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STATISTICAL CORRELATIONS AND INTENSITY SPIKING
IN THE SASE FEL

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Abstract

In the linear regime before saturation, we describe the statistical correlations in the narrow band chaotic output of the SASE FEL. At a fixed position along the undulator axis, we derive joint probability distributions for the intensity in the output pulse to have values I_1 and I_2 at times t_1 and t_2 , and for the spectral intensity to have values \tilde{I}_1 and \tilde{I}_2 at frequencies ω_1 and ω_2 . Probability distributions for the peak values of intensity in the time and frequency domains are also determined.

Key words: self-amplified spontaneous-emission, statistical analysis, joint probability distributions, intensity peaks

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1. Introduction

There is great interest [1,2] in developing a high-peak power, femtosecond x-ray source utilizing a self-amplified spontaneous emission (SASE) free-electron laser (FEL). Recent experimental demonstrations of SASE in the infrared [3], visible [4] and ultraviolet [5] have increased the enthusiasm for the further development of this approach. The theory of high-gain single-pass free-electron lasers has been developing since the late 1970's [6-9]. Because SASE starts from the shot noise in the electron beam, a proper theory requires a statistical treatment of the output radiation [10-20]. Average properties of the output were studied in [10-16] and fluctuations were considered in [17-20]. The spiking of the output radiation (see Fig. 1) was demonstrated by computer simulation in [17].

In ref. [19], it was shown that the intensity fluctuations in the SASE output are described by the exponential distribution, $\exp(-Q)$, where Q is the normalized intensity $I/\langle I \rangle$. In this note [21], we advance the statistical description of SASE by applying the mathematical analysis of random noise developed by Rice [22]. In the time-domain, we determine the joint probability [Eq.(9)] that at a fixed position z along the undulator axis, the normalized intensity in the radiation pulse has the values Q_1 and Q_2 at times t_1 and t_2 . We also find the probability per unit time [Eq. (22)] of observing a spike with maximum normalized intensity Q . In the frequency domain, we derive the joint probability [Eq. (16)] that the normalized spectral intensity has the values \tilde{Q}_1 and \tilde{Q}_2 at frequencies \mathbf{w}_1 and \mathbf{w}_2 . We also find the probability per unit frequency interval [Eq. (28)] of observing a peak in the frequency spectrum with maximum normalized intensity \tilde{Q} .

2. Joint Probability Distributions

We work within the classical, one-dimensional approximation, in the linear regime before saturation. The electron bunch contains N_e electrons and has length, $L_b = cT_b$, long compared to the coherence length of the output radiation. We restrict our attention to the radiated field inside the electron bunch sufficiently far from the back end so that coherent effects [16,23,24] can be neglected. The radiation field has the form,

$$E(z, t) = \frac{1}{2} A(z, t) e^{ik_s z - i\omega_s t} + c.c., \quad (1)$$

with amplitude A given by

$$A(z, t) = \sum_{j=1}^{N_e} e^{i\omega_s t_j} g(z, t - \mathbf{t}_j). \quad (2)$$

The resonant wavenumber $k_s = 2\mathbf{g}_0^2 k_w / (1 + a_w^2)$, where k_w is the undulator wavenumber, $mc^2 \mathbf{g}_0$ is the initial electron energy and a_w is the undulator strength parameter. The arrival time of the j^{th} electron at the undulator entrance ($z=0$) is denoted \mathbf{t}_j , and the random variables \mathbf{t}_j ($j=1, \dots, N_e$) are considered to be uniformly distributed over the interval $0 \leq \mathbf{t}_j \leq T_b$. The Green's function g is known [10,16].

