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# *A Hamiltonian Formulation for Spiral-Sector Accelerators*

**J. Scott Berg**

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**Physics Department/Bldg. 901A**

**Brookhaven National Laboratory**
P.O. Box 5000
Upton, NY 11973-5000
www.bnl.gov

## DISCLAIMER

# A Hamiltonian Formulation for Spiral-Sector Accelerators


J. Scott Berg

*Brookhaven National Laboratory; Building 901A; P. O. Box 5000; Upton, NY 11973-5000; USA*



**Abstract.** I develop a formulation for Hamiltonian dynamics in an accelerator with magnets whose edges follow a spiral. I demonstrate using this Hamiltonian that a spiral FFAG can be made perfectly "scaling." I examine the effect of tilting an RF cavity with respect a radial line from the center of the machine, potentially with a different angle than the spiral of the magnets.



**Keywords:** spiral, Hamiltonian, fixed field alternating gradient accelerator, radio frequency cavity
**PACS:** 29.27.Bd


## INTRODUCTION

When one analyzes motion in an accelerator, one tried to determine what is going on at phase space at a point $s_1$ if the particles started out at a point $s_0$. When one refers to phase space "a point $s$," one imagines a curve parametrized by $s$. There are a series of surfaces passing through that curve. $s$ tells you which surface you are speaking of, where $s$ is the point on the curve through which the plane passes. The phase space coordinates are the position coordinates within the plane, their conjugate momenta, time, and energy. For a synchrotron, these planes are perpendicular to a "reference orbit," and $s$ is the distance along that reference orbit. For a radial-sector cyclotron or FFAG, the planes are oriented along radial lines from the machine center, and $s$ is the length along a circle with some given radius. These are convenient parametrizations, since magnets generally extend from one value of $s$ to a different value of $s$.

The question then arises as to how to handle a spiral machine, similar to what is shown in Fig. 1. One often uses the same parametrization as for the radial-sector cyclotron or FFAG, but the planes now do not even come close to following the magnet edges. Thus, to look at what occurs at the entrance to a magnet, for instance, one cannot look at

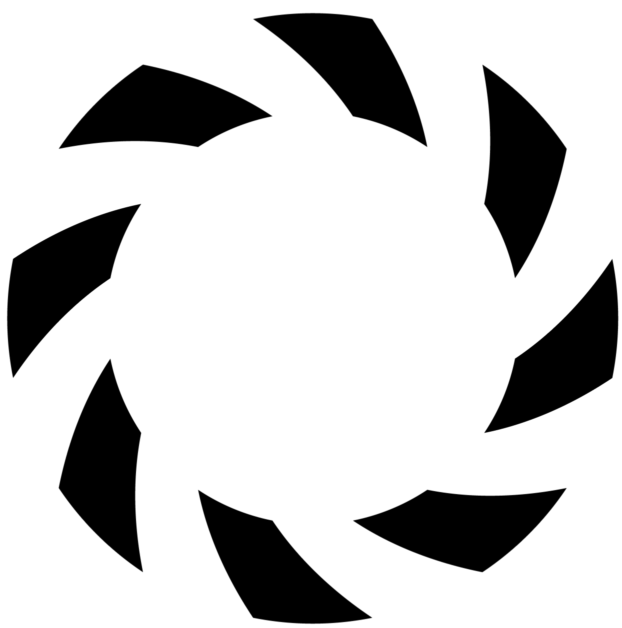

**FIGURE 1.** A diagram of a spiral FFAG accelerator. Black regions are magnets.

motion at a given value for $s$.

Thus, one would like to construct a mathematical formulation so that the planes described above match the spirals of the magnets. The primary purpose of this paper is to present a Hamiltonian description for this situation.

In the first section, I will outline the basic theory of an accelerator in spiral coordinates, focusing particularly on the case of a logarithmic spiral. I will write down a Hamiltonian, a magnetic field expansion, and an RF cavity field expansion in these coordinates. In the following section, I will demonstrate that this Hamiltonian, when used with appropriately defined scaling fields, obeys the usual scaling laws. Finally, I will discuss the effect of having a cavity tilted in the machine.

