Evolution of charge fluctuations and correlations in hydrodynamics

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University of Minnesota

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June 12, 2018

Quantifying the strength of Diffusion:

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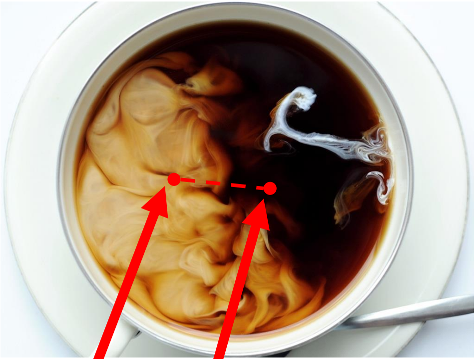

Cups of coffee:

$$\langle \delta n_{\mathrm{milk}}(x)\, \delta n_{\mathrm{coffee}}(x') \rangle_{\text{lots of cups}}$$

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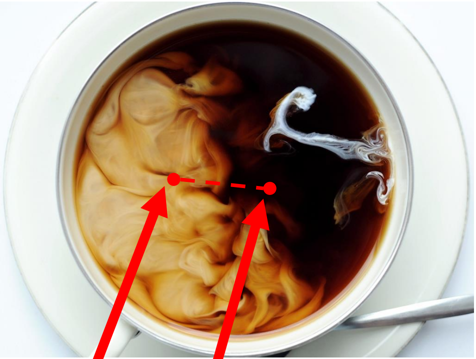

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Heavy-ion collisions: $\langle \delta n_{Q_a}(x)\, \delta n_{Q_b}(x') \rangle_{\text{lots of collisions}}$

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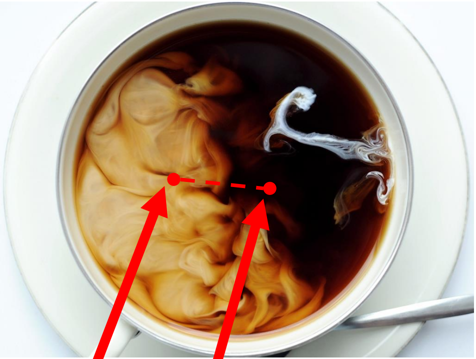

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Heavy-ion collisions: $\langle \delta n_{Q_a}(x) \delta n_{Q_b}(x') \rangle_{\text{lots of collisions}}$

$$\boxed{B_{h^+h^-}(y_1 - y_2) \equiv \left\langle \frac{dN}{dy} \right\rangle_{\mathrm{ev}}^{-1} \left\langle \delta\left(\frac{dN}{dy_1}\right) \delta\left(\frac{dN}{dy_2}\right) \right\rangle_{\mathrm{ev}}}$$

How do we model it?

Today: look at two different approaches

1. Diffusion with fluctuating hydrodynamics ("noise")[1]
2. Diffusion with "Monte-Carlo hydrodynamics"[2]

[1] J. I. Kapusta and CP, Phys. Rev. C **97**, 014906 (2018)
[2] S. Pratt, J. Kim and CP, arXiv:1712.09298 [nucl-th] (accepted by PRC)

Ordinary vs. Fluctuating Hydrodynamics

- **Ordinary (ideal) hydrodynamics**:
  - Quantity of interest: n_Q (Q: electric charge)
  - Current density: $J_Q(x) \equiv n_Q(x)u(x)$
- **Fluctuating hydrodynamics**:
  - Stochastic source term $I(x)$ generates $\delta n_Q(x)$
  - *Fluctuating* current density: $J_Q(x) \equiv n_Q(x)u(x) + \Delta J_Q(x) + I(x)$
- **Modeling diffusion:**
  - Write down/solve hydrodynamic equations of motion: $\partial_\mu J^\mu_Q = 0$
  - Fix $\langle I(x)I(x')\rangle$ to compute $\langle \delta n_Q(x)\delta_Q(x')\rangle$
  - Relate to observable quantities (e.g., $B_{h^+h^-}(y_1 - y_2)$)

Problem: what should we choose for $\langle I(x)I(x')\rangle$?

- White noise: $\left\langle I(x_1) I(x_2) \right\rangle \propto \delta^4(x_1 - x_2)$
    - Corresponds to *ordinary* diffusion: $\left[\partial_t + D_Q \partial_x^2\right] n_Q = 0$
    - Pro: simple to implement
    - Con: acausal signal propagation ($v_Q^2 \to \infty$)

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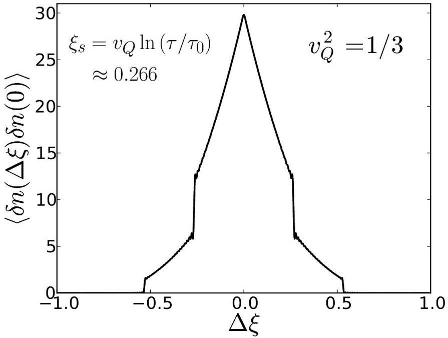
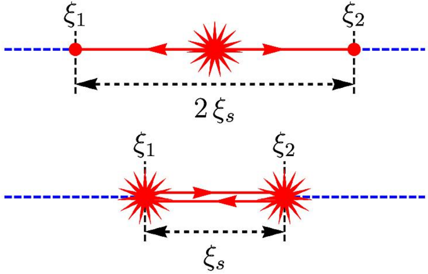
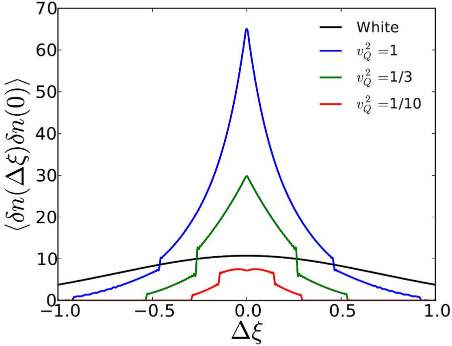
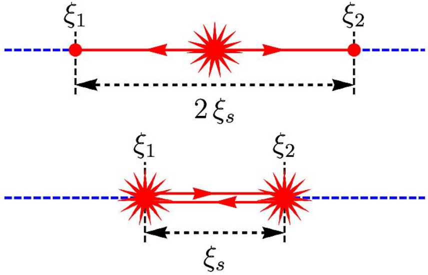
  - Con: acausal signal propagation ($v_Q^2 \to \infty$)
- **Colored** noise: $\langle I(x_1)I(x_2)\rangle \propto \frac{1}{2t_Q}\mathrm{e}^{-|t_1-t_2|/t_Q}\delta^3(\vec{x}_1 - \vec{x}_2)$
  - Corresponds to *causal* diffusion: $\left[\partial_t + \mathbf{t_Q}\boldsymbol{\partial_t^2} + D_Q\partial_x^2\right] n_Q = 0$
  - Pro: causal ($v_Q^2 = D_Q/t_Q = \text{finite}$)
  - Con: more complicated implementation

