Linear Electron Response to An Ion (Analytical Approach)

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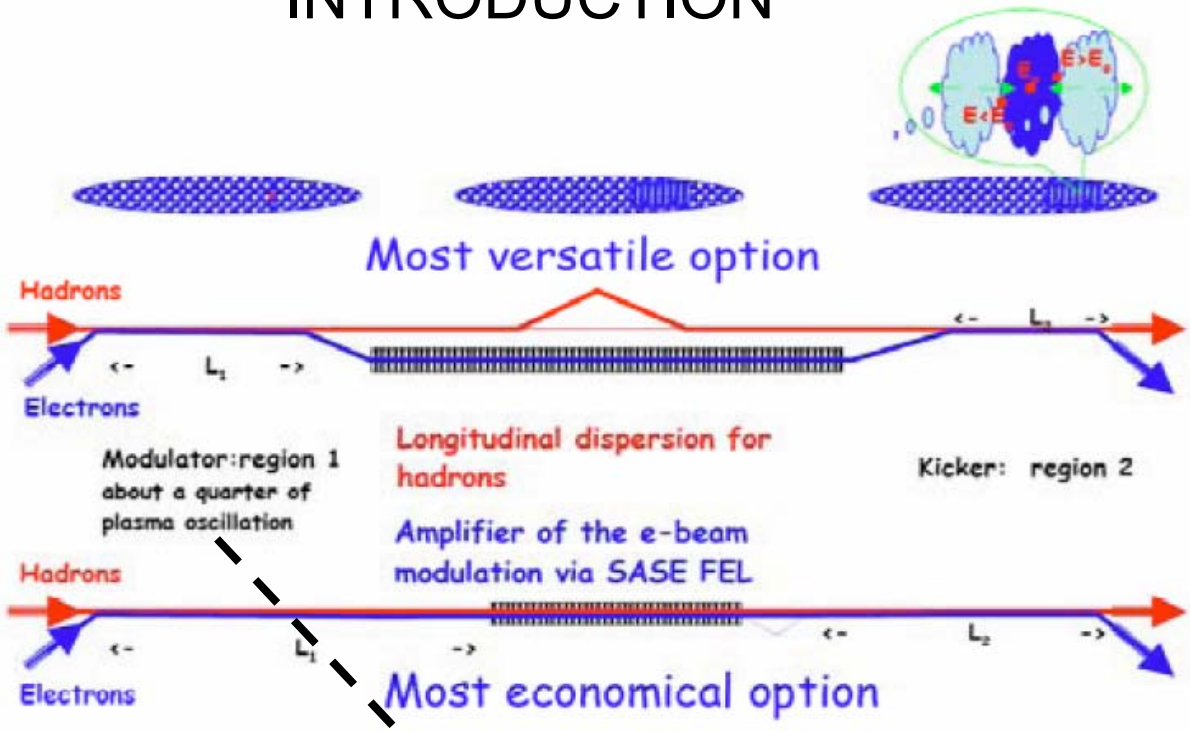
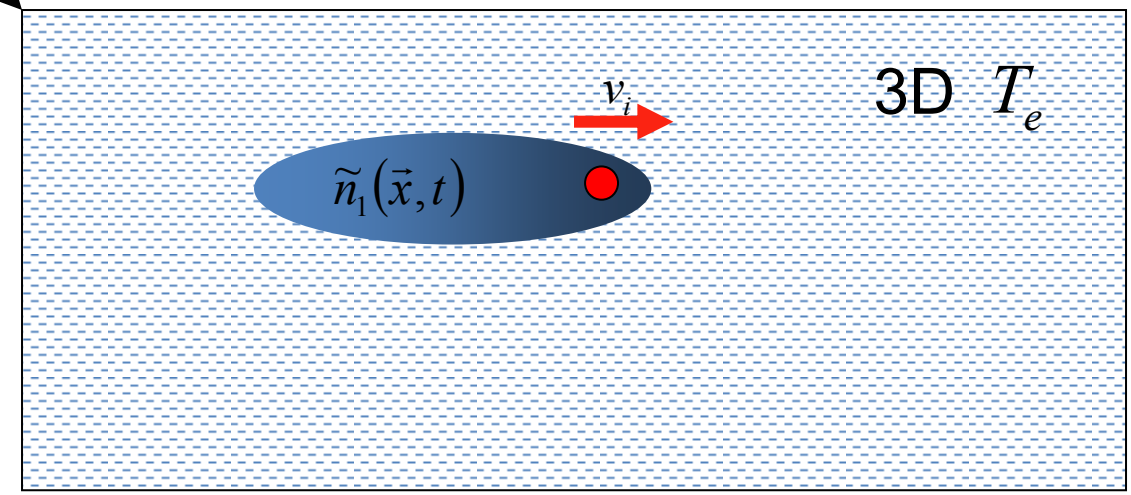
INTRODUCTION



Purpose: Calculate the density distribution of the induced electron clouds and its time dependence.



