_\perp|}$$

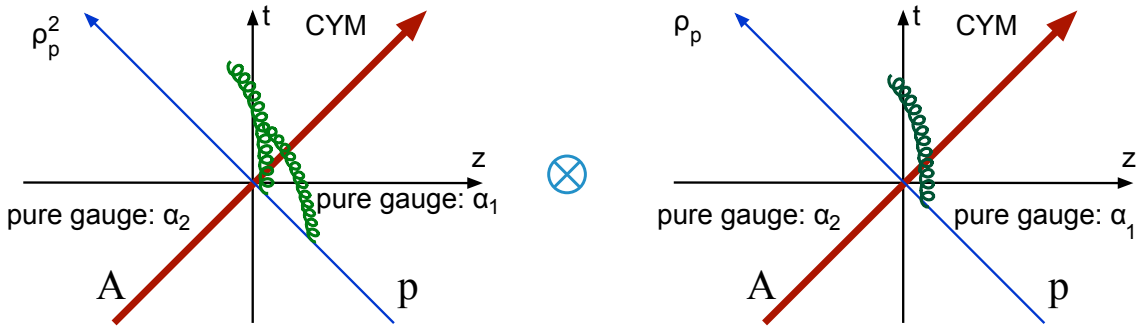
Odd azimuthal anisotropy at NLO

- Performing τ-integration in

$$\frac{E_k}{2}\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp)-\frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right)=\frac{1}{16\pi}{\rm Re}\sum_{\gamma=\perp,\eta}\left(\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp)\right)^*\left[\mathfrak{B}^{(2)}_\gamma(\vec{k}_\perp)+\left(\mathfrak{B}^{(2)}_\gamma(-\vec{k}_\perp)\right)^*\right]$$

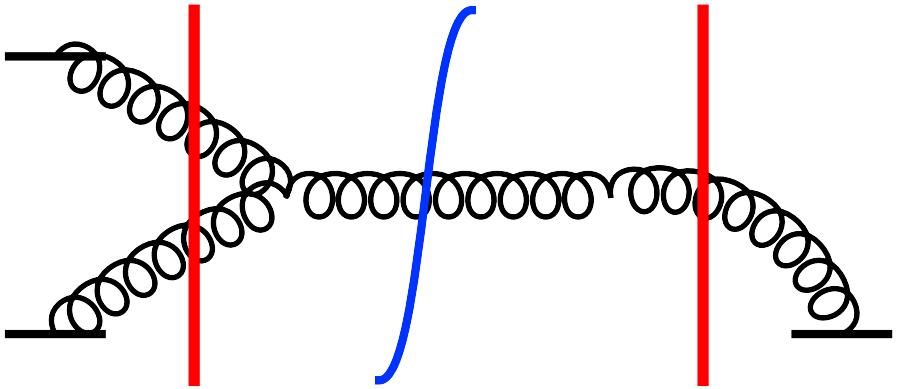
$$={\rm Im}\left\{\frac{2g}{\pi^2k_\perp^2}\int\frac{d^2q}{(2\pi)^2}\frac{\vec{k}_\perp\times\vec{q}_\perp}{|\vec{k}_\perp\times\vec{q}_\perp|}\frac{1}{q_\perp^2|\vec{k}_\perp-\vec{q}_\perp|^2}\,f^{abc}\,\Omega^a_{ij}(\vec{q}_\perp)\,\Omega^b_{mn}(\vec{k}_\perp-\vec{q}_\perp)\,\Omega^{c\star}_{rp}(\vec{k}_\perp)\times\right.$$
$$\left.\left[\left(k_\perp^2\epsilon^{ij}\epsilon^{mn}-\vec{q}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)(\epsilon^{ij}\epsilon^{mn}+\delta^{ij}\delta^{mn})\right)\epsilon^{rp}+2\vec{k}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)\delta^{ij}\epsilon^{mn}\delta^{rp}\right]\right\}$$

Ω represent sources rotated by the target: $\Omega^b_{ij}(\vec{k}_\perp)=g\int\frac{d^2p_\perp}{(2\pi)^2}\frac{p_i(k-p)_j}{p_\perp^2}\rho_a(\vec{p}_\perp)\underbrace{W_{ba}(\vec{k}_\perp-\vec{p}_\perp)}_{\text{ad. Wilson line}}$