Results

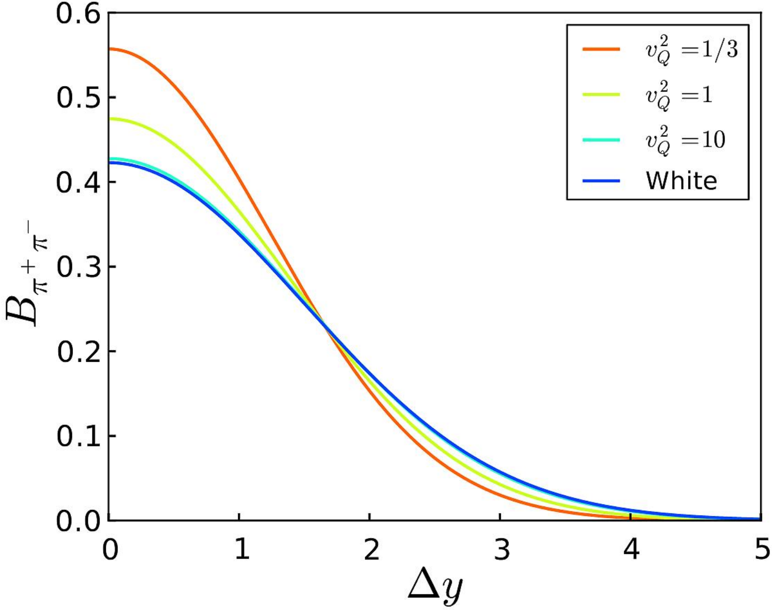

$\xi_s = v_Q \ln(\tau/\tau_0)$
≈ 0.266
$v_Q^2 = 1/3$
$\langle \delta n(\Delta\xi)\delta n(0)\rangle$
$\Delta\xi$
30
25
20
15
10
5
0
−1.0
−0.5
0.0
0.5
1.0


ξ_1
ξ_2
$2\,\xi_s$
ξ_1
ξ_2
ξ_s

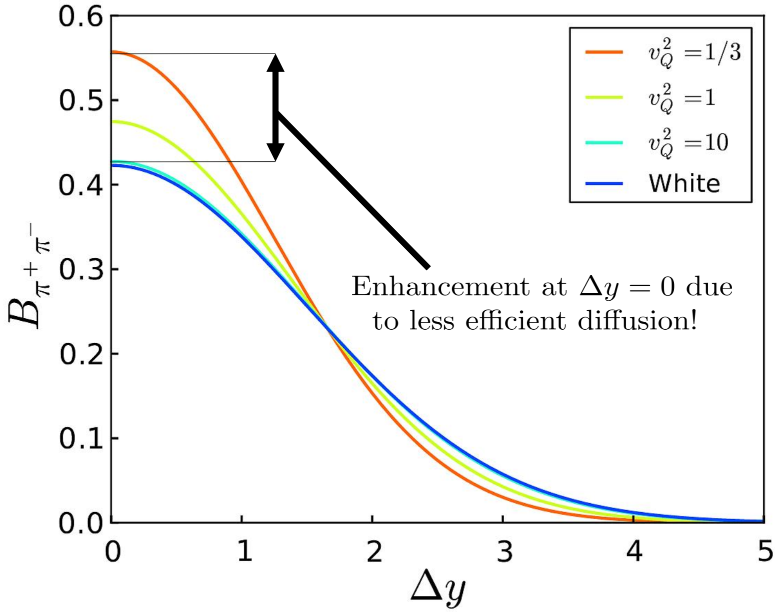




- Both sets of wavefronts reflect essential aspects of causal signal propagation
- Wavefronts travel farther with larger v_Q^2
- *No* wavefronts for $v_Q^2 \to \infty$!



0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
$B_{\pi^+\pi^-}$
Δy
$v_Q^2 = 1/3$
$v_Q^2 = 1$
$v_Q^2 = 10$
White



0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
Δy
$B_{\pi^+\pi^-}$
v_Q^2 =1/3
v_Q^2 =1
v_Q^2 =10
White
Enhancement at $\Delta y = 0$ due
to less efficient diffusion!

Recap:

- Diffusion is an essential aspect of heavy-ion collisions
- Hydrodynamic fluctuations offer a natural framework for modeling diffusion
- White noise leads to violations of relativistic causality
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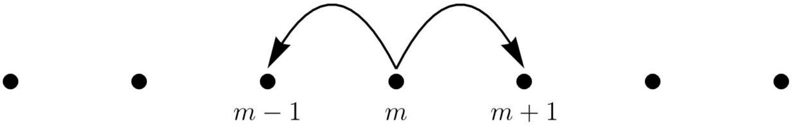
Question: are there other approaches?

Recap:

- Diffusion is an essential aspect of heavy-ion collisions
- Hydrodynamic fluctuations offer a natural framework for modeling diffusion
- White noise leads to violations of relativistic causality
- Colored noise consistently imposes causality on diffusive mechanisms

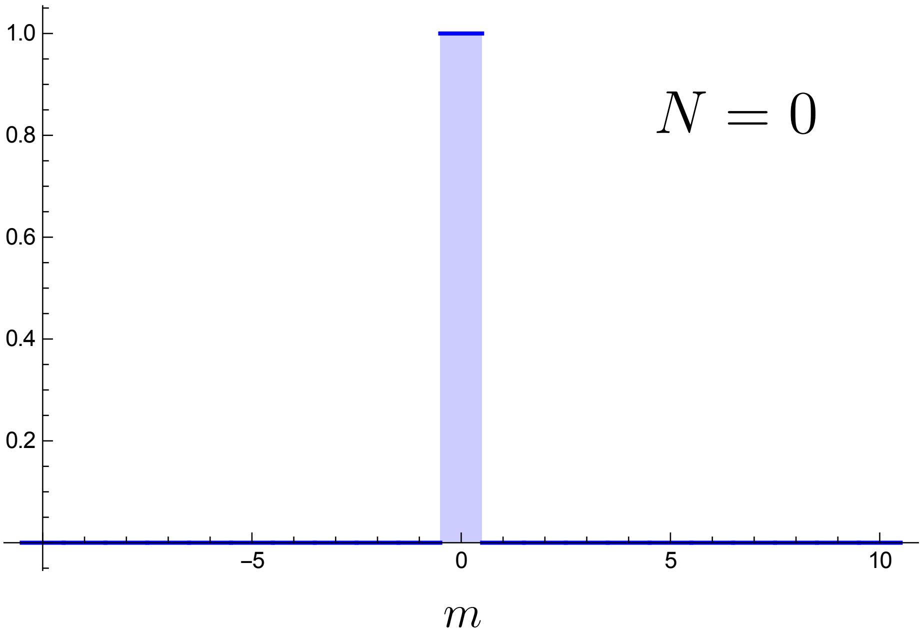
Question: are there other approaches?
Warm-up: *why* are white noise / ordinary diffusion acausal?