Equation Of Motion

- Linearized Vlasov Equation

uniform space distribution

$$\frac{\partial}{\partial t} f_1(\vec{x},\vec{v},t) + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} f_1(\vec{x},\vec{v},t) - \frac{e\vec{E}}{m_e}\frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{x},t)}{\varepsilon_o}$$

$$\rho(\vec{x},t) = eZ\delta(\vec{x}-\vec{v}_0 t) - e\int f_1(\vec{x},\vec{v},t)d\vec{v}$$

Defined as $\tilde{n}_1(\vec{x},t)$

- Fourier transform the equation of motion to k space.

$$\frac{\partial}{\partial t} f_1(\vec{k},\vec{v},t) + i\vec{k}\cdot\vec{v} f_1(\vec{x},\vec{v},t) + i\frac{e}{m_e}\Phi(\vec{k},t)\vec{k}\cdot\frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\Phi(\vec{k},t) = \frac{e}{\varepsilon_o k^2}\left[Z_i - \tilde{n}_1(\vec{k},t)\right]$$

Ion velocity

- Integrating the equation over $\vec{v}$, one gets

$$\tilde{n}_1(\vec{k},t) = \omega_p^2 \int_0^t \left[\tilde{n}_1(\vec{k},t_1) - Z_i\right](t_1 - t)g(\vec{k}(t_1 - t))dt_1$$

$$g(\vec{u}) \equiv \int f_0(\vec{v}) e^{i\vec{u}\cdot(\vec{v}+\vec{v}_0)} d^3v$$

Solution For The Second Power of Lorenzian Distribution

- Equilibrium electron velocity distribution

$$f_0(\vec{v}) = \frac{1}{\pi^2 \sigma_x \sigma_y \sigma_z} \frac{1}{\left(1 + \frac{v_x^2}{\sigma_x^2} + \frac{v_y^2}{\sigma_y^2} + \frac{v_z^2}{\sigma_z^2}\right)^2}$$

which gives

$$g(\vec{u}) = \exp(i\vec{u} \cdot \vec{v}_0) \exp\left(-\sqrt{(u_x\sigma_x)^2 + (u_y\sigma_y)^2 + (u_z\sigma_z)^2}\right)$$

- For Lorenzian distribution, the integral equation reduced to an 2nd order ODE

$$\ddot{H}(\vec{k},t) = -\omega_p^2 H(\vec{k},t) + Z_i \omega_p^2 e^{-\lambda(\vec{k})t}$$

$$H(\vec{k},t) \equiv \tilde{n}_1(\vec{k},t) \exp\left[-\lambda(\vec{k}) \cdot t\right]$$

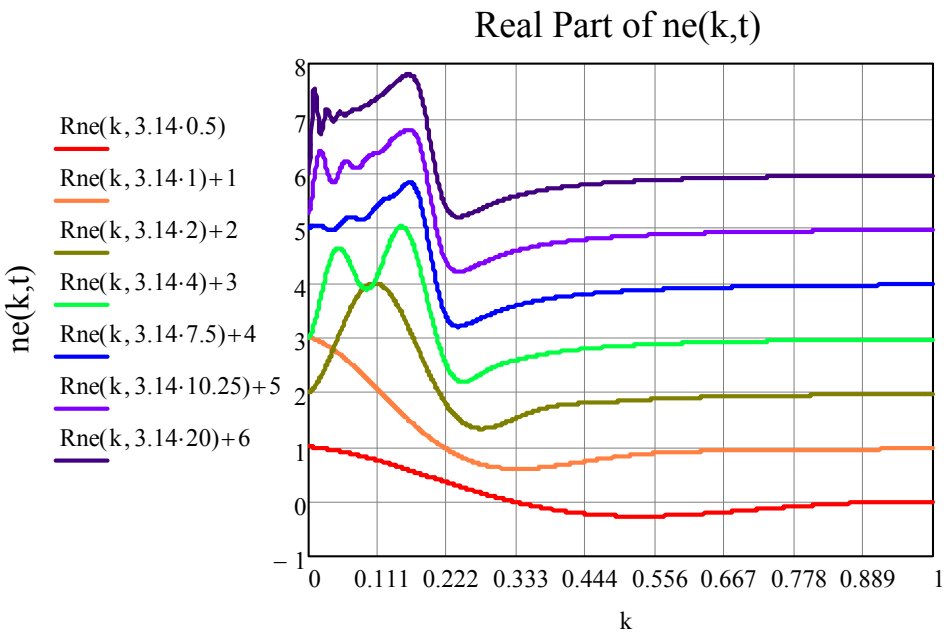
$$\lambda(\vec{k}) \equiv i\vec{k} \cdot \vec{v}_0 - \sqrt{(k_x\sigma_x)^2 + (k_y\sigma_y)^2 + (k_z\sigma_z)^2}$$

- The electron response to a certain wavevector is

$$\tilde{n}_1(\vec{k},t) = \frac{\omega_p^2 Z_i}{\omega_p^2 + \lambda(\vec{k})^2} \left\{1 - \exp\left(\lambda(\vec{k}) \cdot t\right) \cdot \left[\cos(\omega_p t) - \frac{\lambda(\vec{k})}{\omega_p} \cdot \sin(\omega_p t)\right]\right\}$$

