Hydrodynamic fluctuations at finite chemical potential

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Work in collaboration with T. Schäfer
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Motivation

- **Hydrodynamics** has become the ‘workhorse’ of dynamical modeling of ultra-relativistic heavy-ion collisions

- Multiplicity of particles in HIC \( dN/dy \sim O(10^{2-4}) \)
  - Large enough for hydrodynamics to be applicable
  - Sufficiently small that one cannot neglect fluctuations

- In critical dynamics thermal fluctuations become crucial to understand experimental results
Brownian motion and Einstein relation

The brownian motion of a massive particle

$$\frac{d\vec{p}}{dt} = -\alpha_D \vec{p} + \vec{s}(t)$$

Drag coefficient

$$\langle s_i(t) s_j(t') \rangle = \kappa \delta_{ij} \delta(t-t')$$

Particle eventually thermalizes

$$\langle p^2 \rangle = 2 m T$$

Drag coefficient and noise are related via the Einstein relation (fluctuation-dissipation theorem)

$$\kappa = \frac{m}{\alpha_D}$$
A missing element in our hydro models: stochastic hydrodynamics

So far hydrodynamic modeling in HIC tells us about dissipation without including fluctuations
⇒ important for correlations!!!
Linearized hydro fluctuations

Hydrodynamics requires fluctuations (Landau & Lifshitz, 1957)

\[ T^{\mu\nu} = T_b^{\mu\nu} + \delta T^{\mu\nu} + S^{\mu\nu} \]

Evolving background

\[ D_\mu T_b^{\mu\nu} = 0 \]

Fluctuations + noise sources

\[ \delta T^{\mu\nu} = \delta T^{\mu\nu} (\delta \epsilon, \delta u^\mu) \]

\[ D_\mu (\delta T^{\mu\nu} + S^{\mu\nu}) = 0 \]
**Linearized hydro fluctuations**

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\]

\[
D_\mu (\delta T^{\mu\nu} + S^{\mu\nu}) = 0
\]

**White random noise**

\[
\langle S^{\mu\nu} \rangle = 0
\]

\[
\langle \{ S^{\mu\nu}(x), S^{\lambda\delta}(x') \} \rangle = 2T \delta^{(4)}(x - x') \left( \eta \Delta^{\mu\nu \lambda\delta} + \zeta \Delta^{\mu\nu} \Delta^{\lambda\delta} \right)
\]
Challenges with the stochastic hydro approach

A naive discretization of the white noise correlators implies

$$\langle S S \rangle \sim \delta(t - t') \delta^{(3)}(\vec{x} - \vec{x}') \sim (\Delta t a^3)^{-1}$$

Lattice size $a$ limits the spatial extent of hydro modes to propagate

Maximum UV cutoff $\Lambda$

$$|S| \sim (\Delta t a^3)^{-1/2} \sim \frac{\Lambda^{3/2}}{\sqrt{\Delta t}}$$

Noise terms have a large magnitude & numerically difficult to implement
Challenges with the stochastic hydro approach

In Equilibrium

- Fluctuations of hydro variables are related with thermodynamic properties of the system

\[ \langle \frac{\delta p \delta p}{c_s^2} \rangle \sim T^2 c_p \]

- Fluctuations of conjugate hydro variables vanish

\[ \langle \delta p \delta (s/n) \rangle = 0 \]
\[ \langle \delta p \delta u^i \rangle = 0 \]

For rapidly expanding plasmas (out-of-equilibrium) correlations can appear
Hydrokinetics of a charged expanding fluid
Hydrokinetics: basic idea

\[ L \sim c_s \tau \]

\[ \gamma_\eta = \frac{\eta}{\epsilon + p} \]

\[ \gamma_\eta k^2 \hspace{1cm} v.s. \hspace{1cm} (c_s \tau)^{-1} \]

Modes equilibrate if

\[ k \gg k_* = \sqrt{\frac{1}{\gamma_\eta \tau}} \]

Modes deviate from equilibrium for

\[ k \sim \frac{1}{\sqrt{\gamma_\eta \tau}} \]

Effective theory for modes with

\[ k \sim k_* \]