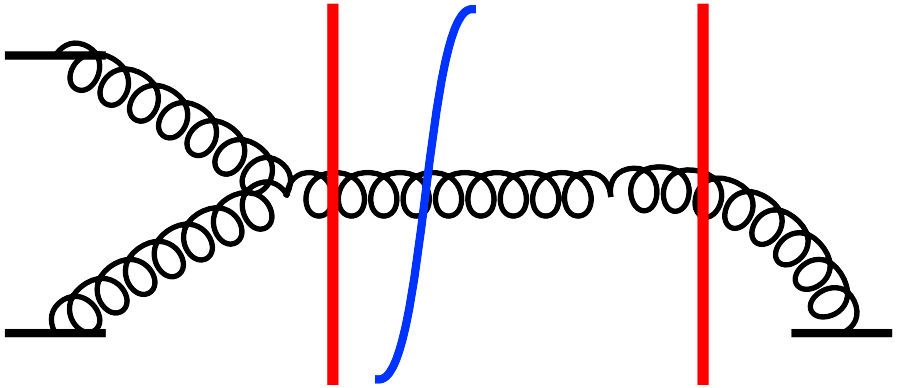


Odd azimuthal anisotropy at NLO: "LCPT"

- An example of a digram that contributes in "LCPT":



- Vanishing odd azimuthal anisotropy (an example):

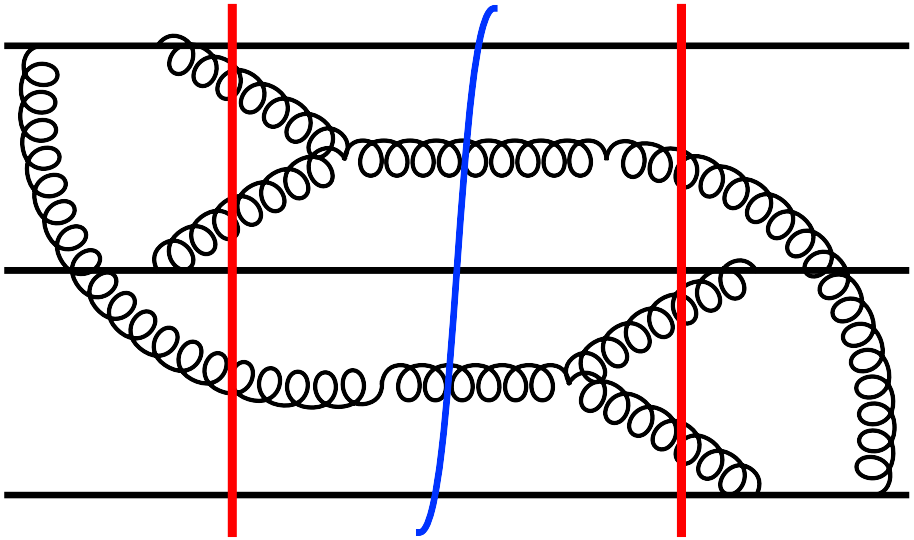


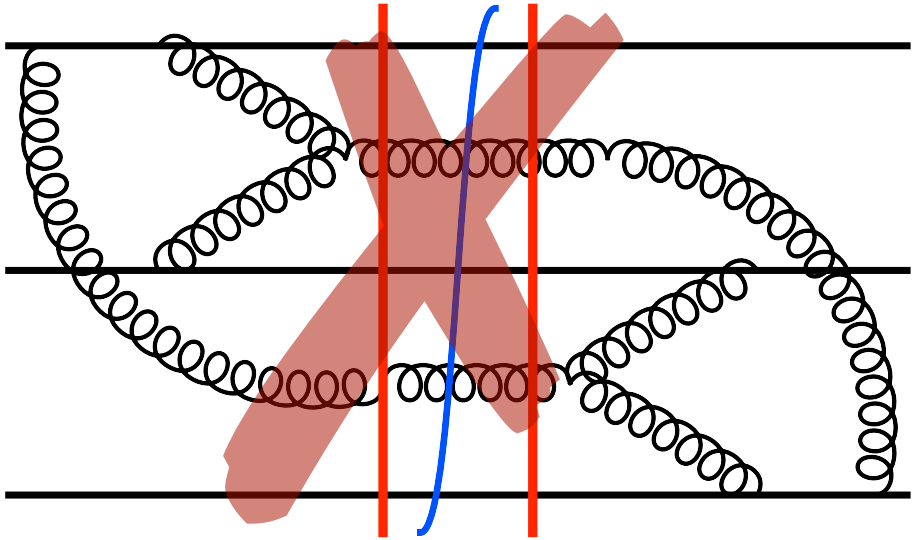
Odd azimuthal asymmetry in double inclusive production