Landau Damping

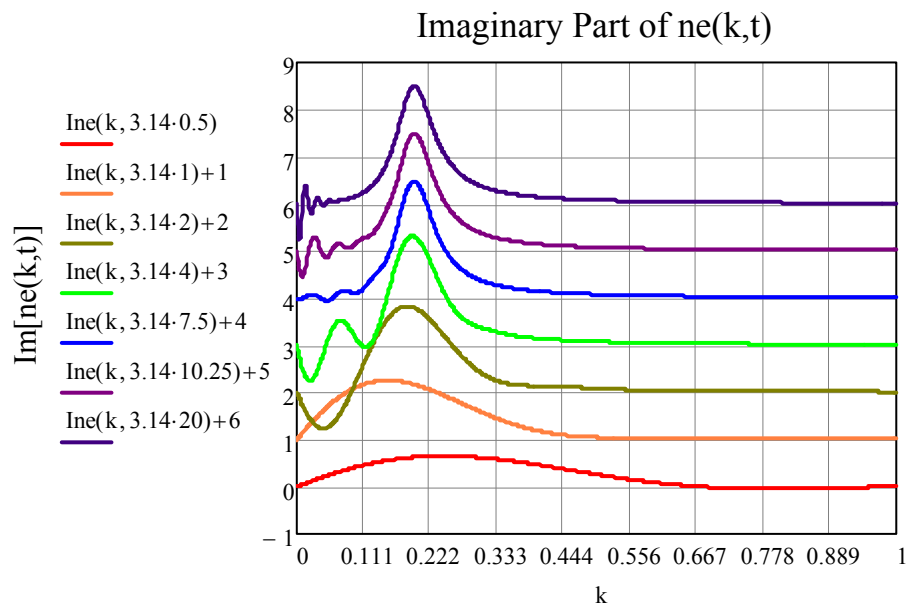
Wavelength Depend. of the Electron Response



sqrt[(kx*rD)**2+(ky*rD)**2+(kz*rD)**2]

The wave vector k is along the moving direction of the ion.

$$\frac{\vec{k}\cdot\vec{v}}{kv}=1$$



sqrt(kx*kx+ky*ky+kz*kz)

$$\bar{k}=\sqrt{(k_x a_x)^2+(k_y a_y)^2+(k_z a_z)^2}$$

Solution in x space

The solution in x space can be obtained by Fourier transform.

$$\dot{\tilde{n}}_1(\vec{x},t) = \frac{Z_i \omega_p t \sin(\omega_p t)}{\pi^2 \sigma_x \sigma_y \sigma_z \left(t^2 + \frac{(x + v_{0x} t)^2}{\sigma_x^2} + \frac{(y + v_{0y} t)^2}{\sigma_y^2} + \frac{(z + v_{0z} t)^2}{\sigma_z^2} \right)^2}$$

If one uses the normalized variable

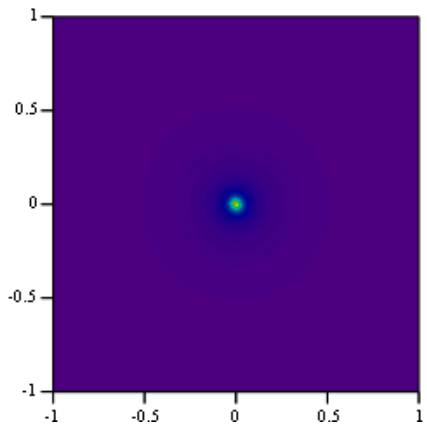
$$\psi \equiv \omega_p t \qquad \bar{v}_{0i} = \frac{\vec{v}_{0i}}{\sigma_i} \qquad \bar{x}_i = \frac{x_i}{r_{Di}}$$

the electron density fluctuation is

$$\boxed{\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 r_{Dx} r_{Dy} r_{Dz}} \int_0^{\psi} \frac{\psi_1 \sin(\psi_1) d\psi_1}{\left(\psi_1^2 + (\bar{x} + \bar{v}_{0x}\psi_1)^2 + (\bar{y} + \bar{v}_{0y}\psi_1)^2 + (\bar{z} + \bar{v}_{0z}\psi_1)^2 \right)^2}}$$

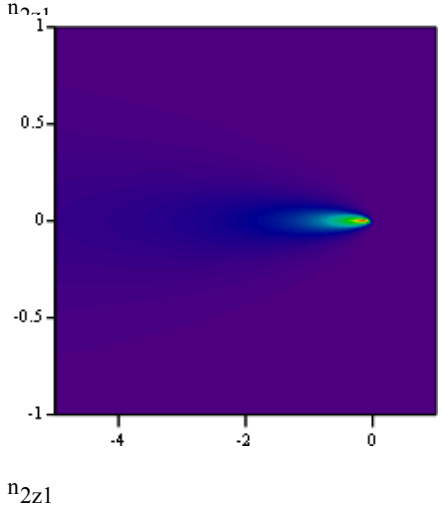
which can be expressed into sum of sine integral.

