Hydrodynamic fluctuations near the QCD critical point

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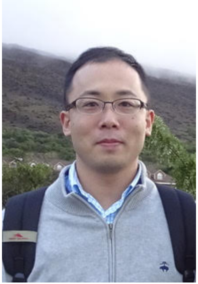
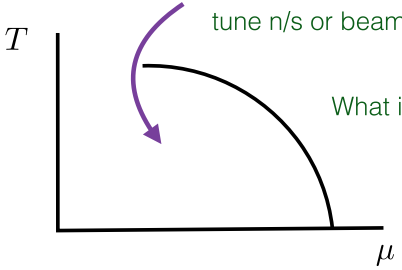


Transits of the critical point: two parameters



tune n/s or beam energy to go through CP

What if we miss it?

Expanding?

- How does missing the critical point regulate the critical flcuts?

$$\Delta_s \equiv \frac{n_c}{s_c}\left(\frac{s}{n} - \frac{s_c}{n_c}\right)$$

- How does the finite expansion rate regulate the critical flucts?

$$\epsilon \equiv \underbrace{\tau_o}_{\text{micro time}} \times \underbrace{\partial_\mu u^\mu}_{\text{expansion rate}} = \frac{\tau_o}{\tau_Q}$$

What should we measure: baryon/entroppy

$$C^{\hat{n}\hat{n}} \equiv \underbrace{\left\langle (\delta n - (n/s)\delta s)^2 \right\rangle}_{\text{flucts of } \delta\hat{n} \equiv s\delta(n/s)}$$

variance in n/s:
not baryon number

- Why? It is a hydro eigenmode, and *always maximally divergent:*

$$\underbrace{\left\langle (\delta n - (n/s)\delta s)^2 \right\rangle}_{\text{what we want}} \propto \underbrace{C_p}_{\text{specific heat}} \propto \underbrace{\chi_{\text{is}}}_{\text{ising susceptibility}}$$

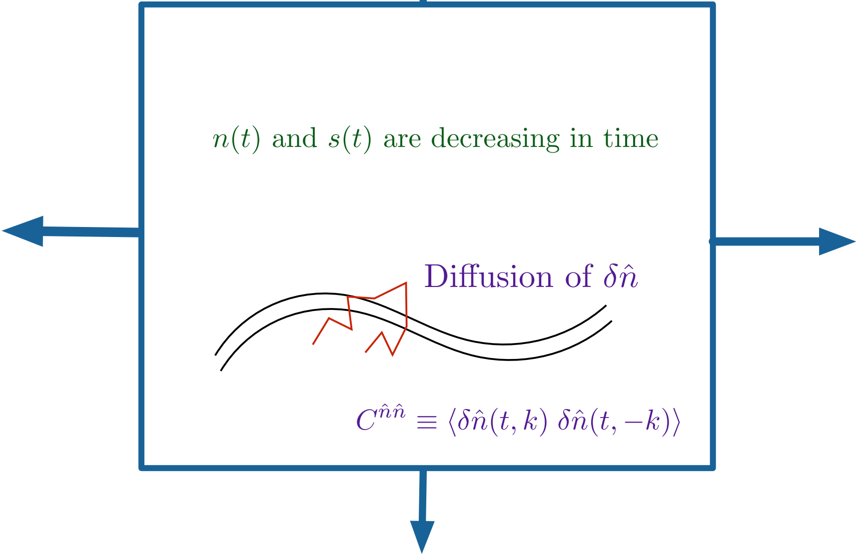
- Which wavelengths diverge? Long wavelengths *don't fluctuate:*

$$C^{\hat{n}\hat{n}}(k) \equiv \langle \delta\hat{n}(k,t)\delta\hat{n}(-k,t) \rangle \propto \underbrace{\frac{\chi_{\text{is}}}{1 + (k\xi)^{2-\eta}}}_{\text{ising } \textit{equilibrium} \text{ prediction}}$$

How do the wavelengths of critical flucts compare to micro lengths?