Akamatsu, Teaney, Mazeliauskas (2016)
Hydrokinetics

Akamatsu et. al. (2016) : non-equilibrium noise contributions to the energy-momentum tensor of a fluid undergoing Bjorken flow

\[
\frac{\langle T_{zz} \rangle}{\epsilon_0 + p_0} = \frac{p_0}{\epsilon_0 + p_0} - \frac{4}{3\tau} \gamma \eta + \frac{\langle G_z^2 \rangle}{\epsilon_0 + p_0} + \frac{8}{9\tau^2} \left( \lambda_1 - \eta_0 \tau_\pi \right)
\]

\[\sim \langle v_z^2 \rangle\]
Hydrokinetics

Akamatsu et. al. (2016) : non-equilibrium noise contributions to the energy-momentum tensor of a fluid undergoing Bjorken flow

$$\frac{\langle T^{zz} \rangle}{\epsilon_0 + p_0} = \frac{p_0}{\epsilon_0 + p_0} - \frac{4}{3\tau} \gamma \eta + \frac{\langle G^2 \rangle}{\epsilon_0 + p_0} + \frac{8}{9\tau^2} \frac{(\lambda_1 - \eta_0 \tau_\pi)}{\epsilon_0 + p_0}$$

First order  3/2 order  2$^\text{nd}$ order

For arbitrary values of $\eta/s$

First order  $>$  3/2 order  $>$  2$^\text{nd}$ order
Hydrokinetics at finite $\mu$

$$\delta \phi_a = (\delta p/c_s, g_i, \delta q) \sim \delta \left( \frac{s}{n} \right)$$

Navier-Stokes-Langevin equations

$$\frac{d}{dt} \delta \phi_a + k A_{ab} \delta \phi_b + k^2 D_{ab} \delta \phi_b = P_{ab} \delta \phi_b + \xi_a$$

The acoustic matrix $A$ has 5 hydro modes + 5 eigenvalues

<table>
<thead>
<tr>
<th>Hydrodynamic eigenmode</th>
<th>Eigenvector</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sound modes $\phi_{\pm}$</td>
<td>$\frac{1}{\sqrt{2}} \left( 1, \pm \hat{k}, 0 \right)$</td>
<td>$\pm i \nu_a$</td>
</tr>
<tr>
<td>Diffusive mode $\phi_d$</td>
<td>$(0, 0, 1)$</td>
<td>0</td>
</tr>
<tr>
<td>Shear modes $\phi_{T_i}$ $i = 1, 2$</td>
<td>$(0, \hat{e}_{T_i}, 0)$</td>
<td>0</td>
</tr>
</tbody>
</table>
Hydrokinetics at finite $\mu$

Deterministic equation for 2 point matrix correlator

$$C_{AB} = \frac{1}{2} \langle \{ \phi_A(t, \mathbf{k}), \phi_B^{\dagger}(t, \mathbf{k}) \} \rangle$$

$$A = \pm, d, T_{1,2}$$

$$\partial_0 C + [A, C] + \{D, C\} = \mathcal{P} C + C \mathcal{P}^{\dagger} + \frac{1}{2} (\mathcal{N} + \mathcal{N}^{\dagger})$$

Evolution + reactive + diffusive = sources + noise correlator

$$C_{AB} =$$

$$\begin{pmatrix}
C_{++} & C_{+-} & C_{+T_1} & C_{+T_2} & C_{+d} \\
C_{-+} & C_{--} & C_{-T_1} & C_{-T_2} & C_{-d} \\
C_{T_1+} & C_{T_1-} & C_{T_1T_1} & C_{T_1T_2} & C_{T_1d} \\
C_{T_2+} & C_{T_2-} & C_{T_2T_1} & C_{T_2T_2} & C_{T_2d} \\
C_{d+} & C_{d-} & C_{dT_1} & C_{dT_2} & C_{dd}
\end{pmatrix}$$
Hydrokinetics at finite $\mu$

Deterministic equation for 2 point matrix correlator

$$C_{AB} = \frac{1}{2} \langle \{ \phi_A(t, k), \phi_B^\dagger(t, k) \} \rangle$$

$$\partial_0 C + [A, C] + \{D, C\} = PC + CP^\dagger + \frac{1}{2} (N + N^\dagger)$$

Evolution + reactive + diffusive = sources + noise correlator

$$C_{AB} = \begin{pmatrix}
C_{++} & \times & \times & \times & \times & \times \\
\times & C_{+-} & \times & \times & \times & \times \\
\times & \times & C_{TT_1} & \times & \times & \times \\
\times & \times & \times & C_{TT_1T_1} & \times & \times \\
\times & \times & \times & \times & C_{TT_2T_2} & \times \\
\times & \times & \times & \times & \times & C_{dd}
\end{pmatrix}$$
Hydrokinetics at finite $\mu$

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$$C_{AB} = \frac{1}{2} \left\langle \{ \phi_A(t, \mathbf{k}), \phi_B^{\dagger}(t, \mathbf{k}) \} \right\rangle$$

$$\partial_0 C + [\mathcal{A}, C] + \{\mathcal{D}, C\} = \mathcal{P} C + C \mathcal{P}^{\dagger} + \frac{1}{2} (\mathcal{N} + \mathcal{N}^{\dagger})$$