Density Response for an Rest Ion



$$\psi = \pi$$

$$\bar{v}_0 = 0$$



$$\psi = \pi$$

$$\bar{v}_0 = (0,0,10)$$

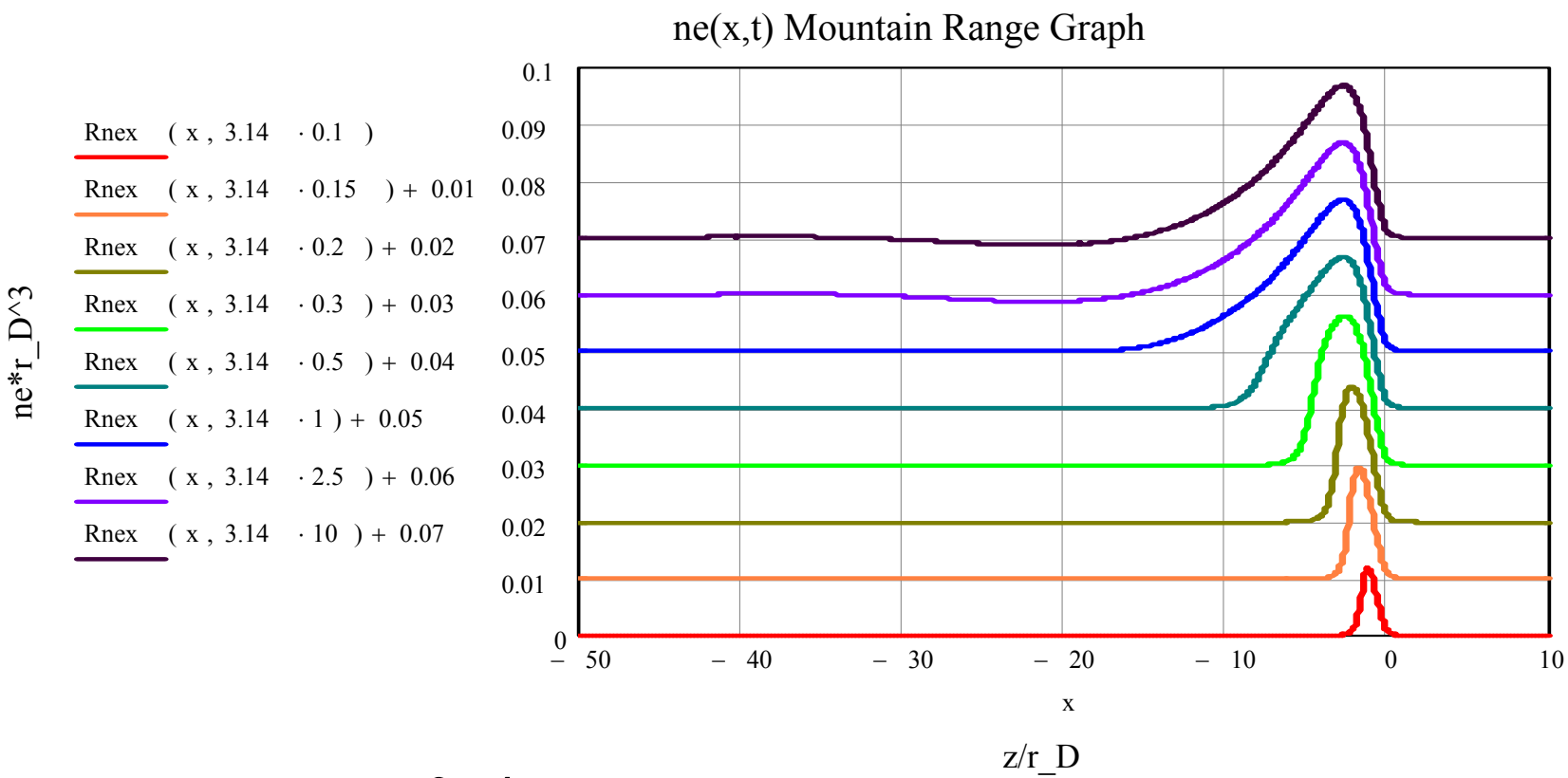
- The solution reduced to the well known Debye screening formula at the condition

$$t \to \infty \quad \vec{v}_0 = 0 \quad r_{Dx} = r_{Dy} = r_{Dz}$$

$$\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 r_D^3} \int_0^\infty \frac{\psi \sin(\psi) d\psi}{\left(\psi^2 + \bar{r}^2\right)^2}$$

$$= \frac{Z_i}{4\pi r_D^2} \cdot \frac{1}{r} \exp\left(-\frac{r}{r_D}\right)$$

Time Dependence of Electron Response



$$x = y = 0.4 r_D$$

Integration over space

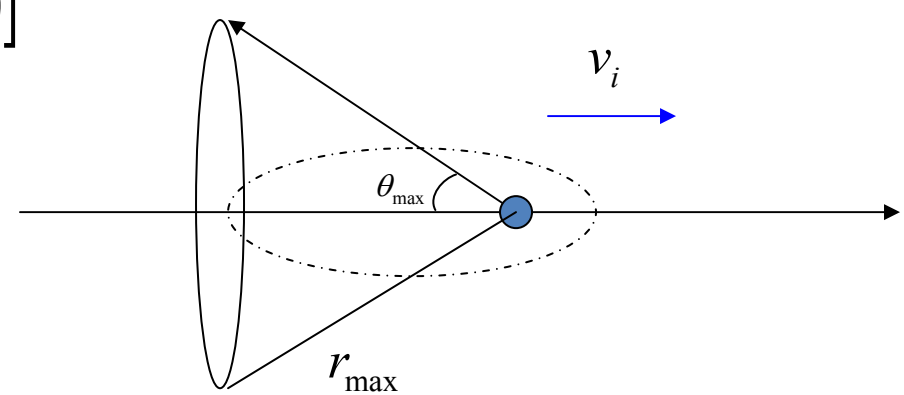
$$\widetilde{N}_1(\theta_{\max}, r_{\max}, t) = 2\pi \int_0^{r_{\max}} \int_{\pi-\theta_{\max}}^{\pi} n(\vec{x}, t)\bar{r}^2 d\theta d\bar{r}$$