## HAMILTONIAN IN SPIRAL COORDINATES

One first transforms into spiral coordinates

$$R = r \qquad Y = y \qquad \Theta = \theta - \int_{r_0}^{r} \frac{\tan\zeta(\bar{r})}{\bar{r}} d\bar{r}, \tag{1}$$

where $\zeta(r)$ is the angle that the spiral faces make with respect to a radial line from the center of the machine, as a function of the radius $r$. For positive $\zeta$, the magnet edges move in the direction of particle motion as the radius increases, assuming that the magnet edges are along lines of constant $\Theta$. Note that $\Theta$ will eventually be the independent variable for the new system, but temporarily we will use time as the independent variable for the purpose of performing these transformations. This change of variables induces a change in the conjugate momenta to

$$p_R = p_r + \frac{p_\theta}{r}\tan\zeta \qquad p_Y = p_y \qquad p_\Theta = p_\theta \tag{2}$$

Note that $p_R$ is different from $p_r$, despite the fact that $R = r$. Furthermore, note that the Hamiltonian with $\theta$ as the independent variable is $-p_\theta$, and the Hamiltonian with $\Theta$ as the independent variable is $-p_\Theta$.

It is important to understand the change of independent variable: it means that the question that one is asking of the equations of motion is changing. With $\theta$ as the independent variable, one is asking about the radius, vertical position, time, and their conjugate momenta *with respect to* $\theta$ at a given value of $\theta$. With $\Theta$ as the independent variable, one is asking about the radius, vertical position, time, and their conjugate momenta *with respect to* $\Theta$ at a given value of $\Theta$. Notice two things have changed in the question: where you are looking, and the nature of the conjugate momenta. Understanding this fact is essential to understanding why the spiral machine behaves differently than the radial sector machine.

The Hamiltonian in these coordinates, with $\Theta$ as the independent variable, is

$$-R\cos\zeta\left[p_R\sin\zeta + qA_\Theta + \sqrt{(E-q\Phi)^2/c^2 - (p_R\cos\zeta - qA_R)^2 - (p_Y - qA_y)^2 - (mc)^2}\right], \tag{3}$$

where

$$A_\Theta = A_\theta\cos\zeta - A_r\sin\zeta \qquad A_R = A_r\cos\zeta + A_\theta\sin\zeta. \tag{4}$$

These are the components of the vector potential perpendicular and parallel to the spirals, respectively.

### Vector Potentials for Magnets

It is most convenient at this point to assume that $\zeta$ is constant, which is required for meeting the scaling condition in an FFAG. Writing the vector potentials in a power series about the midplane

$$A_R(R,Y,\Theta) = \sum_{n=0}^{\infty}\frac{1}{n!}A_{Rn}(R,\Theta)Y^n \qquad A_y(R,Y,\Theta) = \sum_{n=0}^{\infty}\frac{1}{n!}A_{yn}(R,\Theta)Y^n \qquad A_\Theta(R,Y,\Theta) = \sum_{n=0}^{\infty}\frac{1}{n!}A_{\Theta n}(R,\Theta)Y^n, \tag{5}$$

one can obtain a recursion relation for the coefficients from Maxwell's equations using the gauge $\nabla \cdot A = 0$:

$$A_{R,n+2} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{Rn})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{Rn}}{\partial\Theta\partial R} - \frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{Rn}}{\partial\Theta^2} + \frac{2}{R^2}\frac{\partial A_{\Theta n}}{\partial\Theta} \tag{6}$$

$$A_{\Theta,n+2} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{\Theta n})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{\Theta n}}{\partial\Theta\partial R} - \frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{\Theta n}}{\partial\Theta^2} - \frac{2}{R^2}\frac{\partial A_{Rn}}{\partial\Theta} \tag{7}$$

$$A_{y,n+1} = -\frac{\cos\zeta}{R}\frac{\partial}{\partial R}(RA_{Rn}) + \frac{\sin\zeta}{R}\frac{\partial}{\partial R}(RA_{\Theta n}) - \frac{\sec\zeta}{R}\frac{\partial A_{\Theta n}}{\partial\Theta} \tag{8}$$

Starting with the gauge choice $A_{R0}(R,\Theta) = 0$, one also has an equation for $A_{\Theta 0}$ from $B_y(R,0,\Theta)$:

$$B_y(R,0,\Theta) = -\frac{\cos\zeta}{R}\frac{\partial}{\partial R}(RA_{\Theta 0}). \tag{9}$$