The expanding box setup

$$\frac{n}{s} = (1+\Delta_s)\frac{n_c}{s_c} \qquad \epsilon \equiv \frac{\tau_o}{\tau_Q} \equiv \tau_o \partial_\mu u^\mu$$



Slightly miss the critical point: constant n/s trajectories

- Ideal hydro conserves n/s

Entropy conservation: $u^\mu \partial_\mu s = -s\partial_\mu u^\mu \quad \Rightarrow \quad \partial_t s = -\frac{s}{\tau_Q}$

Baryon conservation: $u^\mu \partial_\mu n = -n\partial_\mu u^\mu \quad \Rightarrow \quad \partial_t n = -\frac{n}{\tau_Q}$

- Passing closest to critical point at $t = 0$ with expansion rate $1/\tau_Q$

$$\frac{\Delta n}{n_c} \simeq -\frac{t}{\tau_Q} \qquad \frac{\Delta s}{s_c} \simeq \underbrace{\Delta_s}_{\text{detuning}} - \frac{t}{\tau_Q}$$

where

$$\Delta n \equiv n - n_c \qquad \text{and} \qquad \Delta s \equiv s - s_c$$

The expanding box setup

$$\frac{n}{s} = (1+\Delta_s)\frac{n_c}{s_c}$$

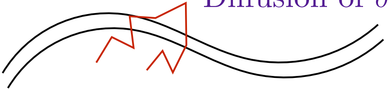
$$\epsilon \equiv \frac{\tau_o}{\tau_Q} \equiv \tau_o \partial_\mu u^\mu$$

$n(t)$ and $s(t)$ are decreasing in time

$$\frac{\Delta n(t)}{n_c} = -\frac{t}{\tau_Q}$$

$$\frac{\Delta s(t)}{s_c} = \Delta_s - \frac{t}{\tau_Q}$$

Diffusion of $\delta\hat{n}$



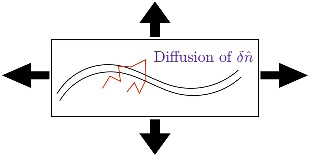
$$C^{\hat{n}\hat{n}} \equiv \langle \delta\hat{n}(t,k)\, \delta\hat{n}(t,-k) \rangle$$

Hydrodynamic equation for $C^{\hat{n}\hat{n}}(k,t) = \langle \hat{n}(k,t)\hat{n}(-k,t)\rangle$

- Start from dissipative hydro with noise

$$\partial_\mu (T^{\mu\nu}_{\rm Ideal} + T^{\mu\nu}_{\rm diss} + \xi^{\mu\nu}) = 0$$

$$\partial_\mu (j^{\mu}_{\rm Ideal} + j^{\mu}_{\rm diss} + \xi^{\mu}) = 0$$



- Can derive time evolution equations for the correlators

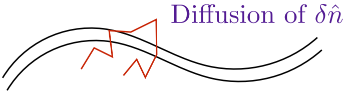
$$C^{ee} = \langle \delta e^*(k,t)\delta e(-k,t)\rangle \quad C^{nn} = \langle \delta n^*(k,t)\delta n(-k,t)\rangle \quad , \text{etc}$$

- From C^{nn}, C^{ee} derive an equation for $C^{\hat{n}\hat{n}} = \langle [\delta n - (n/s)\delta s]^2\rangle$

$$\partial_t C^{\hat{n}\hat{n}} = - \underbrace{\frac{\lambda_{\rm eff} k^2}{C_p}}_{\text{heat diffusion coeff}} (C^{\hat{n}\hat{n}} - \underbrace{C_p}_{\text{specific heat}})$$

From stochastic hydro find that $C^{\hat{n}\hat{n}}$ obeys a relaxation equation

The maximum wavelength that can be equilibrated:



- Equilibration is a *diffusive process*

$$\underbrace{D_0}_{\text{diffusion coef}} \times \underbrace{\tau_Q}_{\text{the total time}} = \underbrace{\ell_{\rm max}^2}_{\text{the longest wavelength}}$$

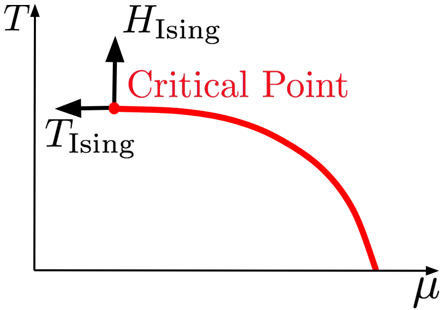
- Here D_0 is the (thermal) diffusion coefficient away from the CP:

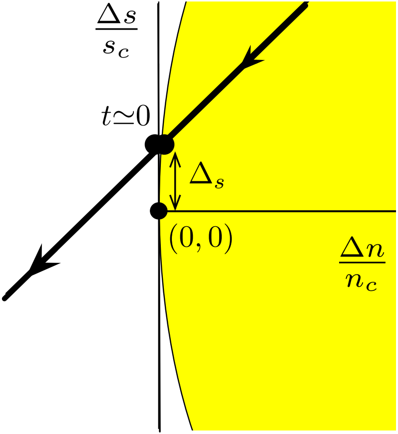
$$D_0 \sim \frac{\ell_o^2}{\tau_0} \qquad \ell_o \equiv \text{micro length}$$

Solve for the upper cuttoff $\ell_{\rm max}$ on the wavelength of critical modes

$$\underbrace{\ell_o}_{\text{microlength}} \ll \underbrace{\ell_{\rm kz}}_{\text{typical critical wavelength}} \ll \underbrace{\ell_o \epsilon^{-1/2}}_{\ell_{\rm max}} \qquad \epsilon \equiv \frac{\tau_0}{\tau_Q}$$

The critical point at last: trajectories in $s - n$ plane





$-\Delta\mu_{\mathrm{QCD}} \longleftrightarrow \Delta T_{\mathrm{Ising}}$

$\Delta T_{\mathrm{QCD}} \longleftrightarrow \Delta H_{\mathrm{Ising}}$

$\Delta s \longleftrightarrow \Delta M_{\mathrm{Ising}}$

$\Delta n \longleftrightarrow \Delta e_{\mathrm{Ising}}$

The *equilibrium* fluctuations versus time and wavelength

- Equilibrium is determined by the correlation length:

$\Delta \equiv 2 - \eta$

$$C_p \propto \chi(t,k) \equiv \frac{\chi_{\text{ising}}}{1 + (k\xi)^{\Delta}} \qquad \chi_{\text{ising}} \propto \xi^{\Delta}$$

- The correlation length is also universal:

$e_{\text{ising}} \equiv$ Ising e-density

$$\xi \propto (\Delta e_{\text{ising}})^{-a\nu} \qquad a \equiv 1/(1-\alpha)$$

With the map $\Delta e_{\text{ising}} \propto \Delta n = t/\tau_Q$, we find time dependence:

$$\chi_{\text{ising}}(t) \propto (\xi(t))^{\Delta} \qquad \xi(t) = \ell_o \left(\frac{t}{\tau_Q}\right)^{-a\nu}$$

The rate of change of $\xi(t)$ is diverging near the CP ($t \to 0$):

$$\underbrace{\frac{\partial_t \xi}{\xi} = -\frac{a\nu}{t}}_{\text{Changing fast! Which modes can keep up?}}$$

Relaxation rate of $C^{\hat{n}\hat{n}}$

- Substitute $\chi_{\rm Ising} = \chi_o \left(\xi/\ell_o\right)^{2-\eta}$ into the relaxation equation

$$\partial_t C^{\hat{n}\hat{n}} = -\underbrace{\frac{\lambda_{\rm eff}}{\chi_o {\ell_o}^2 (\xi/\ell_o)^{4-\eta}}}_{\text{relaxation rate } \Gamma} (k\xi)^2 [C^{\hat{n}\hat{n}} - \chi(k,t)]$$

- Define the relaxation rate

$$\Gamma \equiv \underbrace{\frac{\lambda_{\rm eff}}{\chi_o {\ell_o}^2}}_{1/\tau_o} \times \underbrace{\frac{1}{(\xi(t)/\ell_o)^{4-\eta}}}_{\text{goes to 0 at CP}}$$

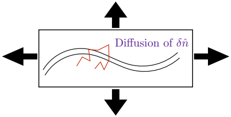
- Simplify Γ with $\xi(t) = \ell_o(\frac{t}{\tau_Q})^{-a\nu}$

$$\Gamma = \frac{1}{\tau_0}\left(\frac{t}{\tau_Q}\right)^{a\nu z} \quad \text{where} \quad z = 4 - \eta.$$

At CP, the hydo fluctuations relax infinitely slowly

Transiting the CP: Kibble-Zurek scales Berdnikov & Rajagopal; Mukherjee, Venugopalan, Yin

- When is this the changing rate of equilibrium comparable to the relaxation rate Γ ?



$$\left|\frac{\partial_t \xi}{\xi}\right| \sim \frac{1}{t} = \frac{(t/\tau_Q)^{a\nu z}}{\tau_o} = \Gamma$$