Diffusion and Random Walks

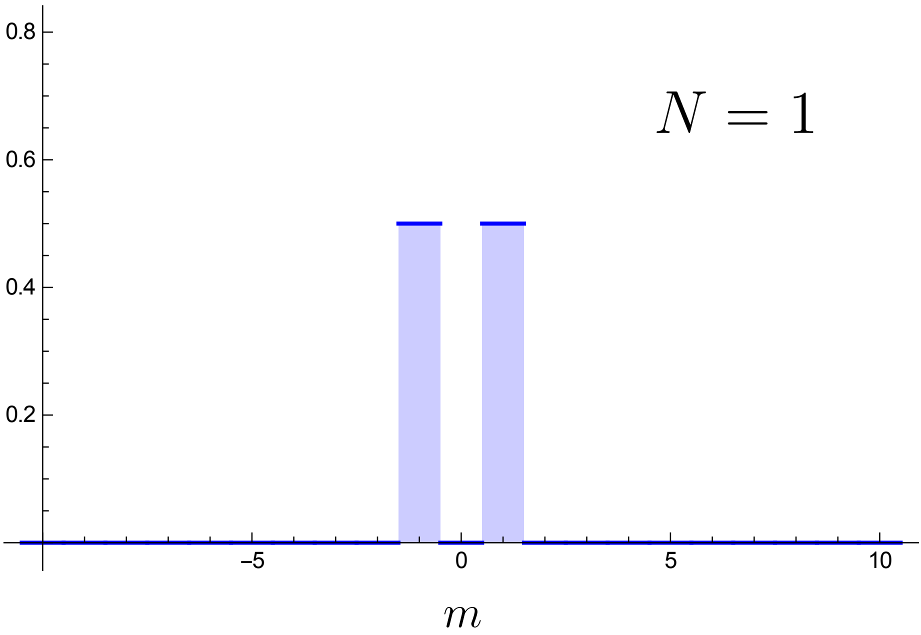


1-D symmetric random walk: $p\left(N,m\right)=\dfrac{2^{-N}N!\Theta\left(N-|m|\right)}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}$

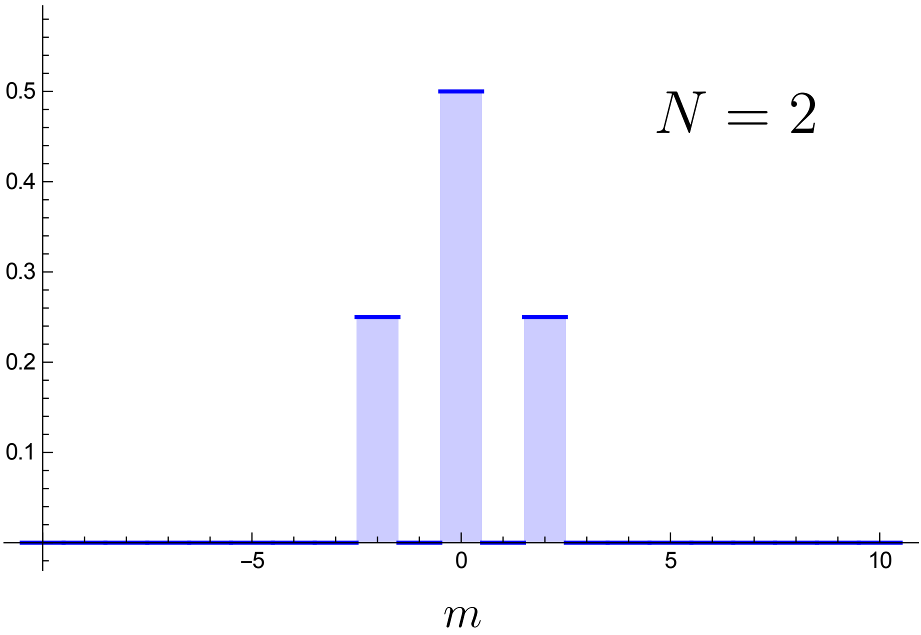
$m:$ distance from origin, $N:$ number of timesteps