If we have scaling fields, where

$$B_y(R,0,\Theta) = B_{y0}(\Theta)(R/r_0)^k, \tag{10}$$

then

$$A_{Rn}(R,\Theta) = \hat{A}_{Rn}(\Theta)(R/r_0)^{k+1-n} \qquad A_{yn}(R,\Theta) = \hat{A}_{yn}(\Theta)(R/r_0)^{k+1-n} \qquad A_{\Theta n}(R,\Theta) = \hat{A}_{\Theta n}(\Theta)(R/r_0)^{k+1-n}, \tag{11}$$

and the recursion relations become

$$\hat{A}_{R,n+2} = -(k+2-n)(k-n)\hat{A}_{Rn} + 2(k+1-n)\tan\zeta\frac{\partial\hat{A}_{Rn}}{\partial\Theta} - \sec^2\zeta\frac{\partial^2\hat{A}_{Rn}}{\partial\Theta^2} + 2\frac{\partial\hat{A}_{\Theta n}}{\partial\Theta} \tag{12}$$

$$\hat{A}_{\Theta,n+2} = -(k+2-n)(k-n)\hat{A}_{\Theta n} + 2(k+1-n)\tan\zeta\frac{\partial\hat{A}_{\Theta n}}{\partial\Theta} - \sec^2\zeta\frac{\partial^2 A_{\Theta n}}{\partial\Theta^2} - 2\frac{\partial A_{Rn}}{\partial\Theta} \tag{13}$$

$$\hat{A}_{y,n+1} = -(k+2-n)\cos\zeta\hat{A}_{Rn} + (k+2-n)\sin\zeta\hat{A}_{\Theta n} - \sec\zeta\frac{\partial A_{\Theta n}}{\partial\Theta} \tag{14}$$

$$B_{y0}(\Theta) = -(k+2)\cos\zeta\hat{A}_{\Theta 0}. \tag{15}$$

## Vector Potentials for Cavities

For cavities, it is best to take the gauge with zero electric scalar potential, in which case the recursion relations for the power series in the midplane become

$$A_{R,n+2} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{Rn})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{Rn}}{\partial\Theta\partial R} - \frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{Rn}}{\partial\Theta^2} + \frac{2}{R^2}\frac{\partial A_{\Theta n}}{\partial\Theta} + \frac{\omega^2}{c^2}A_{Rn} \tag{16}$$

$$A_{\Theta,n+2} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{\Theta n})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{\Theta n}}{\partial\Theta\partial R} - \frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{\Theta n}}{\partial\Theta^2} - \frac{2}{R^2}\frac{\partial A_{Rn}}{\partial\Theta} + \frac{\omega^2}{c^2}A_{\Theta n} \tag{17}$$

$$A_{y,n+1} = -\frac{\cos\zeta}{R}\frac{\partial}{\partial R}(RA_{Rn}) + \frac{\sin\zeta}{R}\frac{\partial}{\partial R}(RA_{\Theta n}) - \frac{\sec\zeta}{R}\frac{\partial A_{\Theta n}}{\partial\Theta}. \tag{18}$$

To start the recursion sequence, we assume there are electric fields $E_{R0}(R,\Theta)\cos(\omega t+\phi)$ and $E_{\Theta 0}(R,\Theta)\cos(\omega t+\phi)$ in the midplane (parallel and perpendicular to the logarithmic spirals respectively), and thus

$$A_{\Theta 0} = -\frac{E_{\Theta 0}(R,\Theta)}{\omega}\sin(\omega t+\phi) \qquad\qquad A_{R0} = -\frac{E_{R0}(R,\Theta)}{\omega}\sin(\omega t+\phi). \tag{19}$$

