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Abstract. In calculating the strengths of depolarizing resonances it may be convenient to reformulate the equations of spin motion in a coordinate system based on the actual trajectory of the particle, as introduced by Kondratenko, rather than the conventional one based on a reference orbit. It is shown that resonance strengths calculated by the conventional and the revised formalisms are identical. Resonances induced by radiofrequency dipoles or solenoids are also treated; with rf dipoles it is essential to consider not only the direct effect of the dipole but also the contribution from oscillations induced by it. .

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Introduction

The dynamics of particle spin in accelerators has been explored extensively [1]. In the process of acceleration beams tend to be depolarized by resonances with perturbing fields. But in a few recent papers [2] the experimental results have led to apparent disagreements between the resonance strengths inferred from the experimental results and theoretical values. Therefore it is felt to be desirable to review and reexamine the theory of depolarizing resonance strengths, especially in cases where a rf dipole or solenoid is present.

The Thomas-BMT equation [3] for the behavior of spin in magnetic fields is customarily written

$$\frac{d\vec{S}}{dt}=\frac{q}{m\gamma}\vec{S}\times[(1+G\gamma)\vec{B}_{\perp}+(1+G)\vec{B}_{\parallel}] \tag{1}$$

where $\vec{B}_{\parallel}=(\hat{v}\cdot\vec{B})\hat{v}$ and $\vec{B}_{\perp}=\vec{B}-\vec{B}_{\parallel}=(\hat{v}\times\vec{B})\times\hat{v}$ are the longitudinal and transverse parts of the magnetic field $\vec{B}$, $\hat{v}$ being the unit vector in the direction of the particle velocity, q and m are its charge and mass, and γ is the Lorentz energy factor.

The conventional analysis employs a Frenet-Serret coordinate system based on a closed *reference orbit* as we consider particles whose motion takes place near (though not exactly on) that orbit. We assume the reference orbit is in a plane and has a circumference $2\pi R$. We define the coordinates to be: s = the distance along the reference orbit from an origin point (arbitrarily chosen) on the reference orbit to the point on the reference orbit closest to the particle; x and z = the horizontal and vertical components of the vector from that point to the particle.

Using s instead of the time t as the independent variable, expressing the components of the magnetic field in terms of the excursions of the particle, and using the spinor-SU2 formalism for analysis of rotation dynamics (as invented by Hamilton 150 years ago), (1) becomes

$$\Phi' = -\frac{i}{2}\begin{pmatrix} -G\gamma / R & \zeta e^{-i\chi} \\ \zeta^* e^{i\chi} & G\gamma / R \end{pmatrix}\Phi \tag{2}$$

where the prime denotes differentiation by s; the complex 2-component spinor Φ describes the components S_1, S_2, S_3 of the spin in the x, s and z directions via $S_1 = \langle\Phi^\dagger\sigma_1\Phi\rangle$; $S_2 = \langle\Phi^\dagger\sigma_2\Phi\rangle$; $S_3 = \langle\Phi^\dagger\sigma_3\Phi\rangle$; $\chi = G\gamma(\Theta - \theta)$, $\Theta = \int_0^s ds'/\rho - x'$ is the turning angle, $\theta = s/R$ is the azimuth along the reference orbit, and the depolarizing term ζ equals

$$\zeta = -(1+G\gamma)z'' - i[G(\gamma-1)z'/\rho - (1+G)z(1/\rho)' + (1+G)B_{sol}/B\rho] \tag{3}$$

The diagonal elements of the matrix in (2) mean that the basic precession frequency about the vertical (spin tune) is $G\gamma$, and the off-diagonal terms produce changes (depolarization) in the spin. The depolarizing term $\zeta e^{-i\chi}$ is a combination of oscillations at various frequencies:

$$\zeta e^{-i\chi} = \frac{1}{R}\sum_r \varepsilon_r e^{-i\kappa_r\theta} \tag{4}$$