$$X \equiv \cos(\pi - \theta_{\max})$$

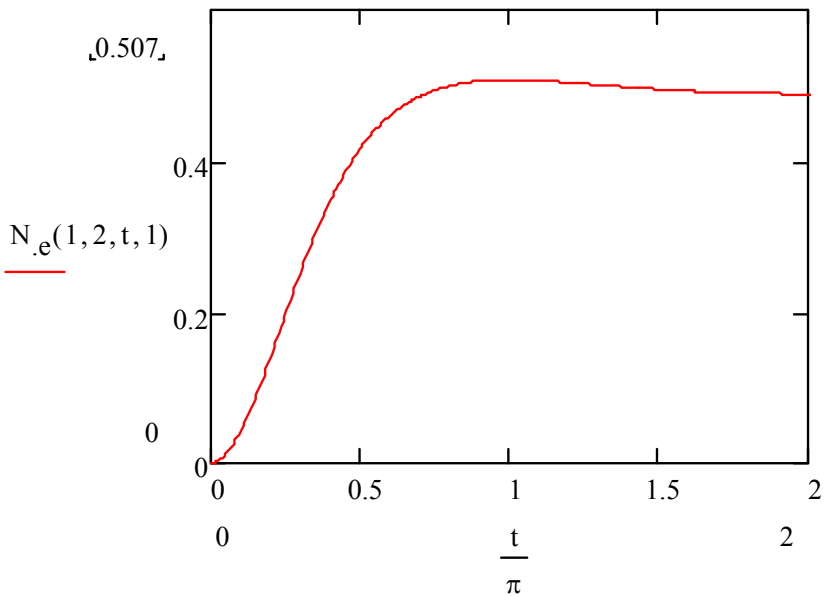
$$= \frac{-2Z_i}{\pi r_{Dx} r_{Dy} r_{Dz}} \int_0^{\psi_1} \psi \sin(\psi)\left[I(X, r_{\max}, t) - I(-1, r_{\max}, t)\right] d\psi$$

$$I(X, r_{\max}, t) = \frac{i}{4\psi^2 \bar{v}_{0z}\sqrt{1+\bar{v}_{0z}^2(1-X^2)}}\left[\bar{r}_- \ln\left(1-\frac{r_{\max}}{\bar{r}_-}\right) - \bar{r}_+ \ln\left(1-\frac{r_{\max}}{\bar{r}_+}\right)\right]$$

$$\bar{r}_\pm(X, t) = \psi\left[-\bar{v}_{0z}X \pm i\sqrt{1+\bar{v}_{0z}^2(1-X^2)}\right]$$