- The odd contribution to single inclusive production is $\propto \rho^3$
- In a Gaussian MV, $\rightsquigarrow \left\langle \frac{E_k}{2}\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right)\right\rangle_\rho = 0$
- However, the asymmetry survives in double inclusive production!

$$\left\langle \frac{E_k E_p}{4} \overbrace{\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right)\frac{dN^{\rm NLO}}{d^3p}(\vec{p}_\perp)}^{\propto \rho^6} \right\rangle_\rho \neq 0$$

- One of "LCPT" diagrams





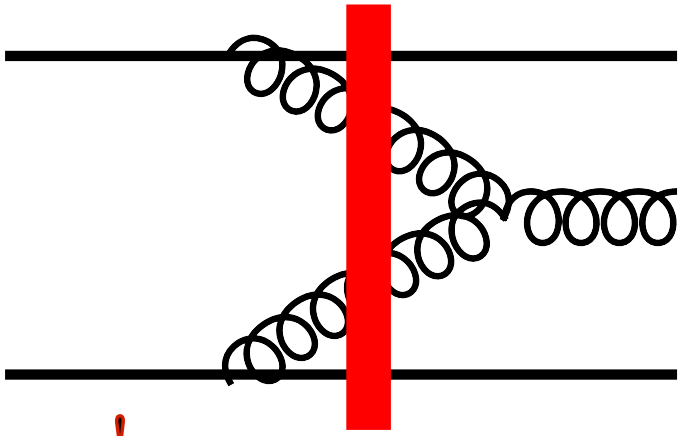
Final result for the antisymmetrized part

$$
\begin{aligned}
\langle\cdots\rangle_{\rho_1} = &\left(\frac{2g}{\pi^2}\right)^2 \frac{1}{p_\perp^2 k_\perp^2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2q'}{(2\pi)^2} \frac{\vec{p}_\perp \times \vec{q}_\perp}{|\vec{p}_\perp \times \vec{q}_\perp|} \frac{\vec{k}_\perp \times \vec{q}'_\perp}{|\vec{k}_\perp \times \vec{q}'_\perp|} \frac{f^{abc}}{q_\perp^2 |\vec{p}_\perp - \vec{q}_\perp|^2} \frac{f^{a'b'c'}}{{q'_\perp}^2 |\vec{k}_\perp - \vec{q}'_\perp|^2} \Big[\Big(p_\perp^2 \epsilon^{ij}\epsilon^{mn} - \vec{q}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)(\epsilon^{ij}\epsilon^{mn} + \delta^{ij}\delta^{mn})\Big)\epsilon^{rp} \\
&+2\vec{p}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)\delta^{ij}\epsilon^{mn}\delta^{rp}\Big]\Big[\Big(k_\perp^2 \epsilon^{i'j'}\epsilon^{m'n'} - \vec{q}'_\perp \cdot (\vec{k}_\perp - \vec{q}'_\perp)(\epsilon^{i'j'}\epsilon^{m'n'} + \delta^{i'j'}\delta^{m'n'})\Big)\epsilon^{r'p'} + 2\vec{k}_\perp \cdot (\vec{k}_\perp - \vec{q}'_\perp)\delta^{i'j'}\epsilon^{m'n'}\delta^{r'p'}\Big] \times \\
&\Big[\Omega^{a,b}_{ij,mn}(\vec{q}_\perp, \vec{k}_\perp - \vec{q}_\perp)\Big(\Omega^{c,a'}_{rp,i'j'}(-\vec{p}_\perp, \vec{q}'_\perp)\Omega^{b',c'}_{m'n',r'p'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{c,b'}_{rp,m'n'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{a',c'}_{i'j',r'p'}(\vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{c,c'}_{rp,r'p'}(-\vec{p}_\perp, -\vec{k}_\perp)\Omega^{a',b'}_{i'j',m'n'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big) \\
&\Omega^{a,c}_{ij,rp}(\vec{q}_\perp, -\vec{p}_\perp)\Big(\Omega^{b,a'}_{mn,i'j'}(\vec{p}_\perp - \vec{p}_\perp, \vec{q}'_\perp)\Omega^{b',c'}_{m'n',r'p'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{b,b'}_{mn,m'n'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{a',c'}_{i'j',r'p'}(\vec{q}'_\perp, -\vec{k}_\perp) + \\
&\Omega^{b,c'}_{mn,r'p'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp)\Omega^{a',b'}_{i'j',m'n'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big) + \Omega^{a,a'}_{ij,i'j'}(\vec{q}_\perp, \vec{q}'_\perp)\Big(\Omega^{b,c}_{mn,rp}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp)\Omega^{b',c'}_{m'n',r'p'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \\
&\Omega^{b,b'}_{mn,m'n'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{c,c'}_{rp,r'p'}(-\vec{p}_\perp, -\vec{k}_\perp) + \Omega^{b,c'}_{mn,r'p'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp)\Omega^{c,b'}_{rp,m'n'}(-\vec{p}'_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big) + \\
&\Omega^{a,b'}_{ij,m'n'}(\vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big(\Omega^{b,c}_{mn,rp}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp)\Omega^{a',c'}_{i'j',r'p'}(\vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{b,a'}_{mn,i'j'}(\vec{p}_\perp - \vec{q}_\perp, \vec{q}'_\perp)\Omega^{c,c'}_{rp,r'p'}(-\vec{p}_\perp, -\vec{k}_\perp) + \\
&\Omega^{b,c'}_{mn,r'p'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp)\Omega^{c,a'}_{rp,i'j'}(-\vec{p}'_\perp, \vec{q}'_\perp)\Big) + \Omega^{a,c'}_{ij,r'p'}(\vec{q}_\perp, -\vec{k}_\perp)\Big(\Omega^{b,c}_{mn,rp}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp)\Omega^{a',b'}_{i'j',m'n'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp) + \\
&\Omega^{b,a'}_{mn,i'j'}(\vec{p}_\perp - \vec{q}_\perp, \vec{q}'_\perp)\Omega^{c,b'}_{rp,m'n'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp) + \Omega^{b,b'}_{mn,m'n'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{c,a'}_{rp,i'j'}(-\vec{p}'_\perp, \vec{q}'_\perp)\Big)\Big].
\end{aligned}
$$