# SCALING LAWS FOR SCALING FFAGS

Now we can see precisely what "scaling" means for a scaling FFAG. Assume that the vector potentials are described by Eq. (5), with the coefficients in those equations given by Eq. (11). The spiral angle $\zeta$ is assumed to be constant. The vector potentials are taken to be independent of time. Then for any of those vector potentials,

$$A(\lambda R,\lambda Y,\Theta) = \lambda^{k+1}A(R,Y,\Theta). \tag{20}$$

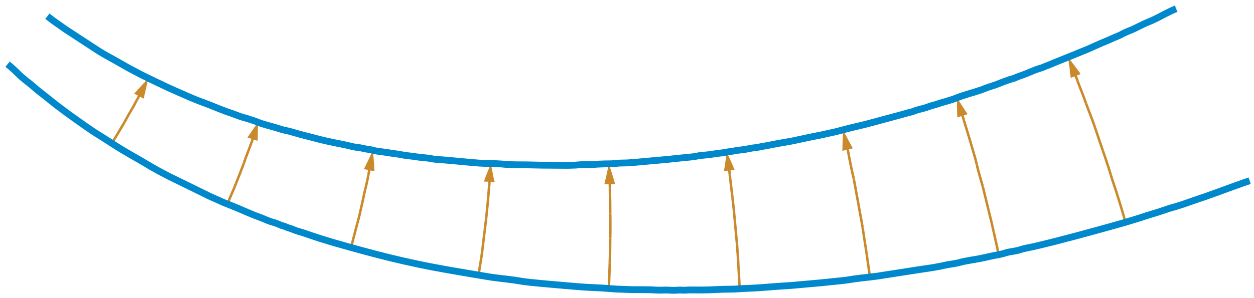

**FIGURE 2.** Cavity walls and assumed electric field lines for a logarithmic spiral cavity shape.

Change variables from $E$ to $\Delta$, where $E = E_0 + \Delta$, and assume that the scalar potential $\Phi$ is zero. The Hamiltonian is then

$$-R\cos\zeta\left[p_R\sin\zeta + qA_\Theta + \sqrt{p_0^2 + 2E_0\Delta/c^2 + \Delta^2/c^2 - (p_R\cos\zeta - qA_R)^2 - (p_Y - qA_y)^2}\right], \tag{21}$$

where $p_0^2 = E_0^2/c^2 - (mc)^2$.

Consider the following transformation

$$\begin{aligned} &\hat{R} = R(\hat{p}_0/p_0)^{1/(k+1)} \quad \hat{p}_R = p_R\hat{p}_0/p_0 \quad \hat{Y} = Y(\hat{p}_0/p_0)^{1/(k+1)} \quad \hat{p}_Y = p_Y\hat{p}_0/p_0 \\ &\hat{\Delta} = \sqrt{\hat{E}_0^2 + 2cE_0(\hat{p}_0/p_0)^2\Delta + (\hat{p}_0/p_0)^2\Delta^2} - \hat{E}_0 \quad \hat{t} = t\frac{p_0}{\hat{p}_0}\frac{\hat{E}_0+\hat{\Delta}}{E_0+\Delta}\left(\frac{\hat{p}_0}{p_0}\right)^{1/(k+1)}, \end{aligned} \tag{22}$$

where $\hat{p}_0^2 = \hat{E}_0^2/c^2 - (mc)^2$. The Hamiltonian which governs the new variables is

$$-\hat{R}\cos\zeta\left[\hat{p}_R\sin\zeta + qA_\Theta + \sqrt{\hat{p}_0^2 + 2\hat{E}_0\hat{\Delta}/c^2 + \hat{\Delta}^2/c^2 - (\hat{p}_R\cos\zeta - qA_R)^2 - (\hat{p}_Y - qA_y)^2}\right], \tag{23}$$

where now $A_R$, $A_y$, and $A_\Theta$ are all evaluated at $\hat{R}$ and $\hat{y}$ instead of $R$ and $y$.

The Hamiltonians for the two sets of variables are clearly identical, with the exception that $p_0$ is replaced by $\hat{p}_0$. The interpretation of this is that if you know the phase space dynamics near a total momentum $p_0$, you can find the phase space dynamics near any other total momentum $\hat{p}_0$ by applying the transformations (22). Several conclusions can be drawn from this:

- Transverse tunes are independent of reference momentum.
- Closed orbits for different momenta are geometrically similar, and their size is proportional to $p_0^{1/(k+1)}$.
- Normalized dynamic aperture in each plane is proportional to $p_0^{(k+2)/(k+1)}$. The shape of the dynamic aperture is independent of momentum, except that the transverse coordinate direction is proportional to to $p_0^{1/(k+1)}$, and the transverse momentum direction is proportional to $p_0$.
- The Courant-Snyder beta functions in normalized coordinates (with units of m/(eV·s)) are proportional to $p_0^{-k/(k+1)}$, and thus the usual beta functions (units of m) are proportional to $p_0^{1/(k+1)}$. The Courant-Snyder alpha function is independent of $p_0$.
- The momentum compaction is $1/(k+1)$, independent of energy.