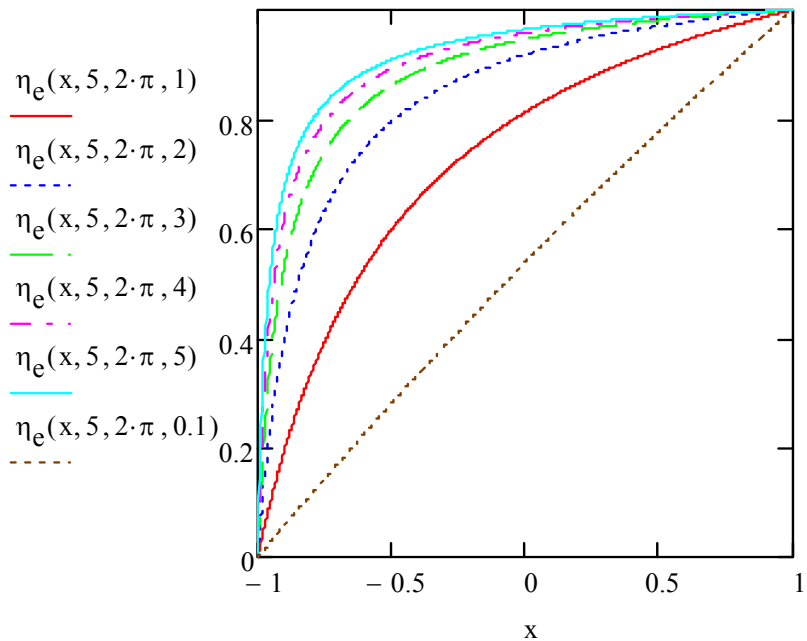


Total Charge Within 2 Debye Radius V.S. Time



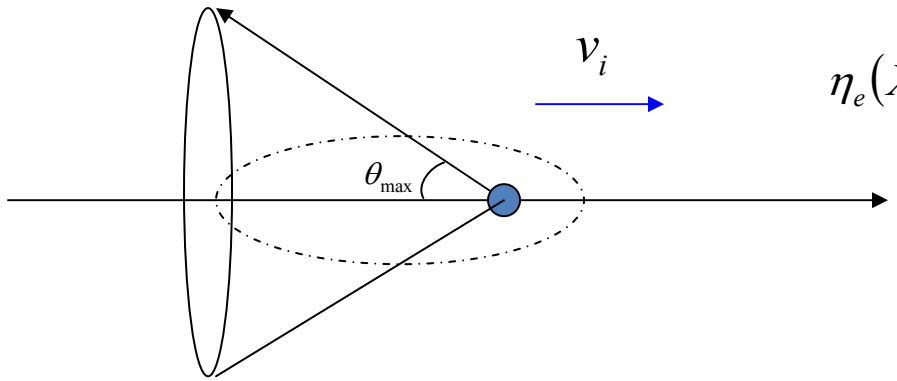
$v_i = \sigma_e$

Percentage of Charge Inside Cone V.S. Solid Angle of Cone



$$X \equiv \cos(\pi - \theta_{\max})$$

$$\eta_e(X, r_{\max}, t, v_i) \equiv \frac{N_e(X, r_{\max}, t, v_i)}{N_e(1, r_{\max}, t, v_i)}$$



Solution For The Third Power of Lorenzian Distribution

Thermal Velocity Distribution

$$f_0(\vec{v}) = \frac{4}{\pi^2 \beta_x \beta_y \beta_z} \left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2}\right)^{-3}$$

Solution in k space

$$\tilde{n}_1(\vec{k},t) = \frac{Z_i}{1+\eta\lambda^2}\Big\{1 + \tilde{c}_1 e^{-(\lambda+\gamma_0)\psi} - e^{-(\lambda-\frac{\gamma_0}{2})\psi} \times \Big[(1+\tilde{c}_1)\cos(\gamma_1\psi) + \frac{\lambda}{\gamma_1}\Big(1 - \frac{\gamma_0}{2\lambda}(1+3\tilde{c}_1)\Big) \times \sin(\gamma_1\psi)\Big]\Big\} \quad (22)$$

$$\bar{k} = \sqrt{(k_x a_x)^2 + (k_y a_y)^2 + (k_z a_z)^2}$$

$$\gamma_0(\bar{k}) \equiv \left(\sqrt{\bar{k}^2 + \frac{1}{27}} + \bar{k}\right)^{\frac{1}{3}} - \left(\sqrt{\bar{k}^2 + \frac{1}{27}} - \bar{k}\right)^{\frac{1}{3}}$$

$$\gamma_1(\bar{k}) \equiv \frac{\sqrt{3}}{2}\left[\left(\sqrt{\bar{k}^2 + \frac{1}{27}} + \bar{k}\right)^{\frac{1}{3}} + \left(\sqrt{\bar{k}^2 + \frac{1}{27}} - \bar{k}\right)^{\frac{1}{3}}\right]$$

$$\eta(\chi) \equiv 1 - \frac{2}{3 - i\chi\bar{v}_0} \qquad \chi \equiv \frac{\vec{k}\cdot\vec{v}_0}{\bar{k}\bar{v}_0}$$

$$\tilde{c}_1(\vec{k}) \equiv -\frac{\lambda^2(1-\eta) + \gamma_0^2}{3\gamma_0^2 + 1} \qquad \bar{v}_{0i} \equiv v_{0i}/\beta_i$$

$$\lambda(\vec{k}) = i\vec{k}\cdot\vec{v}_0 - \bar{k}$$

Fourier inversion will be done numerically.

Second power Lorentzian result:

$$\tilde{n}_1(\vec{k},t) = \frac{\omega_p^2 Z_i}{\omega_p^2 + \lambda(\vec{k})^2}\left\{1 - \exp(\lambda(\vec{k})\cdot t)\cdot\left[\cos(\omega_p t) - \frac{\lambda(\vec{k})}{\omega_p}\cdot\sin(\omega_p t)\right]\right\}$$

Numerical Solution for Maxwellian Plasma

Thermal Distribution

$$f_0(\vec{v}) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_x\sigma_y\sigma_z}\exp\left(-\frac{v_x^2}{2\sigma_x^2} - \frac{v_y^2}{2\sigma_y^2} - \frac{v_z^2}{2\sigma_z^2}\right)$$