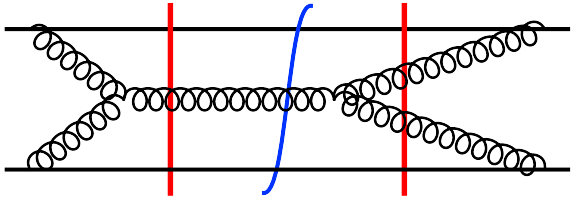
with

$$
\Omega^{a,b}_{ij,lm}(\vec{p}_\perp, \vec{q}_\perp) \equiv \langle \Omega^a_{ij}(\vec{p}_\perp)\Omega^b_{lm}(\vec{q}_\perp)\rangle_{\rho_1} = g^2 \int \frac{d^2u}{(2\pi)^2} \mu^2(\vec{u}_\perp) \frac{u_i(u+p)_j u_l(u-q)_m}{u_\perp^4} \Big[W(\vec{u}_\perp + \vec{p}_\perp)W^\dagger(\vec{u}_\perp - \vec{q}_\perp)\Big]_{ab},
$$

Cross check?!



G. Chirilli, Yu. Kovchegov, D. Wertepny, 1501.03106



!

$$
\begin{aligned}
= & -2\,i\,g^3 \int d^2x_1\, d^2x_2 \int \frac{d^2k}{(2\pi)^2}\,\frac{d^2l}{(2\pi)^2}\,\frac{d^2q_1}{(2\pi)^2}\,\frac{d^2q_2}{(2\pi)^2}\, e^{i\vec{q}_{1\perp}\cdot(\vec{x}_{1\perp}-\vec{b}_{1\perp})+i\vec{q}_{2\perp}\cdot(\vec{x}_{2\perp}-\vec{b}_{2\perp})+i\vec{l}_\perp\cdot(\vec{x}_{2\perp}-\vec{x}_{1\perp})+i\vec{k}_\perp\cdot(\vec{z}_\perp-\vec{x}_{2\perp})} \\
& \times \frac{1}{q_{1\perp}^2\, q_{2\perp}^2}\left(-\vec{q}_{1\perp}\cdot\vec{q}_{2\perp}\,\frac{\vec{\epsilon}_\perp^{\,\lambda *}\times\vec{k}_\perp}{k_\perp^2} + \vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{q}_{1\perp}\,\frac{\vec{q}_{2\perp}\times(\vec{k}_\perp-\vec{l}_\perp)}{(\vec{k}_\perp-\vec{l}_\perp)^2} + \vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{q}_{2\perp}\,\frac{\vec{q}_{1\perp}\times\vec{l}_\perp}{\vec{l}_\perp^{\,2}}\right)\mathrm{Sign}(\vec{k}_\perp\times\vec{l}_\perp) \\
& \times f^{abc}\left[U^{bd}_{\vec{x}_{1\perp}} - U^{bd}_{\vec{b}_{1\perp}}\right]\left[U^{ce}_{\vec{x}_{2\perp}} - U^{ce}_{\vec{b}_{2\perp}}\right]\left(V_{\vec{b}_{1\perp}} t^d\right)_1\left(V_{\vec{b}_{2\perp}} t^e\right)_2 \\
& -2\,g^3 \int d^2x \int \frac{d^2k}{(2\pi)^2}\,\frac{d^2l}{(2\pi)^2}\,\frac{d^2q}{(2\pi)^2}\, e^{i\vec{q}_\perp\cdot(\vec{x}_\perp-\vec{b}_{2\perp})+i\vec{l}_\perp\cdot(\vec{x}_\perp-\vec{b}_{1\perp})+i\vec{k}_\perp\cdot(\vec{z}_\perp-\vec{x}_\perp)}\, f^{abc}\left(V_{\vec{b}_{1\perp}} t^d\right)_1\left(V_{\vec{b}_{2\perp}} t^e\right)_2 \\