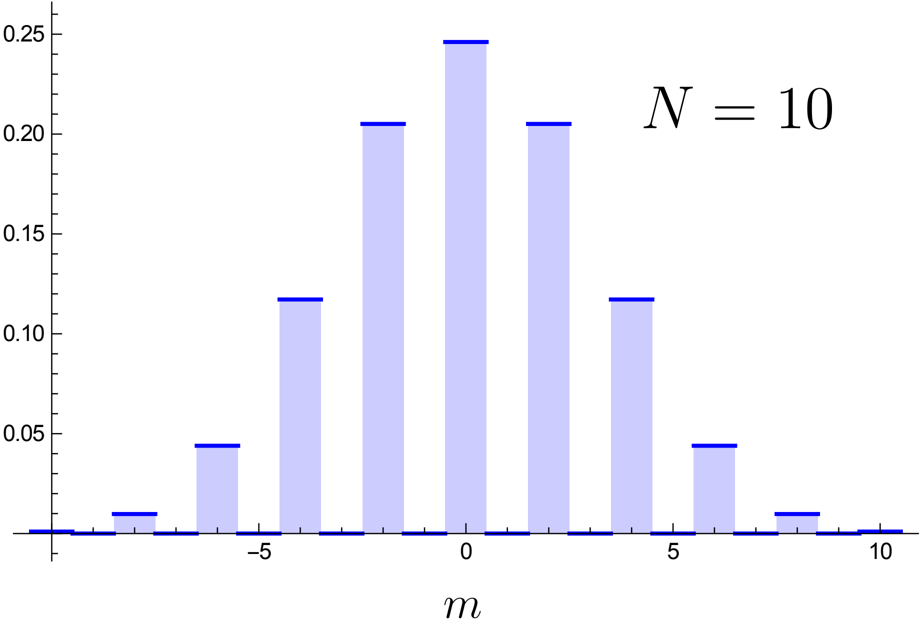
1.0
0.8
0.6
0.4
0.2
–5
0
5
10
$N = 0$
m



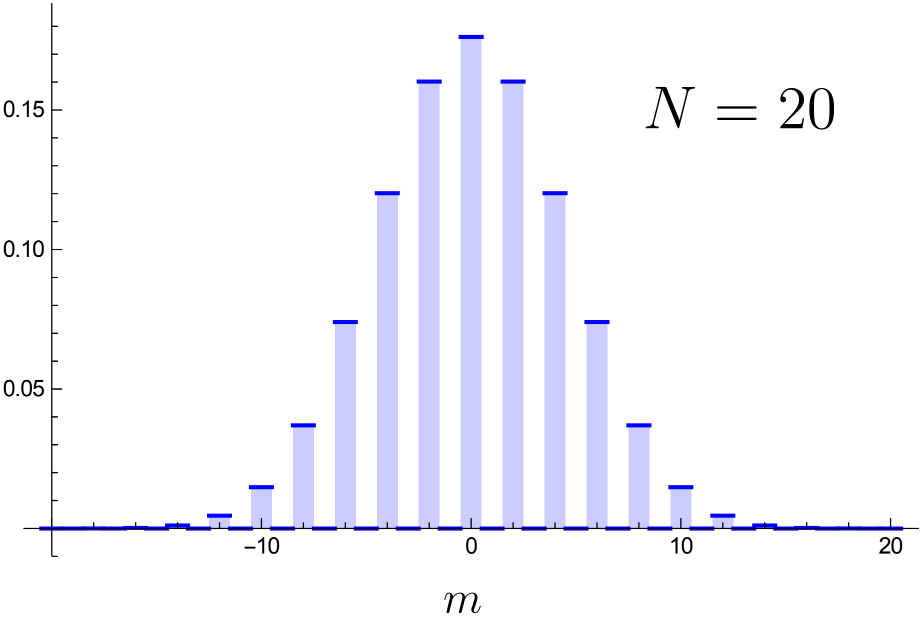
0.8
0.6
0.4
0.2
N = 1
–5
0
5
10
m



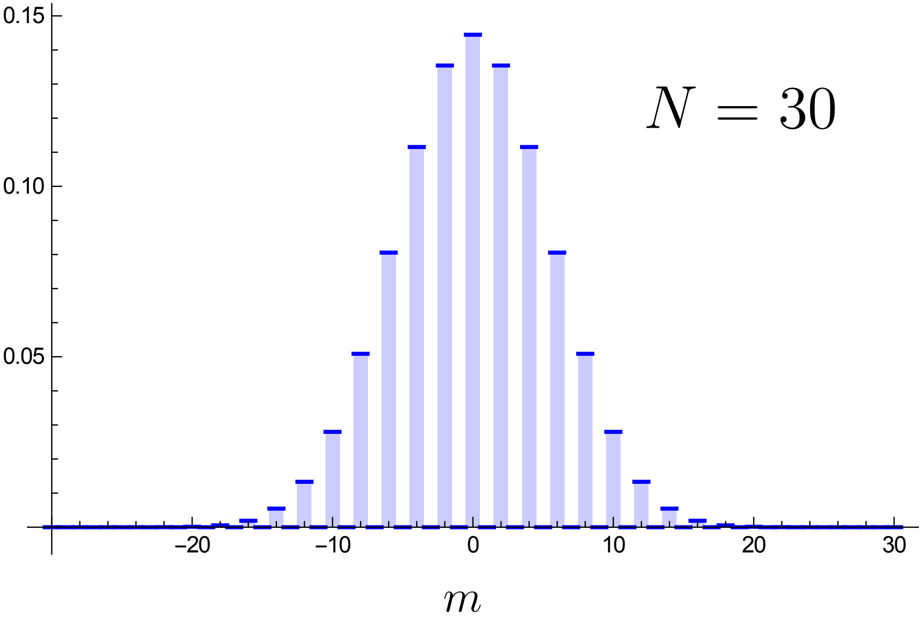
0.5
0.4
0.3
0.2
0.1
–5
0
5
10
m
$N = 2$

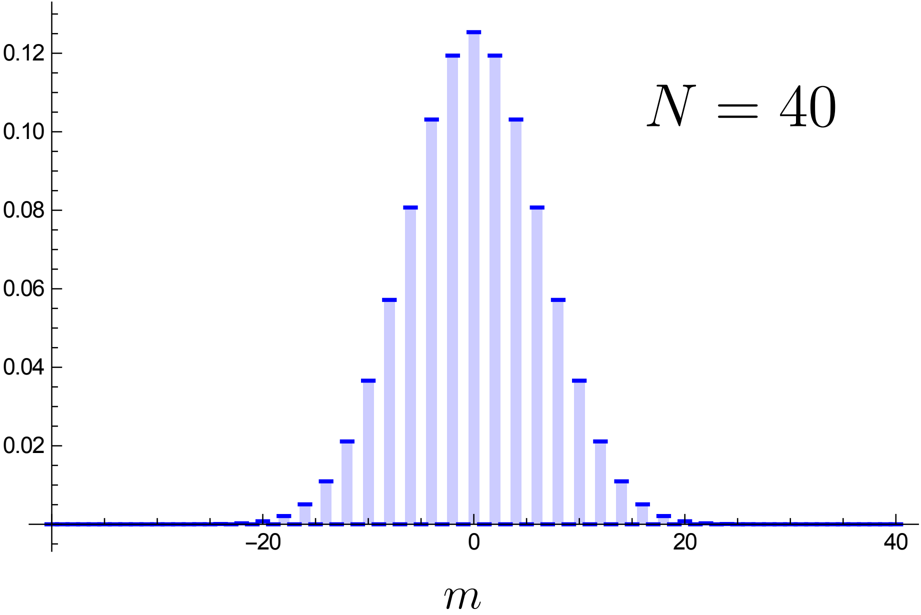


0.25
0.20
0.15
0.10
0.05
–5
0
5
10
$N = 10$
m

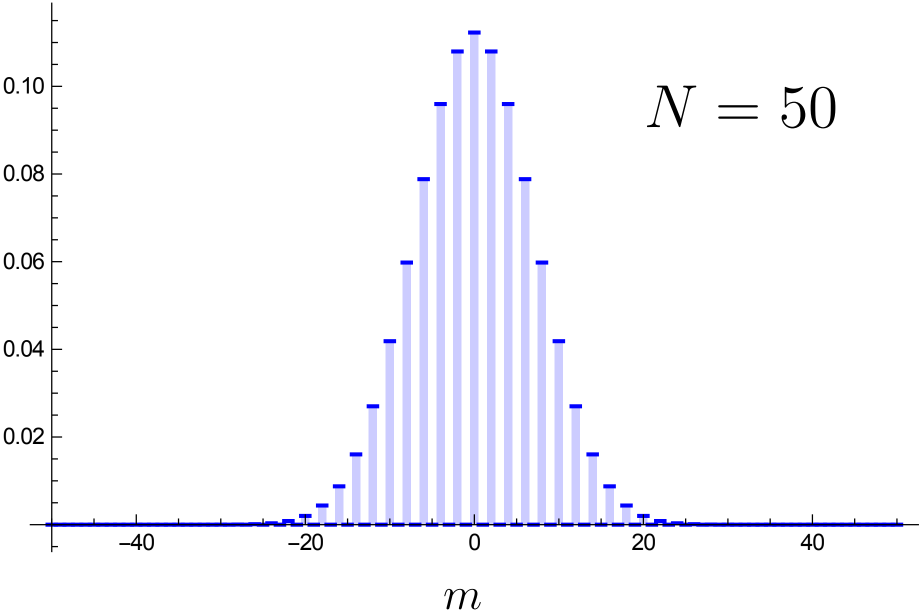


$N = 20$
0.15
0.10
0.05
−10
0
10
20
m


0.15
0.10
0.05
$N = 30$
−20
−10
0
10
20
30
m

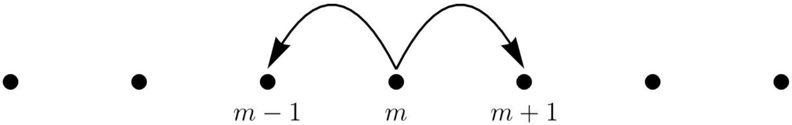


0.12
0.10
0.08
0.06
0.04
0.02
−20
0
20
40
$N = 40$
m



0.10
0.08
0.06
0.04
0.02