We use brackets $\langle \rangle$ to represent an average over the arrival times. For the uniform distribution, $\langle f(\mathbf{t}_j) \rangle = (N_e / T_b) \int_0^{T_b} d\mathbf{t} f(\mathbf{t})$. We define the normalized amplitude,

$$a(z, t) = A(z, t) / \sqrt{\langle |A(z, t)|^2 \rangle}, \quad (3)$$

and introduce the notation $a_1 = a(z, t_1)$ and $a_2 = a(z, t_2)$. Under the approximations we are employing [19],

$$\langle a_1 \rangle = \langle a_2 \rangle = \langle a_1 a_2 \rangle = 0, \quad (4a)$$

$$\langle a_1 a_1^* \rangle = 1, \quad \langle a_1 a_2^* \rangle = u_{12} + i v_{12}, \quad (4b)$$

$$\langle |a_1|^2 |a_2|^2 \rangle = 1 + \mathbf{b}_{12}, \quad (4c)$$

$$\mathbf{b}_{12} = \left| \langle a_1 a_2^* \rangle \right|^2 = u_{12}^2 + v_{12}^2. \quad (4d)$$

In deriving these relations, we retain only the dominant terms characterized by the absence of rapid phase variation, which correspond to keeping only pair-wise equal summation indices from A and A* terms.

In Eq. (2), the amplitude A is represented as a sum of independent random terms; it follows that probability distributions describing the output radiation are determined from the Central Limit Theorem [22]. Let us define

$$a = x + iy = \sqrt{Q} e^{if}. \quad (5)$$

Correlations of x and y are determined from Eqs. (4a,b). The probability $P(x,y)dx dy$ for finding x between x and $x+dx$, and y between y and $y+dy$, is given by

$$P(x, y) = \frac{1}{\mathbf{p}} e^{-x^2-y^2}. \quad (6)$$

The probability $P(Q, \mathbf{f})dQ d\mathbf{f}$ for finding Q between Q and $Q+dQ$, and \mathbf{f} between \mathbf{f} and $\mathbf{f} + d\mathbf{f}$, is given by

$$P(Q, \mathbf{f}) = \frac{1}{2\mathbf{p}} e^{-Q}. \quad (7)$$

As shown by Rice [22], the Central Limit Theorem also enables us to determine the joint probability distribution describing the field amplitudes at two different times.

The required correlations are determined from Eqs. (4), and we find

$$P(x_1, y_1, x_2, y_2) = \frac{1}{4\mathbf{p}^2(1-\mathbf{b}_{12})} \exp\left(\frac{-x_1^2 - y_1^2 - x_2^2 - y_2^2 - 2u_{12}(x_1x_2 + y_1y_2) - 2v_{12}(x_1y_2 - y_1x_2)}{2(1-\mathbf{b}_{12})}\right) \quad (8)$$

Expressing x and y in terms of Q and \mathbf{f} via Eq. (5), and integrating over \mathbf{f}_1 and \mathbf{f}_2 , we obtain the probability $P(Q_1, Q_2)dQ_1 dQ_2$ for finding normalized intensity between Q_1 and Q_1+dQ_1 at time t_1 , and Q_2 and Q_2+dQ_2 at time t_2 [22]:

$$P(Q_1, Q_2) = \frac{e^{\frac{-Q_1}{1-\mathbf{b}_{12}}} e^{\frac{-Q_2}{1-\mathbf{b}_{12}}}}{1-\mathbf{b}_{12}} I_0\left(\frac{2\sqrt{\mathbf{b}_{12}Q_1Q_2}}{1-\mathbf{b}_{12}}\right). \quad (9)$$

I_0 is the Bessel function of imaginary argument of order zero. Note that when $t_1 \rightarrow t_2$, then $\mathbf{b}_{12} \rightarrow 1$ and $P(Q_1, Q_2) \rightarrow e^{-Q_1} \delta(Q_1 - Q_2)$. Also, when $|t_1 - t_2| \rightarrow \infty$, then $\mathbf{b}_{12} \rightarrow 0$ and $P(Q_1, Q_2) \rightarrow e^{-Q_1-Q_2}$. The distribution of Eq. (9) has been used to describe narrow band chaotic light [25], and some mathematical properties of it have been studied in [26].

It is now possible to determine the conditional average $\langle Q_2 \rangle_{Q_1}$ of the intensity at t_2 , given the intensity at t_1 is Q_1 .

$$\langle Q_2 \rangle_{Q_1} = \frac{\int_0^{\infty} dQ_2 Q_2 P(Q_1, Q_2)}{\int_0^{\infty} dQ_2 P(Q_1, Q_2)} = Q_1 + (1 - \mathbf{b}_{12})(1 - Q_1). \quad (10)$$

The corresponding fluctuation is

$$\langle Q_2^2 \rangle_{Q_1} - \langle Q_2 \rangle_{Q_1}^2 = 2Q_1 \mathbf{b}_{12} (1 - \mathbf{b}_{12}) + (1 - \mathbf{b}_{12})^2. \quad (11)$$

We note from Eq. (10) that $\langle Q_2 \rangle_{Q_1}$ is less than Q_1 when $Q_1 > 1$, and is greater than Q_1 when $Q_1 < 1$. This is the statistical basis for the appearance of spikes in the radiation output.