& \times \Bigg[U^{bd}_{\vec{b}_{1\perp}}\left(U^{ce}_{\vec{x}_\perp} - U^{ce}_{\vec{b}_{2\perp}}\right)\left(\frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\,\frac{\vec{l}_\perp\cdot\vec{q}_\perp}{l_\perp^2\, q_\perp^2} - \frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{l}_\perp}{l_\perp^2}\,\frac{(\vec{k}_\perp-\vec{l}_\perp)\cdot\vec{q}_\perp}{(\vec{k}_\perp-\vec{l}_\perp)^2\, q_\perp^2} - \frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{q}_\perp}{2\, l_\perp^2\, q_\perp^2} + \frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\,\frac{(\vec{k}_\perp-\vec{l}_\perp)\cdot\vec{q}_\perp}{(\vec{k}_\perp-\vec{l}_\perp)^2\, q_\perp^2}\right) \\
& - \left(U^{bd}_{\vec{x}_\perp} - U^{bd}_{\vec{b}_{1\perp}}\right) U^{ce}_{\vec{b}_{2\perp}}\left(\frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\,\frac{\vec{l}_\perp\cdot\vec{q}_\perp}{l_\perp^2\, q_\perp^2} - \frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{q}_\perp}{q_\perp^2}\,\frac{(\vec{k}_\perp-\vec{q}_\perp)\cdot\vec{l}_\perp}{(\vec{k}_\perp-\vec{q}_\perp)^2\, l_\perp^2} - \frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{l}_\perp}{2\, l_\perp^2\, q_\perp^2} + \frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\,\frac{(\vec{k}_\perp-\vec{q}_\perp)\cdot\vec{l}_\perp}{(\vec{k}_\perp-\vec{q}_\perp)^2\, l_\perp^2}\right)\Bigg] \\
& + 2g^3 \int d^2x \int \frac{d^2k}{(2\pi)^2}\,\frac{d^2q_1}{(2\pi)^2}\,\frac{d^2q_2}{(2\pi)^2}\, e^{i\vec{q}_{1\perp}\cdot(\vec{x}_\perp-\vec{b}_{1\perp})+i\vec{q}_{2\perp}\cdot(\vec{x}_\perp-\vec{b}_{2\perp})+i\vec{k}_\perp\cdot(\vec{z}_\perp-\vec{x}_\perp)}\,\frac{\vec{q}_{1\perp}\cdot\vec{q}_{2\perp}}{q_{1\perp}^2\, q_{2\perp}^2}\,\frac{\vec{\epsilon}_\perp^{\,\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\,\mathrm{Sign}(b_2^- - b_1^-) \\
& \times \left[U^{ab}_{x_\perp} - U^{ab}_{z_\perp}\right] f^{bde}\left(V_{\vec{b}_{1\perp}} t^d\right)_1\left(V_{\vec{b}_{2\perp}} t^e\right)_2
\end{aligned}
$$