- This timescale is defined to be Kibble-Zurek time t_{kz}

$$t_{kz} = \epsilon^{1/(a\nu z+1)}\tau_Q = \epsilon^{0.26}\tau_Q$$

- Kibble-Zurek length is defined to be

$$\ell_{kz} = \xi(t_{kz}) = \ell_o \epsilon^{-a\nu/(a\nu z+1)} = \ell_o \epsilon^{-0.19}$$

$$\underbrace{\ell_o}_{\text{micro-length}} \ll \underbrace{\ell_o \epsilon^{-0.19}}_{\text{kibble-zurek } \ell_{kz}} \ll \underbrace{\ell_o \epsilon^{-0.5}}_{\text{cutoff } \ell_{\text{max}}}$$

Transiting the critical point: the rescaled equation

- Measure t and k in KZ units:

$$\bar{t} = t/t_{\rm kz} \qquad \bar{k} = k\ell_{\rm kz}$$

$$t_{kz} = \tau_o \epsilon^{-0.74} \qquad \ell_{\rm kz} = \ell_o \epsilon^{-0.19}$$

- $C^{\hat{n}\hat{n}}$ is of order the $\chi(t)$ at $t_{\rm kz}$

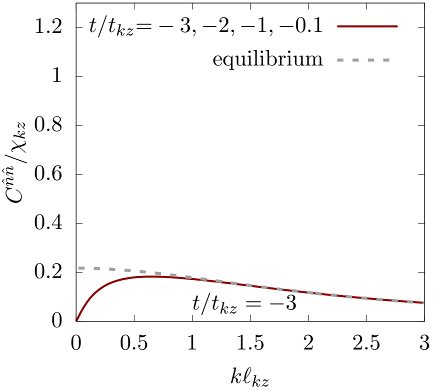
$$\bar{C}^{\hat{n}\hat{n}} = C^{\hat{n}\hat{n}}/\chi_{kz} \qquad \chi_{kz} \equiv \chi_{\rm Is}(t_{kz}) = \chi_o \epsilon^{-0.365}$$

- The rescaled equation becomes

$$\partial_{\bar{t}} \bar{C}^{\hat{n}\hat{n}} = -\frac{\bar{k}^2}{\bar{\chi}} \left(\bar{C}^{\hat{n}\hat{n}} - \bar{\chi}\right) \qquad \bar{\chi} = \frac{\bar{\chi}_{\rm Ising}}{1 + (\bar{k}\bar{\xi})^{2-\eta}}$$

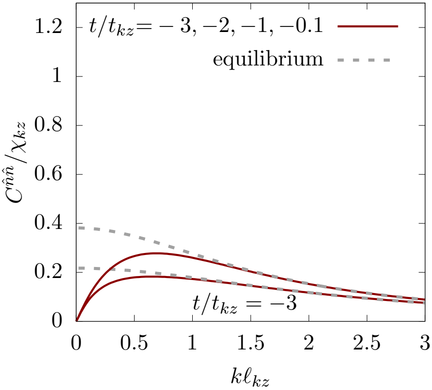
All dimensionful quantities are rescaled into the KZ-units