In the high-gain regime before saturation ,

$$\mathbf{b}_{12} = e^{-\mathbf{p}(t_1 - t_2)^2 / T_{coh}^2}, \quad (12)$$

where T_{coh} [18,19] is the coherence time, which is related to the rms bandwidth \mathbf{s}_w of the output SASE radiation by

$$T_{coh} = \frac{\sqrt{\mathbf{p}}}{\mathbf{s}_w}. \quad (13)$$

The rms fluctuation of the output radiation energy per pulse, W , is [18,19]

$$\frac{\mathbf{s}_w}{W} = \frac{1}{\sqrt{T_b / T_{coh}}}. \quad (14)$$

The analysis just presented in the time-domain can be extended into the frequency-domain. Let $\tilde{A}(z, \Delta \mathbf{w})$ be the Fourier transform of $A(z, t)$, where $\Delta \mathbf{w} = \mathbf{w} - \mathbf{w}_s$. Then

$$\tilde{A}(z, \Delta \mathbf{w}) = G(z, \Delta \mathbf{w}) \sum_{j=1}^{N_s} e^{i \Delta \mathbf{w} t_j}, \quad (15)$$

G is the Fourier transform of g [Eq. (2)], $|G(z, \Delta \mathbf{w})|^2 = |G_0(z)|^2 \exp[-(\Delta \mathbf{w})^2 / 2\mathbf{s}_w^2]$.

We define, $\tilde{a}(\Delta\mathbf{w}) = \tilde{A}(z, \Delta\mathbf{w}) / \sqrt{\langle |\tilde{A}(z, \Delta\mathbf{w})|^2 \rangle}$, $\tilde{Q}(\Delta\mathbf{w}) = |\tilde{a}(\Delta\mathbf{w})|^2$, and $\tilde{Q}_1 = \tilde{Q}(\Delta\mathbf{w}_1)$,

$\tilde{Q}_2 = \tilde{Q}(\Delta\mathbf{w}_2)$. The joint probability $P(\tilde{Q}_1, \tilde{Q}_2) d\tilde{Q}_1 d\tilde{Q}_2$ that the normalized spectral intensities at frequencies ω_1 and ω_2 have values between \tilde{Q}_1 and $\tilde{Q}_1 + d\tilde{Q}_1$, and \tilde{Q}_2 and $\tilde{Q}_2 + d\tilde{Q}_2$, respectively, is determined in a manner similar to the derivation in the time-domain leading to Eq. (9). We find,

$$P(\tilde{Q}_1, \tilde{Q}_2) = \frac{e^{\frac{-\tilde{Q}_1}{1-\mathbf{g}_{12}}} e^{\frac{-\tilde{Q}_2}{1-\mathbf{g}_{12}}}}{1-\mathbf{g}_{12}} I_0 \left(\frac{2\sqrt{\mathbf{g}_{12} \tilde{Q}_1 \tilde{Q}_2}}{1-\mathbf{g}_{12}} \right), \quad (16)$$

where

$$\mathbf{g}_{12} = \langle \tilde{Q}_1 \tilde{Q}_2 \rangle - 1 = \left\langle \tilde{a}_1 \tilde{a}_2^* \right\rangle^2 = \text{sinc}^2 \left(\frac{(\mathbf{w}_1 - \mathbf{w}_2) T_b}{2} \right), \quad (17)$$

and $\text{sinc}(x) = (\sin x)/x$.