−40
−20
0
20
40
$N = 50$
m

Diffusion and Random Walks



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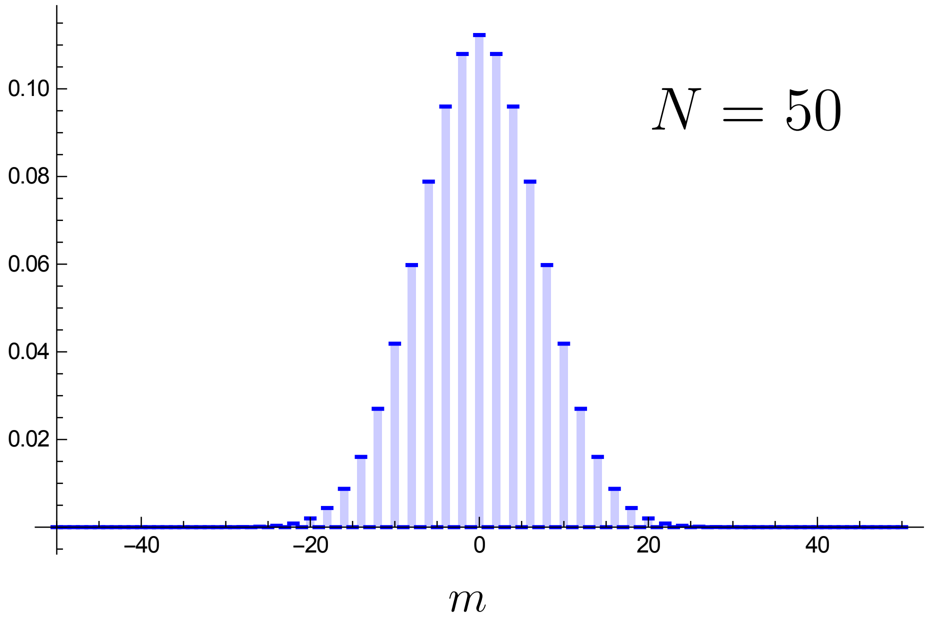
m : distance from origin, N : number of timesteps

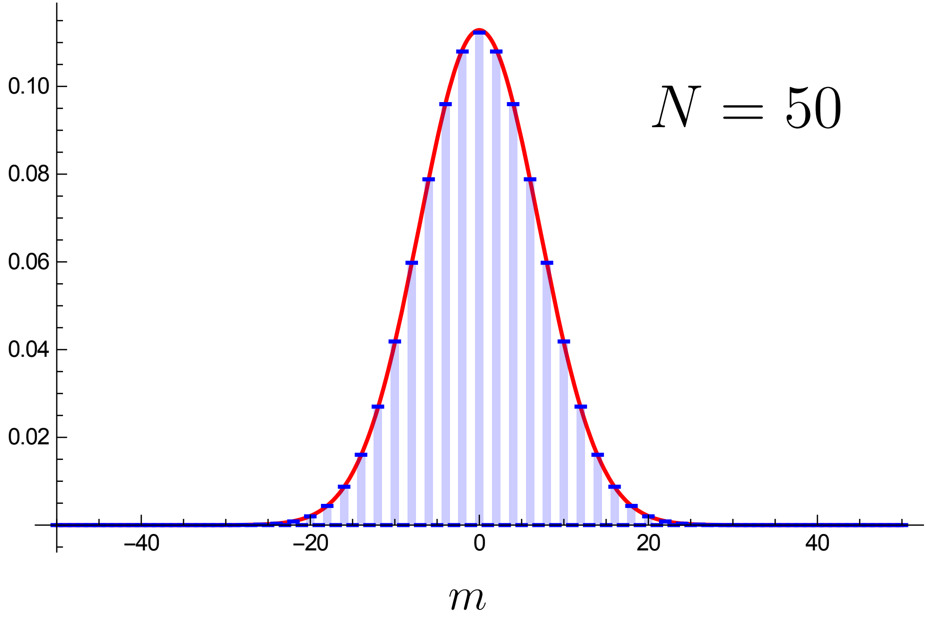
$$\lim_{N\to\infty}p\left(N,m\right)\sim\sqrt{\frac{2}{\pi N}}\exp\left(-\frac{m^2}{2N}\right);$$

i.e., solution to continuum diffusion equation:

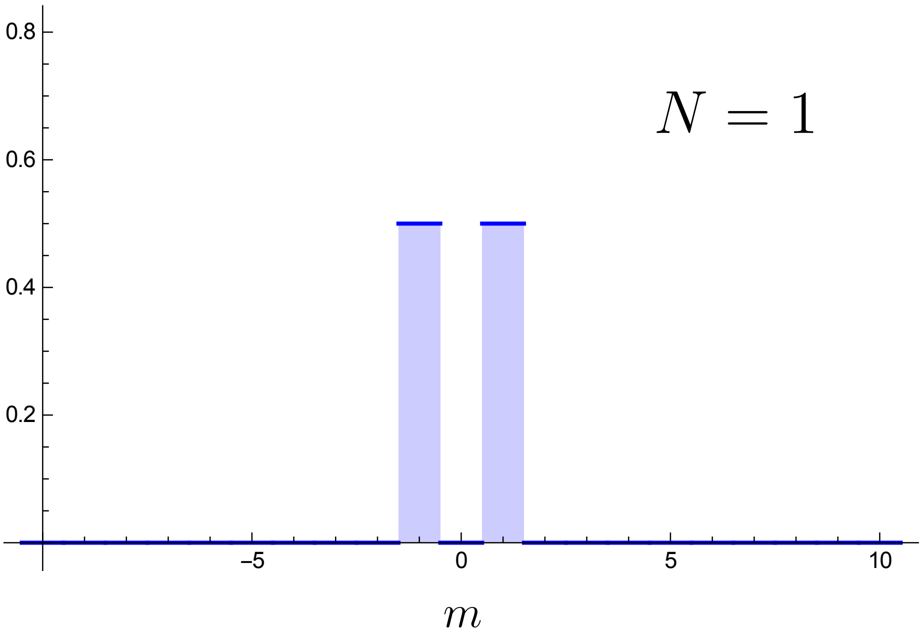
$$\frac{\partial}{\partial t}p\left(t,x\right)=D\frac{\partial^2}{\partial x^2}p\left(t,x\right)\text{ with } t\to N,\ x\to m,\text{ and } D\to 1/2$$