and ε_r is the strength of the depolarizing resonance which occurs at the energy where the spin tune $G\gamma$ equals κ_r. Here κ_r, the r-th resonance value of $G\gamma$, may be imperfection resonances, κ_r= any integer k; "intrinsic" resonances (due to vertical betatron oscillations) $\kappa_r = kP \pm \nu_z$, where ν_z is the vertical betatron tune, P is the periodicity of the magnet structure, k is any integer; broken periodicity resonances $\kappa_r = k \pm \nu_z$ which occur when the structure periodicity P is inexact; RF resonances $\kappa_r = k \pm \omega_{rf} / \omega_{orbit}$ induced by radiofrequency dipoles and/or solenoids placed somewhere on the orbit. When the orbit is the closed orbit produced by imperfections, without betatron oscillations, (4) is a straightforward Fourier series and resonances occur when $G\gamma$ = an integer k; the resonance strengths ε_k can be calculated by Fourier analysis of $\zeta e^{-i\chi}$ as defined in (3):

$$\varepsilon_k = \frac{1}{2\pi}\int_0^{2\pi R} \zeta e^{-i(k\theta+\chi)} ds = \frac{1}{2\pi}\int_0^{2\pi R} \zeta e^{-iG\gamma\Theta} ds \tag{5}$$

If betatron oscillations are present we have intrinsic and/or broken periodicity resonances $\kappa_r = k \pm \nu_z$; we consider z to be the trajectory of betatron oscillations of frequency ν_z. The frequencies κ_i in (4) are not multiples of one frequency; therefore (4) is no longer a simple Fourier series but a general combination of oscillations (an *almost periodic* function), and Fourier analysis (5) does not apply directly. But since $z = z_+(\theta)e^{i\nu_z\theta} + z_-(\theta)e^{-i\nu_z\theta}$, where $z_+(\theta)$ and $z_-(\theta) = z_+^*(\theta)$ are periodic functions, we can separate ζ into parts containing the factors $e^{i\nu_z\theta}$ and $e^{-i\nu_z\theta}$, which we call $\zeta_+ + \zeta_-$; $\zeta_+ e^{-i\nu_z\theta}$ and $\zeta_- e^{i\nu_z\theta}$ are periodic functions, and we perform Fourier analysis for each part, finding, for $G\gamma = \kappa = k \pm \nu_z$,

$$\varepsilon_\kappa = \frac{1}{2\pi}\int_0^{2\pi R}\zeta_\pm e^{-i(\kappa\theta+\chi)}ds = \frac{1}{2\pi}\int_0^{2\pi R}\zeta_\pm e^{-iG\gamma\Theta}ds \tag{6}$$

To evaluate the resonance strengths in any particular case it is necessary to express the integration factors ζ, ζ_+, ζ_- in (5) and (6) in terms of the components of the lattice structure, as is done in [6].

Trajectory-based Coordinate System

The particles do not necessarily travel on the reference orbit, therefore the spin components in the directions $\hat{e}_x,\hat{e}_s,\hat{e}_z$ are not exactly in the directions transverse and longitudinal to the motion of a particle. Since the dynamical equation (1) shows that the transverse and longitudinal spin components behave differently, it may be desirable to formulate equations of motion that maintain this distinction, i.e. describe, for a given particle, the behavior of the components of spin parallel and perpendicular to the direction of motion of that particle, rather than the components parallel and perpendicular to the reference orbit.

In several papers Kondratenko [5a] and Sivers [5b] introduce a "natural" or "local" reference frame based on the actual trajectory of the particle. In their formalism the basis vector $\hat{u}_2$ is taken to be the unit vector $\hat{v}$ in the direction of the instantaneous particle velocity, and the other two are in the local radial and vertical direction orthogonal to $\hat{v}$ and to each other. In what follows we ignore second and higher terms in the excursions x and z of the particle from the reference orbit. In this coordinate system the depolarizing term ζ in the spinor equation (2) becomes

$$\zeta = -G\gamma z'' + i[Gz'/\rho + (1+G)z(1/\rho)' - (1+G)B_{sol}/B\rho] \tag{7}$$

Note that the leading term $-G\gamma z''$ in (7) is different from the leading term $-(1+G\gamma)z''$ in (3), which led some people, including the present author [8], to the erroneous conjecture that resonance strengths in the two frames differ by the factor $G\gamma/(1+G\gamma)$. Kondratenko [5a] shows that this conjecture is incorrect and that the resonance strength is the same in both systems. His derivation, in our terminology, is: The difference between the resonance strength calculated in the two different frames is (using a superscript r for the reference-orbit based system, and t for the trajectory-based system)

$$\varepsilon_\kappa{}^{(r)} - \varepsilon_\kappa{}^{(t)} = \frac{1}{2\pi}\int_0^{2\pi R}\left(\zeta^{(r)} - \zeta^{(t)}\right)e^{iG\gamma\Theta}ds \tag{8}$$