3. Intensity Peaks

Following Rice [22], we now provide a statistical description of the spikes in the output, first in the time-domain and then in the frequency-domain. Let $a^{(m)} = \partial^m a / \partial t^m$ and note that

$$\langle a^{(m)} a^{(n)*} \rangle = 2i^{m-n} b_{m+n}, \quad (18)$$

$$b_m = K(z) \int_{-\infty}^{\infty} \frac{d(\Delta\mathbf{w})}{2\mathbf{p}} |G(z, \Delta\mathbf{w})|^2 (\Delta\mathbf{w})^m. \quad (19)$$

The normalization $K(z)$ is chosen so that $b_0=1/2$. We write $a = x + iy$, $a' = x' + iy'$,

$a'' = x'' + iy''$, where the prime denotes differentiation with respect to time. Using the

Central Limit Theorem, Rice [22] determines $P(x, y, x', y', x'', y'')$. He then introduces

$x = R \cos \mathbf{f}$, $y = R \sin \mathbf{f}$, and takes the first and second time derivatives. By integrating

over $f, f',$ and f'' , Rice determines $P(R, R', R'')$. He then notes that the probability $p(R) dR$ that a maximum of the envelope R falls within the elementary rectangle dR is given by

$$p(R) = -\int_{-\infty}^0 P(R, 0, R'') R'' dR'' . \quad (20)$$

We recall that $Q=R^2$, and we choose the normalization $K(z)$ such that

$$K(z)|G(z, \Delta \mathbf{w})|^2 = \frac{1}{2\sqrt{2ps_w}} e^{-(\Delta \mathbf{w})^2 / 2s_w^2} . \quad (21)$$

The probability $p_t(Q) dQ$ that a maximum of Q with value between Q and $Q+dQ$ is found in time interval dt is determined by [22],

$$p_t(Q) = \frac{1}{4T_{coh}} (2Q)^{1/4} e^{-3Q/2} \sum_{n=0}^{\infty} \frac{(n+1)(Q/2)^{n/2}}{\Gamma(\frac{n}{2} + \frac{7}{4})} . \quad (22)$$

For $Q \gg 1$,

$$p_t(Q) \approx \frac{1}{T_{coh}} e^{-Q} \sqrt{Q} (1 - \frac{1}{2Q}) . \quad (23)$$

The number of peaks per unit time is

$$N_t = \int_0^{\infty} dQ p_t(Q) = \frac{0.711}{T_{coh}} . \quad (24)$$

The probability of finding a maximum with normalized intensity between Q and $Q+dQ$ is $p_t(Q)/N_t dQ$, which is plotted in Fig 2. The average peak height is $\langle Q \rangle = 1.56$ and the rms fluctuation is 1.27.

In the frequency domain,

$$\tilde{a}(\Delta \mathbf{w}) = \frac{1}{\sqrt{N_e}} \sum_{j=1}^{N_e} e^{i\Delta \mathbf{w} t_j} . \quad (25)$$

Defining $\tilde{a}^{(m)} = \partial^m \tilde{a} / \partial \mathbf{w}^m$, we find

$$\left\langle \tilde{a}^{(m)} \tilde{a}^{(n)*} \right\rangle = 2i^{m-n} b_{m+n} , \quad (26)$$

$$b_m = \frac{1}{2T_b} \int_{-T_b/2}^{T_b/2} dt t^m . \quad (27)$$

Following Rice's analysis [22], we find that the probability $p_w(\tilde{Q})d\tilde{Q}$ that a maximum of $\tilde{Q} = \left| \tilde{a} \right|^2$ with value between \tilde{Q} and $\tilde{Q} + d\tilde{Q}$ is observed in frequency interval $d\omega$ is determined by

$$p_w(\tilde{Q}) = \frac{T_b/2\mathbf{p}}{4} \sqrt{\frac{\mathbf{p}}{6}} \tilde{Q}^{1/4} e^{-9\tilde{Q}/4} \sum_{n=0}^{\infty} \frac{(5/4)^{n/2-3/4} \tilde{Q}^{n/2} A_n}{\Gamma\left(\frac{n}{2} + \frac{7}{4}\right)}, \quad (28)$$

with

$$A_0 = 1, \quad (29a)$$

$$A_n = \sum_{m=0}^n \frac{(1/2)(3/2)\cdots(m-1/2)}{m!} (n-m+1)(3/5)^m, \quad (n \geq 1). \quad (29b)$$