IMAGINARY
REAL

G. Chirilli, Yu. Kovchegov, D. Wertepny, 1501.03106
Yu. Kovchegov, V.S., relation to v_3 – work in progress

Double inclusive production

- The obtained result can only be used to extract the antisymmetrized part, or the leading contribution to $v_3\{2\}$.
- **It is not full NLO for double inclusive production!**
- $\sigma(\vec{k}_\perp)$ as a shorthand notation for $E_k\, dN/dk^3$:

$$\sigma(\vec{k}_\perp) = \underbrace{\overset{\text{LO}}{\sigma_2(\vec{k}_\perp)}}_{g^2\rho^2} + \underbrace{\overset{\text{NLO}}{\sigma_3(\vec{k}_\perp)}}_{g^4\rho^3} + \underbrace{\overset{\text{FSC}}{\sigma_4(\vec{k}_\perp)}}_{g^6\rho^4} + \cdots$$

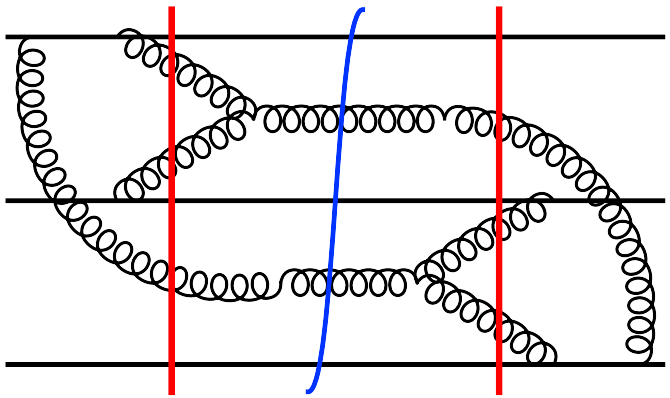
Double inclusive:

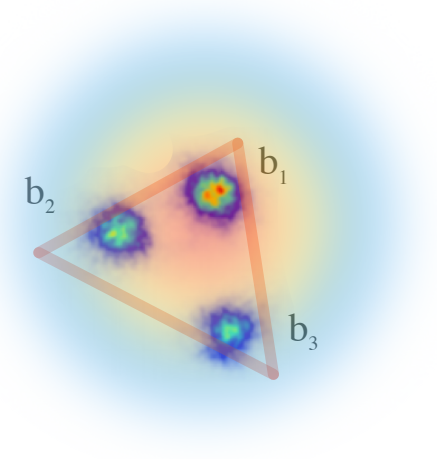
$$\langle\sigma(\vec{k}_\perp)\sigma(\vec{p}_\perp)\rangle = \underbrace{\langle\sigma_2(\vec{k}_\perp)\sigma_2(\vec{p}_\perp)\rangle}_{\text{LO}} + \underbrace{\overbrace{\langle\sigma_3(\vec{k}_\perp)\sigma_3(\vec{p}_\perp)\rangle}^{\text{computed here}} + \overbrace{\langle\sigma_4(\vec{k}_\perp)\sigma_2(\vec{p}_\perp) + \sigma_4(\vec{p}_\perp)\sigma_2(\vec{k}_\perp)\rangle}^{\text{even anisotropy only}}}_{\text{complete NLO}} + \cdots$$

- Complete NLO requires computing the first saturation correction, $\sigma_4(\vec{k}_\perp)$.
- Nevertheless, this represents a first explicit analytic result for the odd azimuthal anisotropy.

Properties of the odd azimuthal anisotropy

- The surface contribution on the light cone gives zero odd azimuthal anisotropy to all orders. It is T-even and can be written in a local form.
- The odd harmonics originate from the time evolution in the forward light cone. They are non-local and not T-even.
- The argument that the double inclusive cross section is even under the inversion of one of the gluon momenta is valid only for the surface contribution (at any order in the classical approximation). The bulk contribution to single inclusive production is not symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$ configuration-by-configuration.
- The result can be used to calculate the leading order $v_3\{2\}$ without solving Classical Yang-Mills numerically.
- Is it related to inital state geometric asymmetry ε_3?





"Why did not you use LCPT?"