The integrand of (8) is

$$(\zeta^{(r)} - \zeta^{(t)})e^{iG\gamma\Theta} = -[z'' + iG\gamma\Theta' z']e^{iG\gamma\Theta} = -\frac{d}{ds}(z' e^{iG\gamma\Theta}) \tag{9}$$

which is a perfect derivative. The integral of a perfect derivative over a period averages to zero. Therefore $\varepsilon_\kappa{}^{(t)} = \varepsilon_\kappa{}^{(r)}$: the resonance strength is independent of the frame in which it is calculated.

It follows that in any particular case where the strengths of resonances are to be calculated one may select the trajectory- based formalism or the reference-orbit formalism, whichever is more convenient. The algorithms given in [6] for calculation of resonance strengths for a given magnet structure, leading to the computer program DEPOL, remain valid.

Resonance Strength with RF Solenoids or Dipoles

RF solenoids or dipoles may be inserted in an accelerator or storage ring in order to deliberately excite spin resonances, either for the purpose of reversing (flipping) the spin, or to enhance intrinsic resonances to a strength where the spin reverses completely rather than partially.

The strength of these induced resonances may be calculated by using the results of the previous sections.

(a) RF Solenoids

First; consider a radiofrequency solenoid with field $B_{sol} = \sqrt{2}B_{rms}\cos\omega_{rf}t$ .at one point in the ring, which we designate as θ=0. It rotates the spin by an angle

$$\Delta_s \cos\omega_{rf}t;\ \Delta_s = (1+G)\frac{\int\sqrt{2}B_{rms}ds}{B\rho} \tag{10}$$

once per revolution, and does not affect the orbit. (The integral over B_{rms} extends over the length of the rf magnet). Therefore we may set $z = z' = 0$, and (8) simplifies to

$$\zeta = -i(1+G)B_{sol} / B\rho = -i\Delta_s\delta_p(\theta)\cos\nu_{rf}\theta \tag{11}$$

where $\delta_p(\theta)$ is the periodic delta function and $\nu_{rf} = \omega_{rf} / \omega_{orbit}$ is the rf frequency normalized to the revolution frequency. Since $\cos\nu_{rf}\theta = (e^{i\nu_{rf}\theta} + e^{-i\nu_{rf}\theta}) / 2$ we can, as before, divide ζ into $\zeta_+ + \zeta_-$ and obtain, for $G\gamma=\kappa=k\pm\nu_z$,

$$\varepsilon_K = \frac{1}{2\pi}\int_0^{2\pi R}\zeta_\pm e^{iG\gamma\Theta}ds = -\frac{i(1+G)}{4\pi}\frac{\int\sqrt{2}B_{rms}ds}{B\rho} \tag{12}$$

in agreement with [7] and [10]. Thus there are two resonances of equal strength and equal phase in each interval of $G\gamma$ between one integer and the next . If ν_{rf} is exactly a half integer these coalesce into a single resonance of twice the strength. Note that, since Δ_s contains the factor $1 / B\rho$ these resonances become weak at high energy; therefore rf solenoids are primarily useful for low-energy rings such as IUCF or COSY.

(b) RF Dipoles

An alternative is to use rf dipoles with <u>transverse</u> fields. We assume that we have a radiofrequency dipole with radial horizontal field $\sqrt{2}B_{rms}\cos\omega_{rf}t$ at one point in the ring, which we designate as θ=0. The beam deflection it produces is

$$\Delta\cos\omega_{rf}t;\quad \Delta = \frac{\int\sqrt{2}B_{rms}ds}{B\rho} \tag{13}$$

The spin rotation associated with this deflection is, according to eq. (1), just $(1+G\gamma)$ times the deflection, and this leads to the naïve equation

$$\varepsilon = (1+G\gamma)\frac{\Delta}{4\pi} = (1+G\gamma)\frac{\int \sqrt{2}B_{rms}ds}{4\pi B\rho} \tag{14}$$

for the resonance strength at $G\gamma = k \pm \nu_{rf}$ [7,10]. But this is clearly incomplete and therefore incorrect; thus any disagreement of experimental results [2] with (14) is of no significance. The dipole induces a forced vertical oscillation in the whole ring, which in turn also affects the spin just like any other vertical oscillation, and this effect must also be considered, as recognized in the text of [10] and by other authors [11].