Evolution + reactive + diffusive = sources + noise correlator

$$C_{AB} = \begin{pmatrix}
C_{++} & \cancel{C_{+-}} & \cancel{C_{+T_1}} & \cancel{C_{+T_2}} & \cancel{C_{+d}} \\
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\cancel{C_{T_1+}} & \cancel{C_{T_1-}} & C_{T_1 T_1} & \cancel{C_{T_1 T_2}} & \cancel{C_{T_1 d}} \\
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\cancel{C_{d+}} & \cancel{C_{d-}} & C_{dT_1} & C_{dT_2} & C_{dd}
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Hydrokinetics at finite $\mu$

Deterministic equation for 2 point matrix correlator

$$C_{AB} = \frac{1}{2} \langle \{ \phi_A(t, k), \phi_B^\dagger(t, k) \} \rangle$$

$$\partial_0 C + [A, C] + \{D, C\} = \mathcal{P} C + C \mathcal{P}^\dagger + \frac{1}{2} (\mathcal{N} + \mathcal{N}^\dagger)$$

Evolution + reactive + diffusive = sources + noise correlator

Close to equilibrium (asymptotic regime)

$$C_{AA} = C_{eq} \left( 1 + \frac{\#}{D_{AA} k^2 \tau} \right)$$

$$C_{dT_i} = \# \frac{C_{eq}}{(D_0 + \gamma_\eta) k^2 \tau}$$
Hydrokinetic contributions

We restrict here to the conformal case.

- The hydrodynamic fluctuating contributions to the longitudinal component of the particle current is

\[ \langle J^z \rangle = \frac{D_0}{\alpha_2} \mathcal{E}^z + \frac{1}{\epsilon_0 + p_0} \left( \langle \frac{\delta p}{\nu^2} g^z \rangle - \langle \delta q \ g^z \rangle \right) \]

\[ \sim \int^\Lambda d^3k \ \underbrace{(C_{++}(t,k) - C_{--}(t,k))}_{\#/(\gamma_n \ k^2 \tau)} \]

\[ \sim \int^\Lambda d^3k \ \underbrace{(C_{dT_1}(t,k) + C_{dT_2}(t,k))}_{\#/([D + \gamma_n] \ k^2 \tau)} \]

Linearly divergent integrals which are regularized · Martinez and Schaefer (2018)
Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

\[
\langle J^z \rangle \overline{E} \approx D_0 + \alpha_2 \frac{T}{w^2} \frac{\Lambda}{\pi^2} \left( \frac{1}{8 \eta_0} + \frac{T c_p}{3 \bar{H}(D_0 + \gamma \eta_0)} \right) \\
- \frac{T \tau}{w^2} \left( \frac{0.04282}{(\gamma \eta_0 \tau)^{3/2}} + \frac{T c_p}{\bar{H}} \frac{0.008}{[(D_0 + \gamma \eta_0) \tau]^{3/2}} \right)
\]

- Universal low frequency behaviour, i.e. modes with \( k < \Lambda \).
- Renormalized diffusion coefficient coincides with the static limit (diagrammatic approach)
- Non-universal high frequency behaviour, i.e. modes with \( k > \Lambda \).
- Long time tails \( O(\tau^{-3/2}) \)
Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

\[
\frac{\langle \vec{J}^z \rangle}{\mathcal{E}^z} = D_0 + \alpha_2 \frac{T}{\bar{w}^2} \frac{\Lambda}{\pi^2} \left( \frac{1}{8\gamma \eta_0} + \frac{T c_p}{3\bar{H}(D_0 + \gamma \eta_0)} \right) - \frac{T \tau}{\bar{w}^2} \left( \frac{0.04282}{(\gamma \eta_0 \tau)^{3/2}} + \frac{T c_p}{\bar{H}} \frac{0.008}{[(D_0 + \gamma \eta_0)\tau]^{3/2}} \right)
\]

If one run some numbers one finds that effect of fluctuations is 10% of the first order gradient expansion
Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

\[ \langle J^\tau \rangle = \bar{n}(\Lambda) + \frac{1}{4\pi^2} \frac{T}{\bar{w}} \Lambda^3 + \frac{T}{\bar{w}} \frac{0.04808}{(\gamma_\eta \tau)^{3/2}} \]

- Low frequency behaviour, i.e. modes with \( k < \Lambda \).
  - Renormalized particle density is the same as in the static case

- Non-universal high frequency behaviour, i.e. modes with \( k > \Lambda \).
  - Long time tails \( O(\tau^{-3/2}) \)
Conclusions

- We studied the role of hydrodynamic fluctuations on different energy, momentum and density correlation functions.

- Hydrokinetics has been generalized for rapidly expanding fluids at finite chemical potential.