For $\tilde{Q} \gg 1$,

$$p_w(\tilde{Q}) \approx \frac{T_b}{2\mathbf{p}} \sqrt{\frac{\mathbf{p}}{3}} e^{-\tilde{Q}} \sqrt{\tilde{Q}} \left(1 - \frac{1}{2\tilde{Q}} \right). \quad (30)$$

The number of peaks per unit frequency is

$$N_w = \int_0^{\infty} d\tilde{Q} \tilde{Q} p_w(\tilde{Q}) = 0.641(T_b/2\mathbf{p}). \quad (31)$$

The probability of finding a maximum with normalized spectral intensity between \tilde{Q} and $\tilde{Q} + d\tilde{Q}$ is $p_w(\tilde{Q})/N_w d\tilde{Q}$, which is plotted in Fig 3. The average spectral peak height is

$\langle \tilde{Q} \rangle = 1.66$ and the rms fluctuation is 1.29.

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Figure Caption

Fig. 1. A typical simulation result for the output SASE intensity in the central region of a long pulse.

Fig. 2. $p_t(Q)/N_t dQ$ is the probability of a peak in the time-domain having normalized intensity between Q and $Q+dQ$.

Fig. 3. $p_\omega(\tilde{Q})/N_\omega d\tilde{Q}$ is the probability of a peak in the frequency-domain having normalized intensity between \tilde{Q} and $\tilde{Q}+d\tilde{Q}$.