The equation of orbit motion for the forced oscillations is

$$\frac{d^2 z}{ds^2} + K(s)z = \frac{\Delta}{R}\delta_p(\theta)\cos\nu_{rf}\theta \tag{15}$$

where $K(s)$ is the focusing gradient function, $\theta = s/R$ is the normalized azimuth, $\delta_p(\theta)$ is the periodic delta function, and $\nu_{rf} = \omega_{rf}/\omega_{orbit}$. We assume the solution of the homogeneous equation corresponding to (15) is known; the vertical tune is ν_z and the orbit functions are $\beta_z(s)$, $\alpha_z(s)$. The solution of (15) is

$$z = \frac{\nu_z\beta_z\Delta}{4\pi}\sum_{k=-\infty}^{\infty}\left(\frac{e^{i(k+\nu_{rf})\varphi}}{\nu_z^2-(k+\nu_{rf})^2} + \frac{e^{i(k-\nu_{rf})\varphi}}{\nu_z^2-(k-\nu_{rf})^2}\right) \tag{16}$$

The equations of spin dynamics expressed in terms of the trajectory excursion $z(s)$ remain valid . As in the discussion leading to eq. (6) the depolarizing term is $\zeta = \zeta_+ + \zeta_-$, where, in the trajectory-based frame, with $\kappa_\pm = k \pm \nu_{rf}$,

$$\zeta_\pm = \frac{\nu_z\Delta}{4\pi}\sum_{k=-\infty}^{\infty}\frac{e^{i\kappa_\pm\varphi}}{\nu_z^2-\kappa_\pm^2}\left[\frac{1+G\gamma}{\beta_z}\left(K\beta_z^2 - 1 + \frac{\kappa_\pm^2}{\nu_z^2}\right) + G\frac{\gamma-1}{\rho}\left(i\alpha_z + \frac{\kappa_\pm}{\nu_z}\right) + i(1+G)\beta_z\left(\frac{1}{\rho}\right)'\right] \tag{17}$$

The resonance strengths are, analogously with (6), for $G\gamma = \kappa = k \pm \nu_{rf}$,

$$\varepsilon_\kappa = \frac{1}{2\pi}\int_0^{2\pi R}\zeta_\pm e^{i\kappa\varphi}ds = \frac{\nu_z}{2\pi}\int_0^{2\pi}\beta_z\zeta_\pm e^{i\kappa\varphi}d\varphi \tag{18}$$

In the simple case of uniform focusing,

$$\beta_z = R/\nu_z;\ \alpha_z = 0;\ K\beta_z^2 = 1;\ \rho = R;\ \varphi = \theta,\ (1/\rho)' = 0$$

and (17) becomes

$$\zeta_\pm = \frac{\Delta}{4\pi R}\sum_{k=-\infty}^{\infty}\frac{e^{i\kappa_\pm\varphi}}{\nu_z^2-\kappa_\pm^2}\left[(1+G\gamma)\kappa_\pm^2 + G(\gamma-1)\kappa_\pm\right] \tag{19}$$

For each k only a single term of the infinite series contributes to (20): For $\kappa = G\gamma = n + \nu_{rf}$ (n any integer) this is the term $k = -n$, $\kappa_- = -n - \nu_{rf} = -G\gamma$ in ζ_-, obtaining

$$\zeta_{\kappa_-} = \frac{\Delta}{4\pi R} \frac{e^{-iG\gamma\varphi}}{\nu_z^2 - (G\gamma)^2} \left[(1+G\gamma)(G\gamma)^2 - G(\gamma-1)G\gamma \right]$$

$$\varepsilon_\kappa = \frac{R}{2\pi} \int_0^{2\pi} \zeta_{\kappa-} e^{iG\gamma\varphi} d\varphi = \frac{\int \sqrt{2} B_{rms} ds}{4\pi B\rho} \frac{(G\gamma)^3 + G^2\gamma}{\nu_z^2 - (G\gamma)^2} \qquad (20)$$

and the same expression for $\kappa = G\gamma = n - \nu_{rf}$. Kondratenko, Kondratenko and Filatov [12] have found the same result for the case of uniform focusing.

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