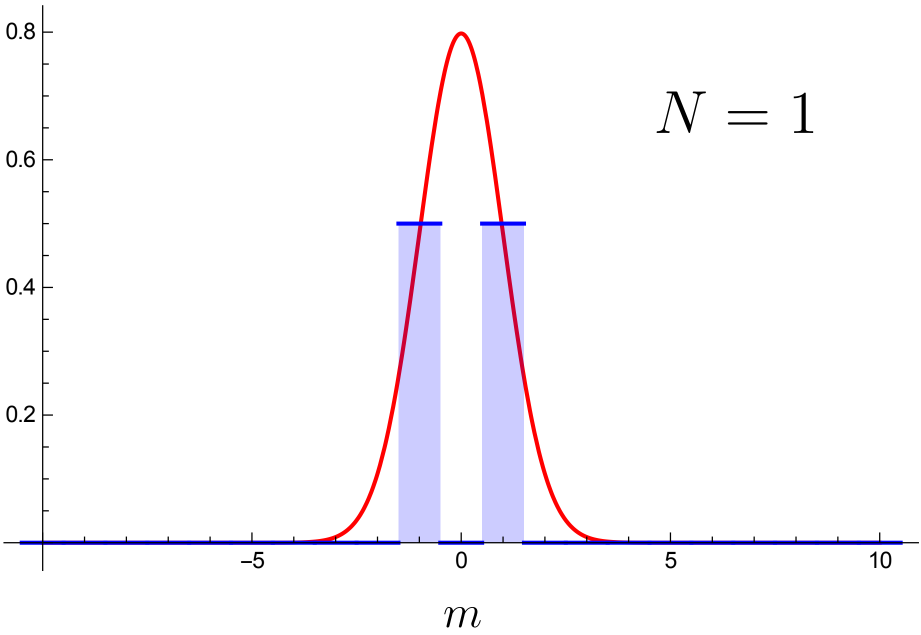
0.10
0.08
0.06
0.04
0.02
−40
−20
0
20
40
m
$N = 50$



0.10
0.08
0.06
0.04
0.02
-40
-20
0
20
40
m
$N = 50$



0.8
0.6
0.4
0.2
–5
0
5
10
m
$N = 1$


0.8
0.6
0.4
0.2
–5
0
5
10
m
N = 1

The moral of the story:

1. Continuum limit approximates random walk *only* in long-time limit
2. Acausality results from applying continuum limit outside regime of validity
3. Restore causality by reverting to random walk on short timescales

Random walks in heavy-ion collisions?

Basic idea: *replace density fluctuations with quarks on a random walk*

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- Consider the density correlator between quark flavors a and b:

$$\begin{aligned} C_{ab}(\vec{r}_1, \vec{r}_2, t) &\equiv \langle \delta n_a(\vec{r}_1, t)\, \delta n_b(\vec{r}_2, t)\rangle \\ &= \underbrace{\chi_{ab}(\vec{r}_1)\, \delta(\vec{r}_1 - \vec{r}_2)}_{\text{short-range}} - \underbrace{C'_{ab}(\vec{r}_1, \vec{r}_2, t)}_{\text{long-range}} \end{aligned}$$

  where $\chi_{ab} \equiv \langle Q_a Q_b\rangle / V$ and $a, b \in \{u, d, s\}$

- Require local charge conservation: $\partial_t \delta n_a(\vec{r}, t) + \nabla \cdot \delta \vec{j}_a(\vec{r}, t) = 0$

$$\begin{aligned} \Longrightarrow \quad & -\partial_t C'_{ab}(\vec{r}_1, \vec{r}_2, t) \\ = \quad & -\nabla_1 \cdot \left\langle \delta \vec{j}_a(\vec{r}_1, t)\, \delta n_b(\vec{r}_2, t)\right\rangle \\ - \quad & \nabla_2 \cdot \left\langle \delta n_a(\vec{r}_1, t)\, \delta \vec{j}_b(\vec{r}_2, t)\right\rangle \\ + \quad & S_{ab}(\vec{r}_1, t)\, \delta(\vec{r}_1 - \vec{r}_2) \end{aligned}$$

Random walks in heavy-ion collisions?

Basic idea: *replace density fluctuations with quarks on a random walk*

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$$\begin{aligned} C_{ab}\left(\vec{r}_1,\vec{r}_2,t\right) &\equiv \left\langle \delta n_a\left(\vec{r}_1,t\right)\delta n_b\left(\vec{r}_2,t\right)\right\rangle \\ &= \underbrace{\chi_{ab}\left(\vec{r}_1\right)\delta\left(\vec{r}_1-\vec{r}_2\right)}_{\text{short}-\text{range}} - \underbrace{C'_{ab}\left(\vec{r}_1,\vec{r}_2,t\right)}_{\text{long}-\text{range}} \end{aligned}$$

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$$\begin{aligned} \Longrightarrow\quad & -\partial_t C'_{ab}\left(\vec{r}_1,\vec{r}_2,t\right) \\ =\quad & -\nabla_1\cdot\left\langle \delta\vec{j}_a\left(\vec{r}_1,t\right)\delta n_b\left(\vec{r}_2,t\right)\right\rangle \\ -\quad & \nabla_2\cdot\left\langle \delta n_a\left(\vec{r}_1,t\right)\delta\vec{j}_b\left(\vec{r}_2,t\right)\right\rangle \\ +\quad & S_{ab}\left(\vec{r}_1,t\right)\delta\left(\vec{r}_1-\vec{r}_2\right) \end{aligned}$$

$$\begin{aligned} S_{ab}\left(\vec{r},t\right) &\equiv -\partial_\mu\left(u^\mu\chi_{ab}\right) \\ \delta\vec{j}_a\left(\vec{r},t\right) &\equiv -D\nabla\delta n_a\left(\vec{r},t\right) \end{aligned}$$

Monte-Carlo hydrodynamics

The recipe:

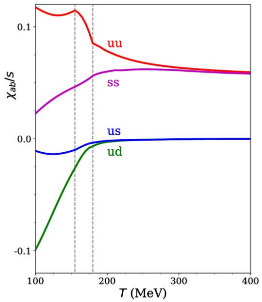
- Solve the ordinary (non-fluctuating) hydrodynamic EoMs:

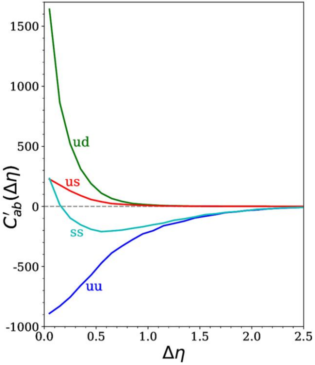
$$\partial_\mu T^{\mu\nu} = 0, \quad \text{2+1D solution using iEBE-VISHNU}^{3}$$

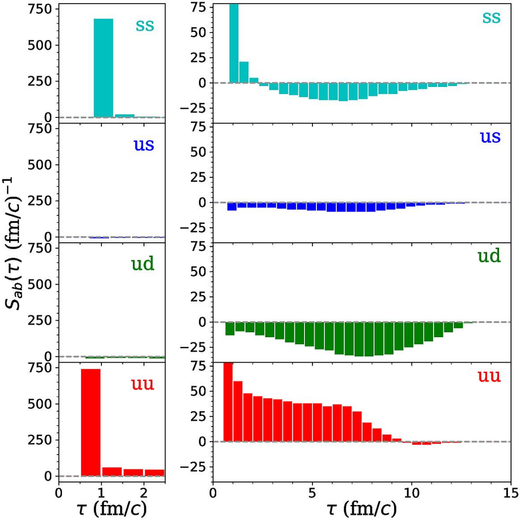
- Use space-time information ($u^\mu(x)$, $T(x)$, etc.) together with $\chi_{ab}(T)$ (from lattice + HRG EoS) to compute $S_{ab}(x) \rightarrow$ yields effective rate of MC quark pair production
- Create quark pairs in accordance with $S_{ab}(x)$ and trace their interactions explicitly $\rightarrow$ yields MC representation of $C'_{ab}(\vec{r}_1, \vec{r}_2, t)$!
- Freeze out at T_{dec} and project all charges onto final-state hadrons *via* Cooper-Frye $\rightarrow$ yields final-state correlations between measurable particles