- The mix of the mix shear-diffusive mode as well as the sound modes modify the tails of the particle current.

- We determine the universal short length behaviour of the hydrodynamic fluctuations which renormalize the particle density and diffusion coefficient.
Outlook

• Non-conformal fluid at finite chemical potential
  Martinez and Schaefer 19xx.xxxxx

• Role of critical fluctuations in the vicinity of a second order phase transition:
  ⇒ enhancement of bulk viscosity and diffusion coefficient near critical point

• Kibble-Zurek dynamics at finite chemical potential
Backup slides
Noise correlators

\[ \langle S^\mu_\nu (x_1^0, x_1) S^{\alpha_\beta} (x_2^0, x_2) \rangle = 2T \left[ 2 \eta_0 \Delta^{\mu_\nu, \alpha_\beta} \right] \times \delta (x_1^0 - x_2^0) \delta^{(3)} (x_1 - x_2), \]

\[ \langle I^\mu (x_1^0, x_1) I^{\nu} (x_2^0, x_2) \rangle = 2T \sigma_0 \Delta^{\mu_\nu} \]

\[ \delta (x_1^0 - x_2^0) \delta^{(3)} (x_1 - x_2), \]

\[ \langle S^{\mu_\nu} (x_1^0, x_1) I^\lambda (x_2^0, x_2) \rangle = 0. \]
Hydrodynamical variables fluctuate (Landau & Lifshitz, 1957)

\[ \langle \delta v_i(t, \vec{x}) \delta v_j(t, \vec{x}') \rangle = \frac{T}{\rho} \delta^{(3)}(\vec{x} - \vec{x}') \]

Linearized hydrodynamics propagates fluctuations of different modes, e.g., shear and sound modes

 shear

\[ \langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{2T}{\rho} (\delta_{ij} - \hat{k}_i \hat{k}_j) \frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \]

 sound

\[ \langle \delta v_i^L \delta v_j^L \rangle_{\omega,k} = \frac{2T}{\rho} \hat{k}_i \hat{k}_j \frac{\omega k^2 \Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2 \Gamma)^2} \]

\[ v = v_T + v_L: \quad \nabla \cdot v_T = 0, \quad \nabla \times v_L = 0 \]

\[ \nu = \eta/\rho, \quad \Gamma = \frac{4}{3} \nu + \ldots \]
Hydrokinetics at finite $\mu$: Bjorken flow

- For the Bjorken case the equations of motion of the equal time symmetric correlators are

$$\partial_0 C + [A, C] + \{D, C\} = P C + C P^\dagger + \frac{1}{2} (N + N^\dagger)$$

\[
\begin{align*}
\partial_\tau \tilde{C}_{\pm\pm} + \frac{4}{3} \gamma_\eta K^2 \tilde{C}_{\pm\pm} &= \tilde{N}_{\pm\pm} - \left(\frac{2 + v_a^2 + \cos^2 \theta_K}{\tau}\right) \tilde{C}_{\pm\pm} + \frac{\hat{K} \cdot \mathcal{E}}{w} \left(1 + v_a^2\right) \tilde{C}_{\pm\pm}, \\
\partial_\tau \tilde{C}_{T_1 T_1} + 2 \gamma_\eta K^2 \tilde{C}_{T_1 T_1} &= \tilde{N}_{T_1 T_1} - \frac{2}{\tau} \tilde{C}_{T_1 T_1}, \\
\partial_\tau \tilde{C}_{T_2 T_2} + 2 \gamma_\eta K^2 \tilde{C}_{T_2 T_2} &= \tilde{N}_{T_2 T_2} - \frac{2 \left(1 + \sin^2 \theta_K\right)}{\tau} \tilde{C}_{T_2 T_2}, \\
\partial_\tau \tilde{C}_{dd} + 2 D K^2 \tilde{C}_{dd} &= \tilde{N}_{dd} - \frac{2}{\tau} \tilde{C}_{dd}, \\
\partial_\tau \tilde{C}_{dT_1} + (\gamma_\eta + D) K^2 \tilde{C}_{dT_1} &= -\frac{2}{\tau} \tilde{C}_{dT_1} + \frac{1}{w} \hat{e}_{T_1} \cdot \mathcal{E} \left(\tilde{C}_{dd} - v_a \tilde{C}_{T_1 T_1}\right), \\
\partial_\tau \tilde{C}_{dT_2} + (\gamma_\eta + D) K^2 \tilde{C}_{dT_2} &= -\frac{2 + \sin^2 \theta_K}{\tau} \tilde{C}_{dT_2} + \frac{1}{w} \hat{e}_{T_2} \cdot \mathcal{E} \left(\tilde{C}_{dd} - v_a \tilde{C}_{T_1 T_2}\right)
\end{align*}
\]