Solutions for $C^{\hat{n}\hat{n}}/\chi_{kz}$



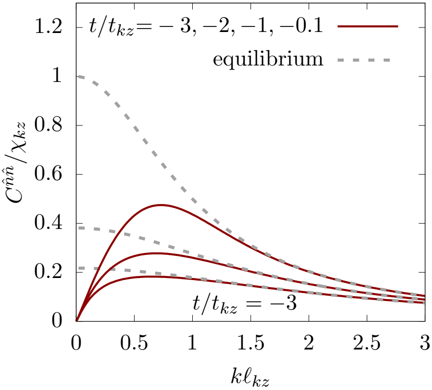
Before critical point

Solutions for $C^{\hat{n}\hat{n}}/\chi_{kz}$



Before critical point

Solutions for $C^{\hat{n}\hat{n}}/\chi_{kz}$



Before critical point

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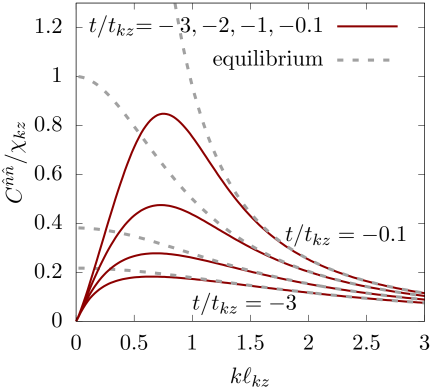
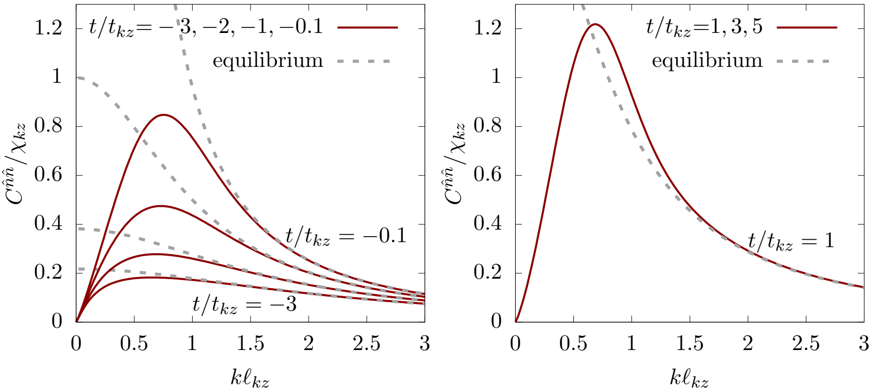


Before critical point

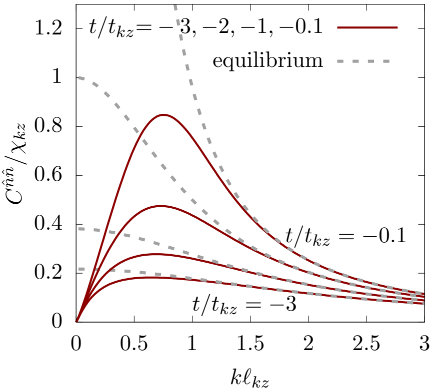
Solutions for $C^{\hat{n}\hat{n}}/\chi_{kz}$



Before critical point

After critical point

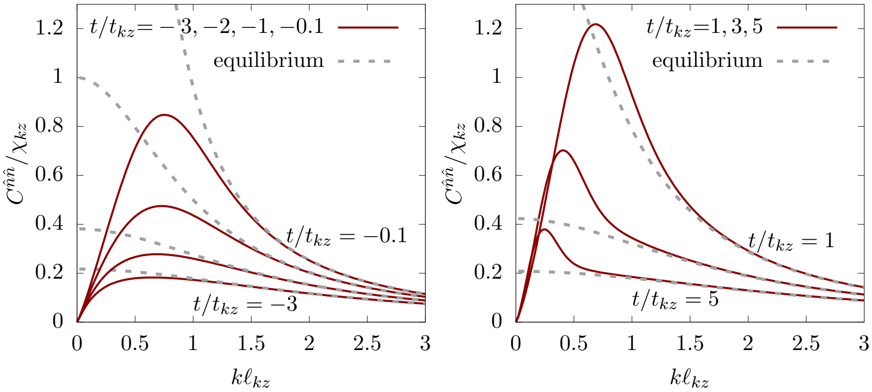
Solutions for $C^{\hat{n}\hat{n}}/\chi_{kz}$



Before critical point

After critical point

Solutions for $C^{\hat{n}\hat{n}}/\chi_{kz}$



Before critical point

After critical point

What have we learnt so far?

- We have obtained order-1 plots after rescaling with KZ units

$$C^{\hat{n}\hat{n}}/\chi_{kz} \sim 1 \qquad k l_{kz} \sim 1 \qquad t/t_{kz} \sim 1$$

- The typical critical wavelength is ℓ_{kz}

$$\underbrace{\ell_o}_{\text{micro-length}} \ll \underbrace{\ell_o \epsilon^{-0.19}}_{\text{kibble-zurek } \ell_{kz}} \ll \underbrace{\ell_o \epsilon^{-0.5}}_{\text{cutoff } \ell_{\max}}$$

Numerically these evaluate to with $\epsilon = 1/5$ and $\ell_0 = 1.2\,\text{fm}$

$$1.2\,\text{fm} \ll 1.6\,\text{fm} \ll 2.7\,\text{fm}$$

So the correlation is at most twice the interparticle spacing!
And the fluctuations are 80% larger than baseline:

$$\frac{C^{\hat{n}\hat{n}}}{\chi_0} \sim \left(\frac{\ell_{\rm kz}}{\ell_0}\right)^{2-\eta} = \epsilon^{-0.365} \sim 1.8$$