The presented results could have been obtained using LCPT, but

- the amount of diagrams is enormous; ensuing expressions are bulky and cumbersome
- LCPT does not provide any intuition about the properties of different contributions, e.g. that the surface contribution gives no odd azimuthal anisotropy
- the key point was to compute single inclusive production on configuration-by-configuration basis with explicit ρ.

Outlook: Numerics

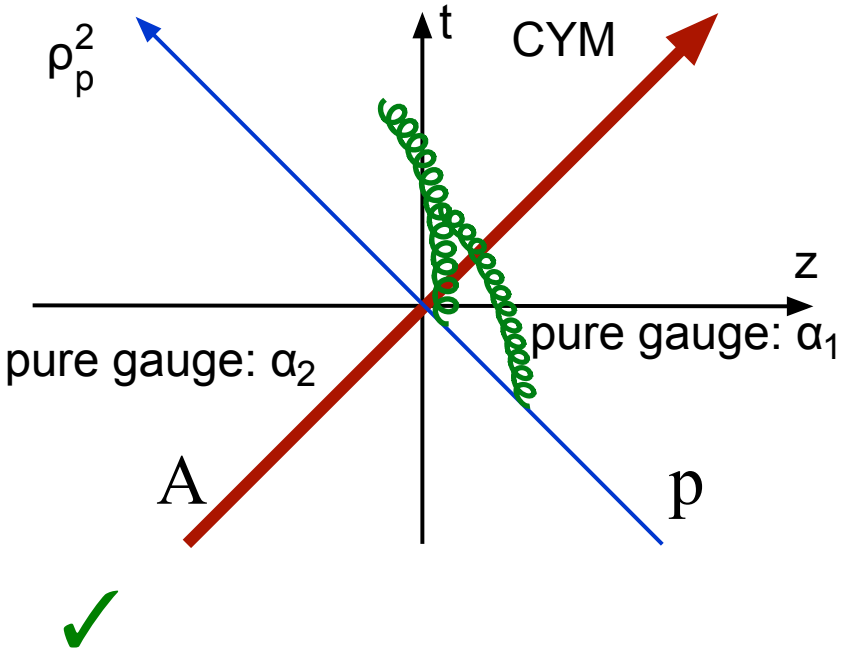
- Generate (target and projectile) sources according to MV
- Compute Wilson lines for the target
- Rotate the projectile sources by the target Wilson lines $\rightsquigarrow \Omega$
- Compute single inclusive asymmetry on conf.-by-conf. basis

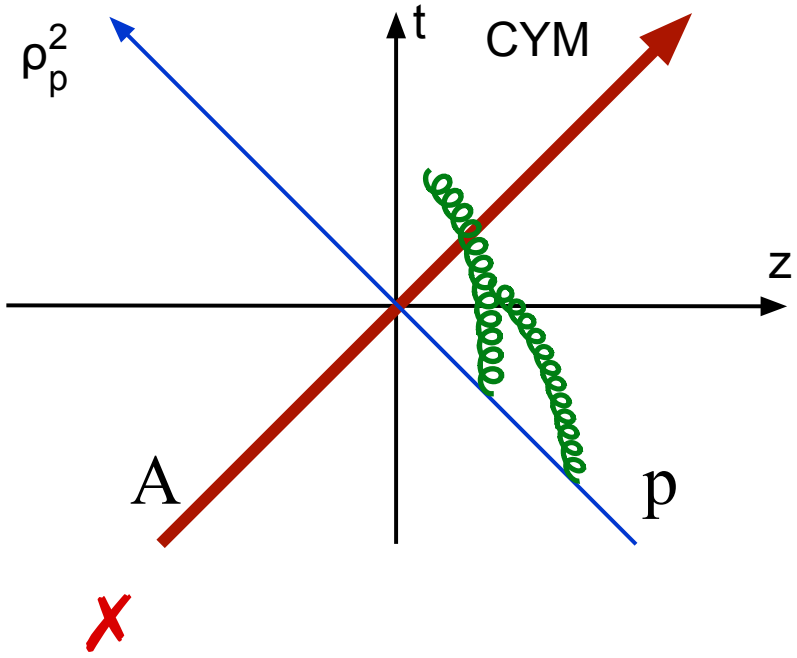
$$\frac{E_k}{2}\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right) = {\rm Im}\left\{\frac{2g}{\pi^2 k_\perp^2}\int\frac{d^2q}{(2\pi)^2}\frac{\vec{k}_\perp\times\vec{q}_\perp}{|\vec{k}_\perp\times\vec{q}_\perp|}\frac{1}{q_\perp^2|\vec{k}_\perp-\vec{q}_\perp|^2}\, f^{abc}\,\Omega^a_{ij}(\vec{q}_\perp)\,\Omega^b_{mn}(\vec{k}_\perp-\vec{q}_\perp)\,\Omega^{c\star}_{rp}(\vec{k}_\perp)\times \left[\left(k_\perp^2\epsilon^{ij}\epsilon^{mn} - \vec{q}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)(\epsilon^{ij}\epsilon^{mn}+\delta^{ij}\delta^{mn})\right)\epsilon^{rp} + 2\vec{k}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)\delta^{ij}\epsilon^{mn}\delta^{rp}\right]\right\}$$