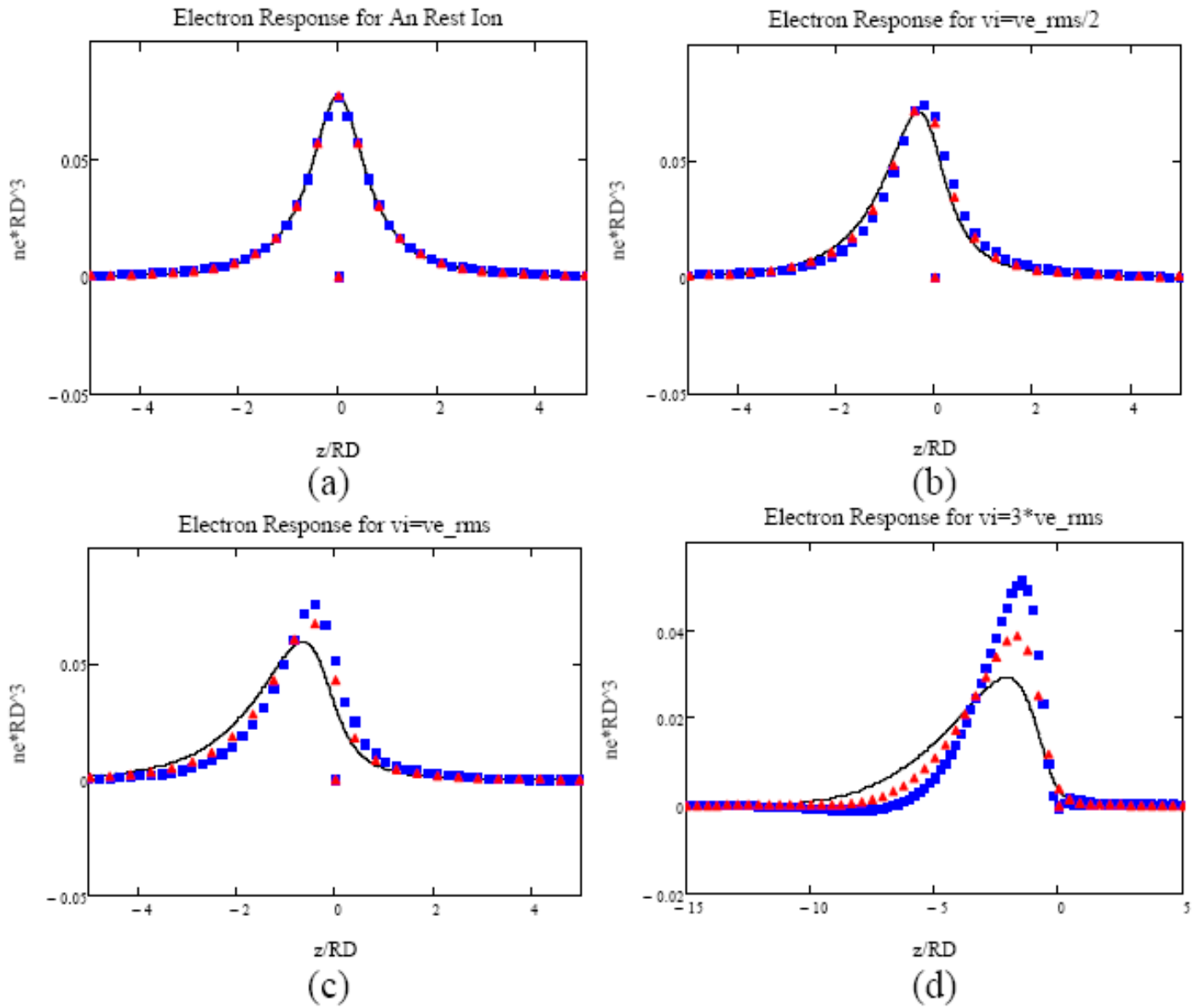
Volterra equation of the second type

$$\tilde{H}_1(\vec{k},t) = \int_0^{\psi} \tilde{H}_1(\vec{k},\tau)W(\psi-\tau)d\tau + G(\psi)$$

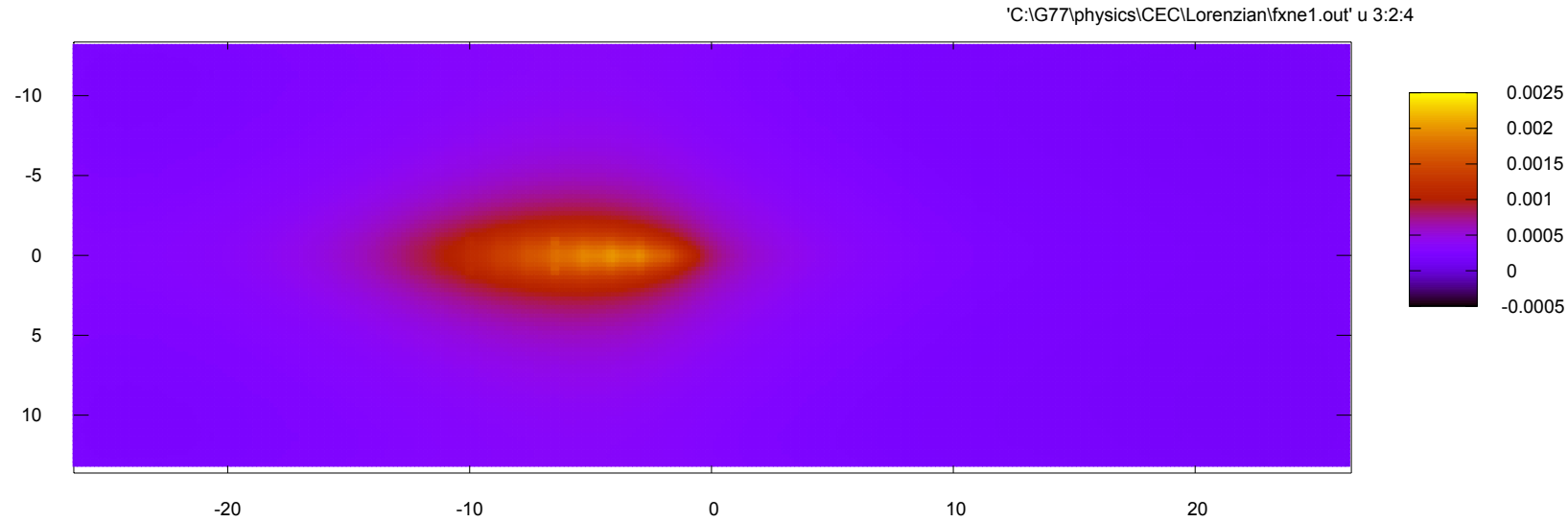
$$\tilde{H}_1(\vec{k},t) \equiv \tilde{n}_1(\vec{k},t)e^{-i\vec{k}\cdot\vec{v}_0 t} \qquad W(\tau) \equiv -\tau e^{-\frac{(\bar{k}\tau)^2}{2}}$$