[3] C. Shen, et al. , Comput. Phys. Commun. **199**, 61 (2016)

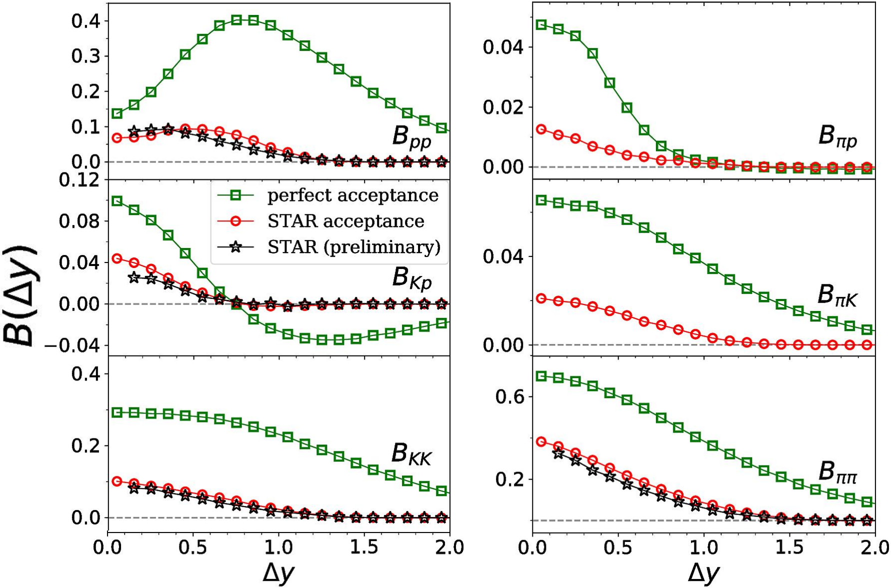
Quarks in central Au+Au collisions at $\sqrt{s_{NN}} = 200\,A$ GeV







Balance functions[4]



[4]H. Wang, arXiv:1304.2073 [nucl-ex]

Where we stand:

- Charge balance functions probe diffusion and chemical evolution in heavy-ion collisions
- Several approaches to modeling with hydrodynamics:
  - Fluctuating hydrodynamics
  - Monte-Carlo hydrodynamics
  - Probably others
- Essential lessons and challenges:
  - Retaining relativistic causality
  - Eliminating "trivial" self-correlations

Future/ongoing work:

- Incorporate into 3+1D hydrodynamic simulations
- Add in hadron cascade
- Generalize to finite isospin
- Fully differential balance functions

Consequences for the BES program

→ What happens at lower $\sqrt{s_{NN}}$?

- Finite net baryon physics
- Loss of boost invariance

⇒ Need 3+1D hydrodynamics: $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu_{Q_a} = 0$

→ What happens near a critical point?

- Critical slowing down with Hydro+[5]
- Divergence of correlation length, enhanced fluctuations[6]
- Cf. also previous talk by Teaney

→ What we can learn:

- Probe T-dependence of χ_{ab}, S_{ab}
- Test chemical equilibration at early times
- Extract diffusion coefficient for light quarks?
- Additional constraints on presence/absence of critical phenomena

→ Going forward:

- Wider range of hadronic correlations (e.g., $B_{\pi K}$ and $B_{\pi p}$)
- Fully differential $B_{hh'}$ in Δy, $\Delta\phi$, etc.
- Wider rapidity coverage/acceptance

---

[5] M. Stephanov and Y. Yin, arXiv:1712.10305 [nucl-th]

[6] S. Pratt, Phys. Rev. C **96**, 044903 (2017)

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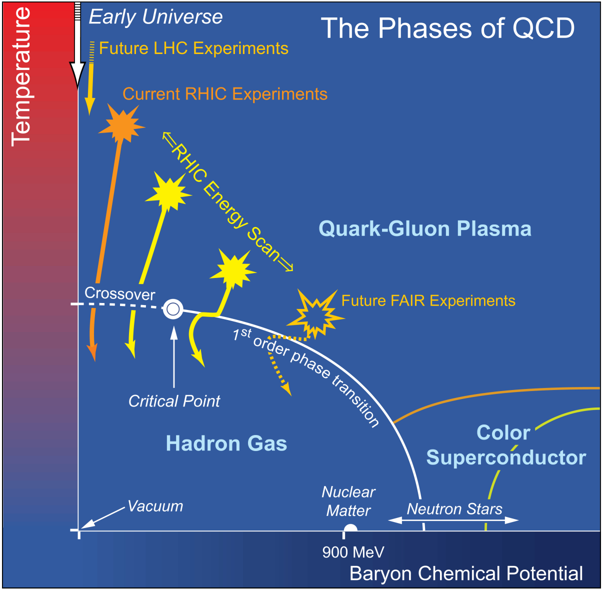
[6] S. Pratt, Phys. Rev. C **96**, 044903 (2017)

Backup slides


The Phases of QCD
Temperature
Early Universe
Future LHC Experiments
Current RHIC Experiments
⇐RHIC Energy Scan⇒
Quark-Gluon Plasma
Future FAIR Experiments
Crossover
1st order phase transition
Critical Point
Hadron Gas
Color Superconductor
Vacuum
Nuclear Matter
Neutron Stars
900 MeV
Baryon Chemical Potential

Steps to solve $\partial_\mu J_Q^\mu = 0$ (with $\tau_Q \neq 0$):

$$\partial_\mu J_Q^\mu = 0 \quad \Longrightarrow$$

$$\frac{\partial^2}{\partial \tau^2}(\tau \delta \tilde{n}) + \left[\frac{1}{\tau_Q} - \frac{\partial}{\partial \tau} \ln \left(\frac{\chi_Q T D_Q}{\tau} \right) \right] \frac{\partial}{\partial \tau}(\tau \delta \tilde{n}) + \frac{D_Q k^2}{\tau_Q \tau^2}(\tau \delta \tilde{n})$$
$$= -iks \left[\frac{\partial \tilde{f}}{\partial \tau} + \left(\frac{1}{\tau_Q} - \frac{1}{\tau} - \frac{\partial}{\partial \tau} \ln \left(\frac{\chi_Q T D_Q}{\tau} \right) \right) \tilde{f} \right].$$

T is the temperature, D_Q is the electric charge diffusion coefficient, σ_Q is the electric charge conductivity, and $\chi_Q = \sigma_Q / D_Q$ is the electric charge susceptibility. Finally, the quantity k is Fourier-conjugate to the spatial rapidity ξ; for any quantity X, we define

$$X(\xi, \tau) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\xi} \tilde{X}(k, \tau)$$

Subtracting self-correlations

White noise density correlator:

$$\langle \delta n\left(\xi_1, \tau_f\right) \delta n\left(\xi_2, \tau_f\right)\rangle = \frac{\chi_Q\left(\tau_f\right) T_f}{A\tau_f}\left[\delta\left(\xi_1 - \xi_2\right) - \frac{1}{\sqrt{\pi w^2}} e^{-\left(\xi_1-\xi_2\right)^2/w^2}\right]$$

- First term: "self-correlations"
  - Represent trivial correlations of a particle with itself
  - Not measured experimentally
- Second term: diffusive correlations
  - Represent physical, non-trivial correlations of distinct particles
  - Are actually what we care about

$\rightarrow$ Self-correlations need to be subtracted out to compare with experiment!