- Compute double inclusive asymmetry on conf.-by-conf. basis
- Average w.r.t. $W[\rho_1]$ and $W[\rho_2]$
- P.S. Yes, $v_3\{2\}$ is positive

V. S. work in progress

Outlook: Beyond classical limit





Some degree of final state interaction is needed?!

- Is there odd azimuthal anisotropy from the genuine initial wave-function beyond classical limit?
- Using the projectile wave function describing quantum high energy evolution (JIMWLK), we were able to show that there is!

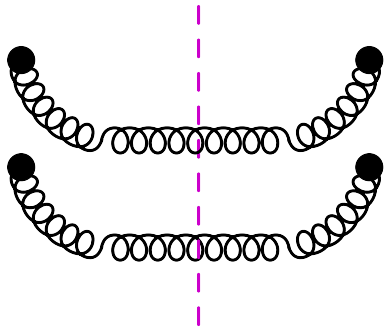
A. Kovner, M. Lublinsky and V. S., 1612.07790

Outlook: CME background from initial state physics

- Projectile wave function has intrinsic parton correlation built in;
  EIC: dedicated machine to probe these correlations

- Scattering frees partons; some correlations are preserved

T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and M. Lublinsky, 1503.07126,
A. Kovner, M. Lublinsky, V. S., 1612.07790,
A. Kovner, M. Lublinsky, V. S., 1706.02330



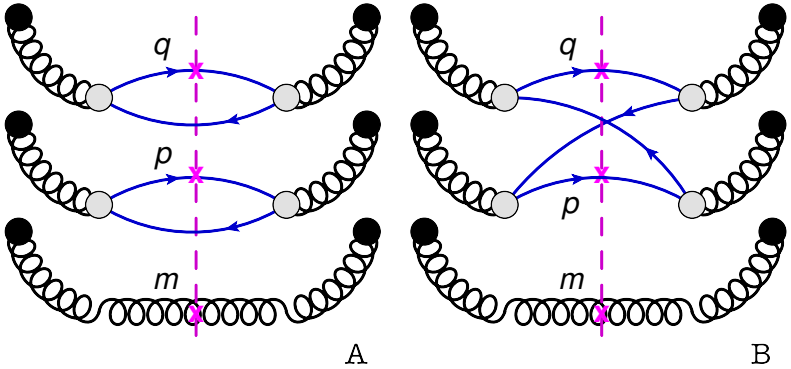
Origin of Glasma-like correlations in projectile WF

- Origin of Glasma graph:
  "Bose enhancement of gluons in the projectile leads to azimuthal collimation of ..."

T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and M. Lublinsky, 1503.07126

- Same-charge correlations, two contributions of the same order $g^4(g^6\rho^4)$
- Rapidity independent pedestal, $\mathtt{A}$, $\rightsquigarrow \gamma_S < 0$
- Pauly blocking term, $\mathtt{B}$, $\rightsquigarrow \gamma_S(\Delta\eta \to 0) > 0$

  $\rightsquigarrow \gamma_S(\Delta\eta \to \infty) < 0$
- Oposite charge correlations: leading order is rapidity independent and positive



A. Kovner, M. Lublinsky, V. S., 1706.02330; appeared two weeks ago

Summary and Conclusions

- $v_3\{2\} \neq 0$ in CGC calculations!
- Explicit expression for the odd azimuthal anisotropy;
  interpretations in terms of LCPT.
- First correction is sufficient to have some $v_3\{2\}$.
  It is a quantitative question whether it is comparable to dense-dense limit.