$$\begin{aligned} G(\psi) \;=\; & \frac{Z_i}{\bar{k}^2}\Bigg\{e^{-i\bar{k}\bar{v}_0\chi\psi}\left[1 + i\sqrt{\frac{\pi}{2}}\bar{v}_0\chi\cdot w\left(\frac{\bar{v}_0\chi}{\sqrt{2}}\right)\right] \\ & -e^{-\frac{(\bar{k}\psi)^2}{2}}\left[1 + i\sqrt{\frac{\pi}{2}}\bar{v}_0\chi\cdot w\left(i\sqrt{2}\bar{k}\psi + \frac{\bar{v}_0\chi}{\sqrt{2}}\right)\right]\Bigg\} \end{aligned}$$

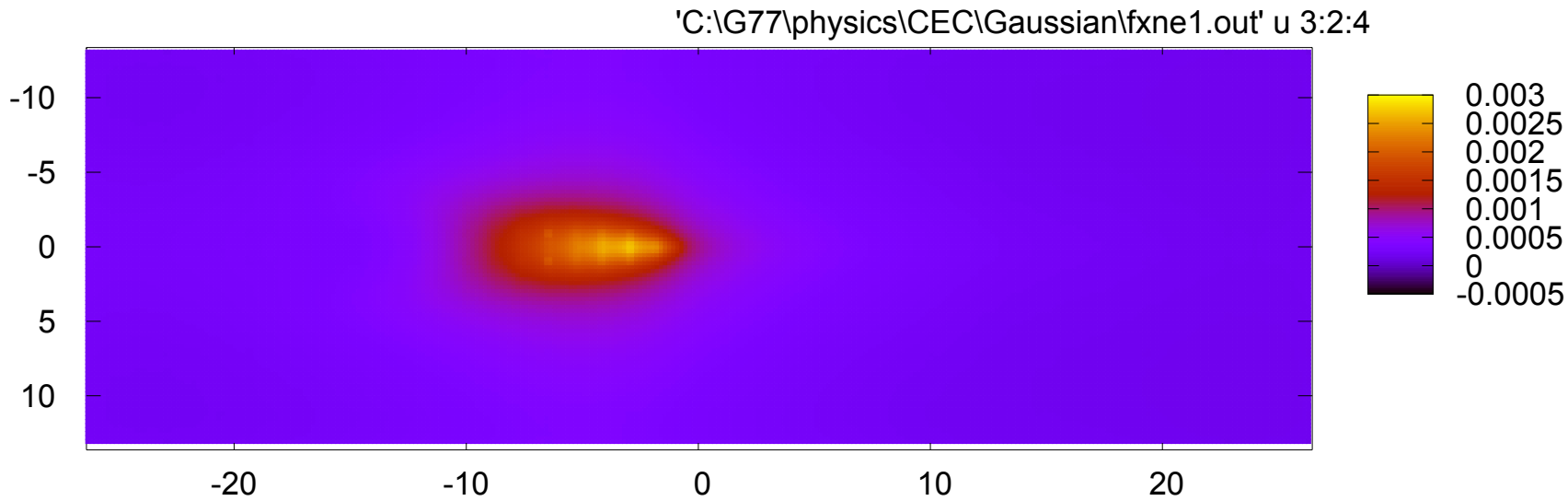
$$w(z) \equiv e^{-z^2}Erfc(-iz)$$


Electron Response for An Rest Ion
ne*RD^3
z/RD
(a)
Electron Response for vi=ve_rms/2
ne*RD^3
z/RD
(b)
Electron Response for vi=ve_rms
ne*RD^3
z/RD
(c)
Electron Response for vi=3*ve_rms
ne*RD^3
z/RD
(d)

2D Mapping of Electron Density



Lorenzian v_z=5sigma, x=y=0.41rD



Summary

- The analytical result reproduces the Debye screening formula for an rest ion and for slow ions($v_{0i} < \sigma_i$) it is close to the numerical Gaussian result. For fast ions, the Lorentzian approximation gives smoother result than Gaussian distribution. However, the solution for third power Lorentzian distribution behaves much closer to the Maxwellian plasma than the second power Lorentzian distribution.
- The Local electron density around the ion reaches its equilibrium after half plasma oscillation. (The plasma oscillation for small wavelength damps while the oscillation for long wavelength remains at large t.)
- The numerical Gaussian solution might be relatively fast compared with the PIC simulation and can be used for a quick estimation.