$C^{\hat{n}\hat{n}}$ has length scale ℓ_{kz} and has limited growth of 80%

Slightly miss the critical point: scalings

- From Ising scaling, $\xi(\Delta e_{\rm Ising}, \Delta M_{\rm Ising})$, scales

$$\xi = \ell_o (\Delta e_{\rm Is})^{-a\nu} f_\xi(\underbrace{\Delta e_{\rm Is}/\Delta M_{\rm Is}^{\frac{1-\alpha}{\beta}}}_{\text{scaling var}})$$

- Translating to QCD

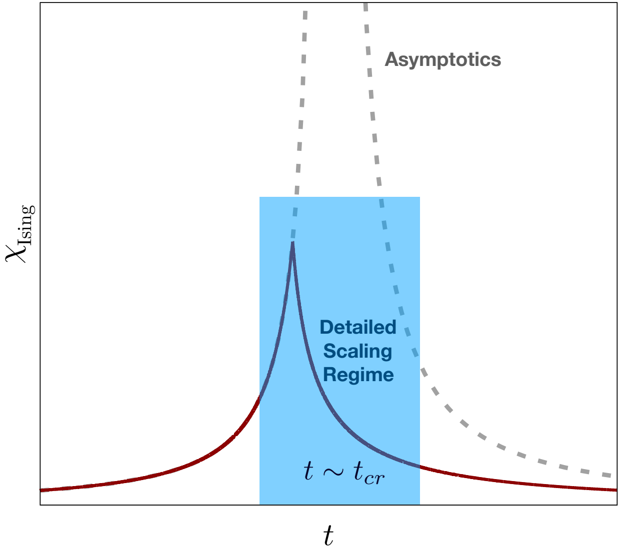
$$\Delta e_{\rm Ising} \leftrightarrow \frac{\Delta n}{n_c} = -\frac{t}{\tau_Q} \qquad \Delta M_{\rm Ising} \leftrightarrow \frac{\Delta s}{s_c} \sim \Delta_s$$

- The scaling of the Ising EOS implies a scaling in time

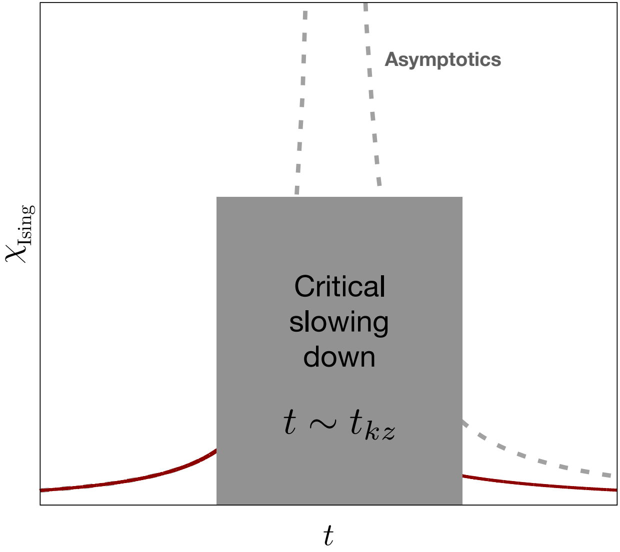
$$\xi = \ell_o \left(\frac{t}{\tau_Q}\right)^{-a\nu} \times \underbrace{f_\xi(t/t_{cr})}_{\text{scaling func}} \qquad t_{cr} \equiv \Delta_s^{\frac{1-\alpha}{\beta}} \tau_Q$$