## ANALYSIS OF CAVITY PLACEMENT

Start with a cavity which makes an angle of $\zeta_C$ with respect to radial lines. To be able to produce some analytic results, I assume $\zeta_C$ to be constant. The cavity thus has a logarithmic spiral shape, unless $\zeta_C = 0$. The center of the cavity is given by

$$\theta = \theta_C + \tan\zeta_C \ln(r/r_C). \tag{24}$$

Assume that the midplane electric fields in the cavity are perpendicular to lines that make an angle $\zeta_C$ with respect to radial lines (see Fig. 2). We will define the cavity fields in coordinates which are along the logarithmic spirals making angle $\zeta_C$ with respect to radial lines and which are along curves which are perpendicular to those spirals. We define these coordinates to be $r_1$ and $\theta_1$ as follows:

$$\theta_1 = \theta - \theta_C - \tan\zeta_C \ln(r/r_C) \qquad r_1 = r^{\cos^2\zeta_C} r_C^{\sin^2\zeta_C} \exp[(\theta-\theta_C)\cos\zeta_C \sin\zeta_C] \tag{25}$$

Now, assume that the magnitude of the electric field in the midplane is of the form

$$c(r_1)d(\theta_1) \tag{26}$$

Integrating in $\theta_1$ to find the on-crest energy gain in the cavity (ignoring the time dependence of the electric field), one finds the voltage to be

$$r_1 c(r_1)\int d(\theta_1)\exp(-\theta_1 \sin\zeta_C \cos\zeta_C)\,d\theta_1. \tag{27}$$

If one wishes the energy gain to be independent of the line along which you integrated, then $c(r_1) \propto r_1^{-1}$.

Thus, in terms of $r$ and $\theta$, one can write the electric field component in the $\theta_1$ direction to be

$$(r/r_C)^{-\cos^2\zeta_C}\exp[-(\theta-\theta_C)\sin\zeta_C \cos\zeta_C]\,d\big(\theta-\theta_C-\tan\zeta_C\ln(r/r_C)\big). \tag{28}$$

Redefining $d$ to eliminate the exponential, one can rewrite this as

$$\frac{r_C}{r}E_0\big(\theta-\theta_C-\tan\zeta_C\ln(r/r_C)\big) \tag{29}$$

On performing the integral (27), we find the voltage to be

$$r_C\cos\zeta_C\int E_0(\theta)d\theta. \tag{30}$$

In terms of the spiral coordinates for the Hamiltonian and the electric field components used earlier (and taking $r_C = r_0$ and $\zeta$ constant),

$$E_\Theta(R,\Theta) = \frac{r_0}{R}\cos(\zeta-\zeta_C)E_0\big(\Theta-\theta_C+(\tan\zeta-\tan\zeta_C)\ln(R/r_0)\big) \tag{31}$$

$$E_R(R,\Theta) = \frac{r_0}{R}\sin(\zeta-\zeta_C)E_0\big(\Theta-\theta_C+(\tan\zeta-\tan\zeta_C)\ln(R/r_0)\big) \tag{32}$$

Note that if $\zeta=\zeta_C$, then $E_R=0$ (thus $A_R=0$ as well), and the only $R$ dependence remaining in $E_\Theta$ (and therefore $A_\Theta$) is an inverse dependence in $R$. Thus, such a cavity following the spiral coordinates will give terms in the Hamiltonian which do not depend on the transverse variables, and which should therefore minimize longitudinal-transverse coupling.

## DISCUSSION AND CONCLUSIONS

I have developed a Hamiltonian formulation for dynamics in a spiral machine. In particular, it appears that having any RF cavities follow the spiral of the magnets will minimize longitudinal-transverse coupling effects.

However, this latter conclusion is still somewhat speculative. There is longitudinal-transverse coupling that arises from the having dispersion in the RF cavities. It is conceivable that giving the cavity a different angle would be able to reduce this coupling. However, it initially appears that the two effects do not come into the Hamiltonian in the same way. However, to verify this, the longitudinal-transverse coupling due to finite dispersion in the cavities should be computed.

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