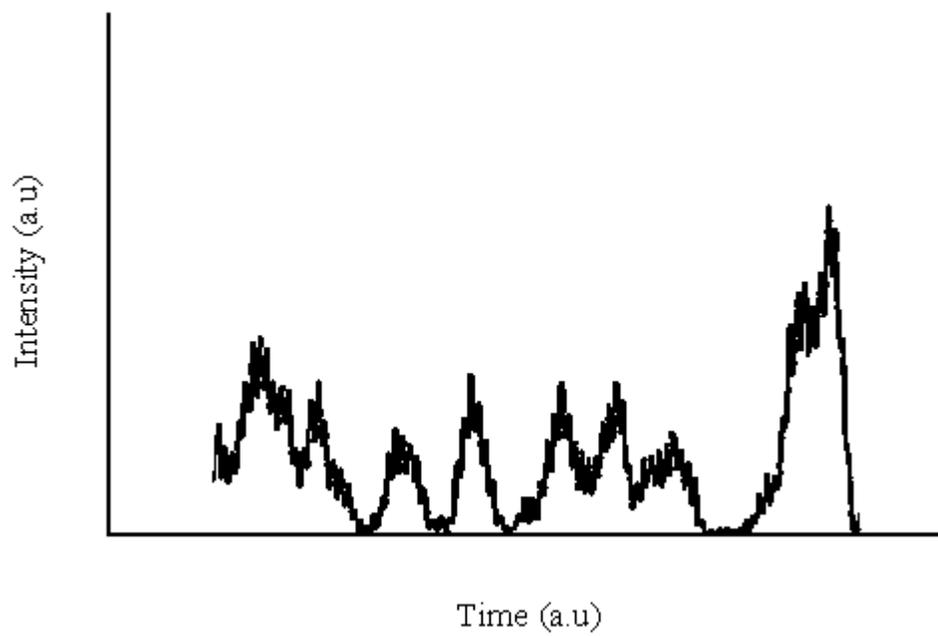


Figure 1

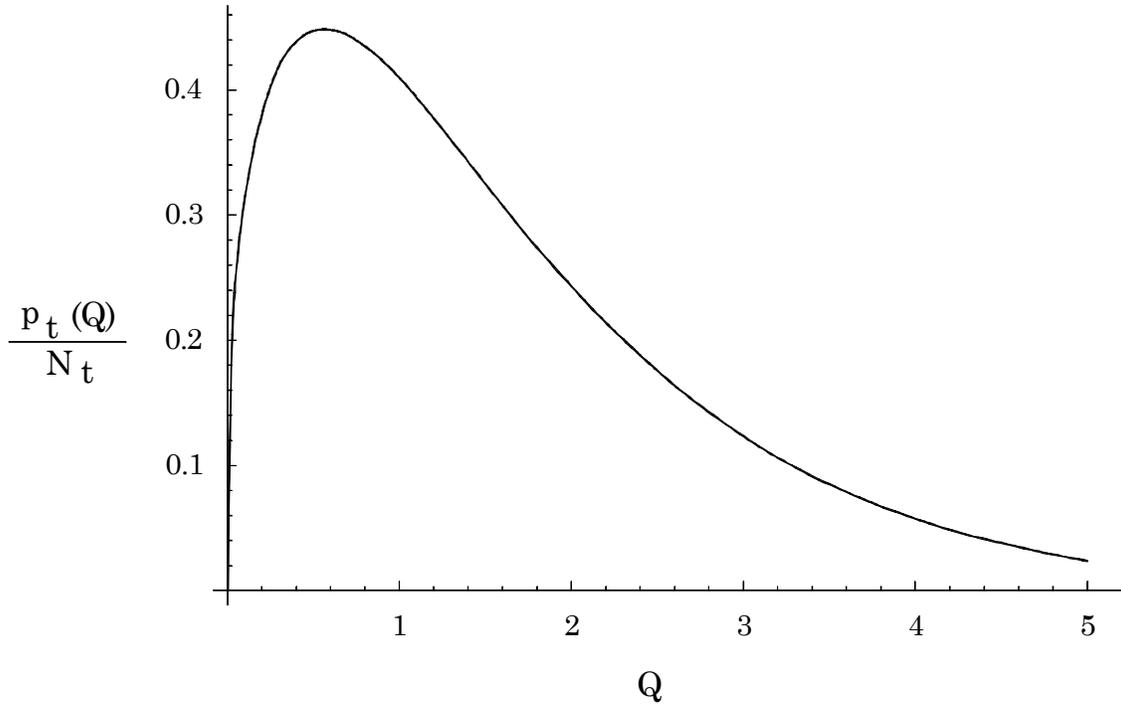


Figure 2

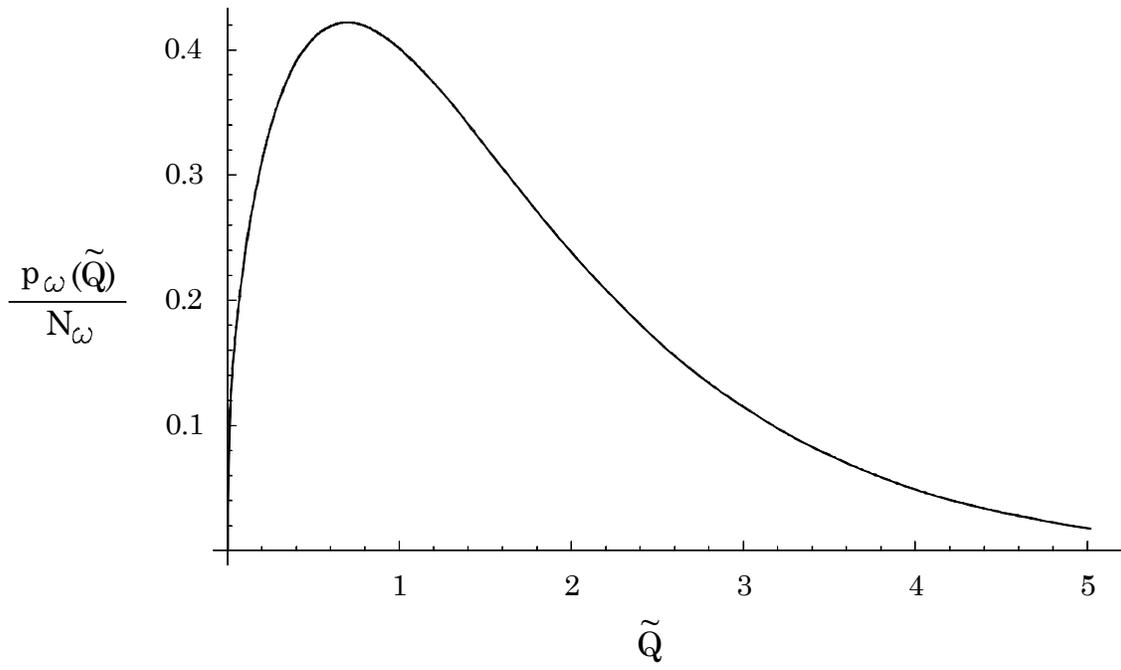


Figure 3