t_{cr} is a new time scale that quantifies the missing of CP

Detailed scaling regime happens when $t \sim t_{kz}$

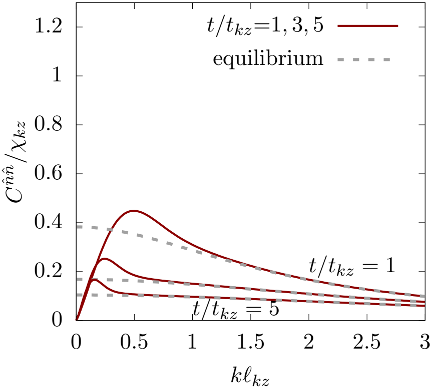


Detailed scaling regime is obscured by KZ dynamics if $t_{kz} \gg t_{cr}$



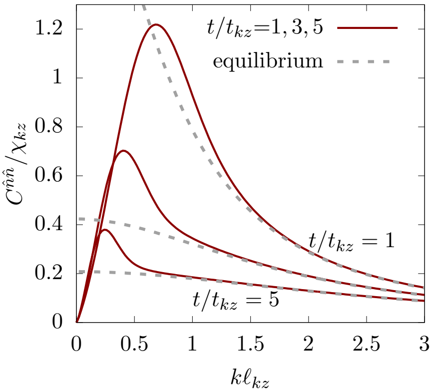
Missing the critical point further limits the flucts

Missing the CP



$t_{cr}/t_{kz} = 1$

Right through the CP



$t_{cr}/t_{kz} = 0$

Conclusions

- There are two scales t_{kz} and t_{cr}, they compete with each other

$$t_{kz} = \epsilon^{0.26}\tau_Q \qquad \text{vs} \qquad t_{cr} = {\Delta_s}^{2.72}\tau_Q$$

Numerically with $\tau_Q = 10$ fm, $\epsilon = 0.2$, $\Delta_s = 0.3$

$$t_{kz} = 6.58 \text{ fm} \qquad \gg \qquad t_{cr} = 0.38 \text{ fm}$$

So Kibble-Zurek dynamics is more important than detailed scaling

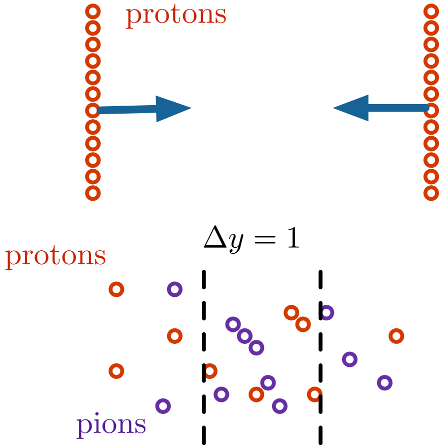
$$\epsilon \gg 0.005 \left(\frac{\Delta_s}{0.6}\right)^{10.4}$$

- $C^{\hat{n}\hat{n}}$ is a non-flow, and is quite local near CP

$$\ell_o \qquad \ll \qquad \ell_o\epsilon^{-0.19} \qquad \ll \qquad \ell_o\epsilon^{-0.5} \qquad \ll \qquad R$$

$$\underbrace{1.2\,\text{fm}}_{\text{micro-length } \ell_o} \qquad \underbrace{1.6\,\text{fm}}_{\text{Kibble-Zurek } \ell_{kz}} \qquad \underbrace{2.7\,\text{fm}}_{\text{cutoff } \ell_{\text{max}}} \qquad \underbrace{10\,\text{fm}}_{\text{nucleus size}}$$

Do current kurtosis measurements probe the CP?



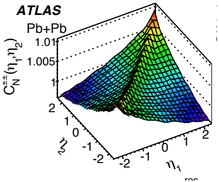
Measure $P(n_B|n_\pi)$ in Δy bin and characterize with kurtosis.
Reflects stopping not critical point.

I would *first* fix n_B/n_π in Δy bin, and then study *non-flow* correlations

Measure correlations at fixed n_B/s

$$C(\eta_1, \eta_2) = \frac{\left\langle \frac{dN}{d\eta_1} \frac{dN}{d\eta_2} \right\rangle}{\left\langle \frac{dN}{d\eta_1} \right\rangle \left\langle \frac{dN}{d\eta_2} \right\rangle}$$

Correlation function



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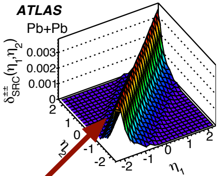
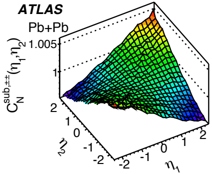
Short Range = "Non-flow"



Find the CP in here
at lower energy

+

Long range rapidity flucts



Look for short range in η, baryon/entropy correlations (for fixed $\bar{n}/\bar{s}$) with momentum scale

$$\Delta p \sim 50\,\mathrm{MeV} \sim \frac{\hbar}{\ell_{kz}} \sim \frac{\hbar}{2\,\mathrm{fm}}$$