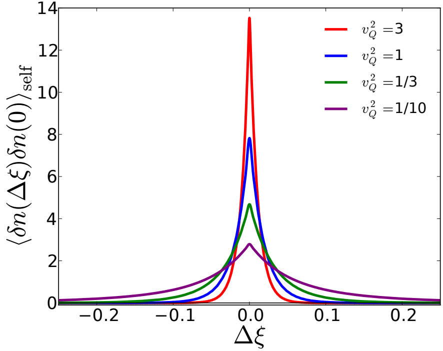
$\rightarrow$ Not so hard to do for white noise...

$\rightarrow$ ...but highly non-trivial for colored noise!

Subtracting self-correlations

Colored noise density self-correlations ($v_Q \gg 1$):

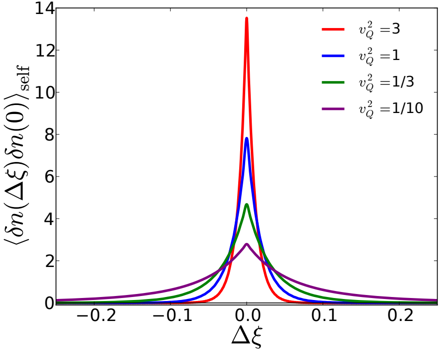
$$\langle \delta n\left(\xi_1, \tau_f\right) \delta n\left(\xi_2, \tau_f\right)\rangle_{\mathrm{self}} \approx \frac{\chi_Q\left(\tau_f\right) T_f}{A\tau_f} \frac{v_Q \tau_f}{2D_Q} \exp\left(-\frac{v_Q \tau_f}{D_Q}\left|\xi_1 - \xi_2\right|\right)$$



Subtracting self-correlations

Colored noise density self-correlations ($v_Q \gg 1$):

$$\langle \delta n(\xi_1, \tau_f)\, \delta n(\xi_2, \tau_f) \rangle_{\rm self} \approx \frac{\chi_Q(\tau_f)\, T_f}{A\tau_f} \frac{v_Q \tau_f}{2D_Q} \exp\left(-\frac{v_Q \tau_f}{D_Q} |\xi_1 - \xi_2| \right)$$

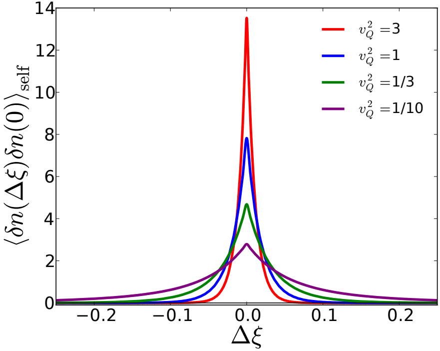


- "Adiabatic limit" ($v_Q \gg 1$) reduces to exponential form on quasi-static background

Subtracting self-correlations

Colored noise density self-correlations ($v_Q \gg 1$):

$$\langle \delta n\left(\xi_1, \tau_f\right) \delta n\left(\xi_2, \tau_f\right) \rangle_{\mathrm{self}} \approx \frac{\chi_Q\left(\tau_f\right) T_f}{A \tau_f} \frac{v_Q \tau_f}{2 D_Q} \exp\left(-\frac{v_Q \tau_f}{D_Q}\left|\xi_1 - \xi_2\right|\right)$$



- "Adiabatic limit" ($v_Q \gg 1$) reduces to exponential form on quasi-static background
- "Instantaneous limit" ($v_Q \ll 1$) just takes all correlations to zero

See these references for more detail:

- Ling, Springer, and Stephanov [PRC **89**, 064901 (2014)]
- Kapusta and CP [PRC **97**, 014906 (2018)]

Holographic considerations

Key idea: matching colored noise to holographic dispersion relations yields estimates for D_Q, τ_Q
Gurtin-Pipkin noise:

$$\begin{aligned} \frac{\partial}{\partial t} - D_Q \nabla^2 + \tau_1 \frac{\partial^2}{\partial t^2} + \tau_2^2 \frac{\partial^3}{\partial t^3} - \tau_3 D_Q \frac{\partial}{\partial t} \nabla^2 &= 0 \\ \Rightarrow \tau_2^2 \omega^3 + i\tau_1 \omega^2 - \left(1 + \tau_3 D_Q k^2\right) \omega - i D_Q k^2 &= 0 \end{aligned}$$

Holography[7,8] yields Kaluza-Klein-type tower of poles in holographic dispersion relation:

$$\omega(k=0) = (\pm n - in) 2\pi T,\, n = 0, 1, 2, \ldots$$

Match GP noise onto three lowest frequency poles

$$\Rightarrow D_Q = \tau_Q = \tau_1 = \frac{1}{2\pi T},\, \tau_2 = \tau_1/\sqrt{2}, \tau_3 = \tau_1/2$$

[7] Nunez and Starinets [PRD **67**, 124013 (2003)]
[8] Policastro, Son and Starinets [JHEP **0209**, 043 (2002)]

Hadronic correlations and balance functions

Hadronic correlators:

$$
\begin{aligned}
\delta N_h &= n_h \chi^{-1}_{ab} q_{ha} \delta Q_b \\
\Longrightarrow C'_{hh'}(\Delta\eta) &= \int d\eta d\eta' \delta\left(\Delta\eta - \left|\eta - \eta'\right|\right) \\
&\qquad \times \left\langle \left(N_h(\eta) - N_{\bar{h}}(\eta)\right)\left(N_{\bar{h}'}(\eta') - N_{h'}(\eta')\right)\right\rangle \\
&= \sum_{ab} K_{hh';ab} C'_{ab}(\Delta y), \\
K_{hh';ab} &= -4 n_h n_{h'} q_{hc} {q'}_{h'd} \chi^{-1}_{ca} \chi^{-1}_{db} \\
\Longrightarrow S_{hh'}(x) &\equiv \sum_{ab} K_{hh';ab} S_{ab}(x), \;\; \chi_{hh'}(x) \equiv \sum_{ab} K_{hh';ab} \chi_{ab}(x)
\end{aligned}
$$

Balance functions:

$$
\begin{aligned}
B_{hh'}(\Delta y) \; = \; \frac{1}{\langle N_h + N_{\bar{h}} \rangle} &\int dy dy' \delta\left(\Delta y - \left|y - y'\right|\right) \\
&\times \left\langle \left(N_h(y) - N_{\bar{h}}(y)\right)\left(N_{\bar{h}'}(y') - N_{h'}(y')\right)\right\rangle
\end{aligned}
$$

Hadrons in central Au+Au collisions at $\sqrt{s_{NN}} = 200\,A$ GeV