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Linear Algebraic Method for Non-Linear Map Analysis

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Abstract

We present a newly developed method to analyze some non-linear dynamics problems such as the Henon map using a matrix analysis method from linear algebra. Choosing the Henon map as an example, we analyze the spectral structure, the tune-amplitude dependence, the variation of tune and amplitude during the particle motion, etc., using the method of Jordan decomposition which is widely used in conventional linear algebra.[1]

INTRODUCTION

The question of the long term behavior of charged particles in storage rings has a long history. To gain understanding, one would like to analyze particle behavior under many iterations of the one turn map. The most reliable numerical approach is the use of a tracking code with appropriate local integration methods. For analysis, however, one would like a more compact representation of the one turn map out of which to extract relevant information. The main representations that have been studied have been either in terms of Lie operators[1], or power series[2]. Here, we use a third representation, a square matrix constructed out of the power series map. We find that using linear algebra methods on this matrix, the relevant information can be extracted with relative ease.[2] In particular, we might consider our method as a *normal form* algorithm [4],[5],[6], which provide tunes as functions of phase space. Our method is distinguished from other normal form approaches in that a measure of the lack of integrability may be also be calculated, thus giving an analytical approach to what has previously only been available numerically[3].

We illustrate and apply these ideas using the Henon map as an example.

REPRESENTATION OF NON-LINEAR MAPS USING SQUARE MATRICES

The power series for the Henon map is given by

$$x = x_0 \cos \mu + p_0 \sin \mu p_0 + \epsilon x_0^2 \sin \mu \qquad (1)$$
$$p = -x_0 \sin \mu + p_0 \cos \mu + \epsilon p_0^2 \cos \mu$$

This represents a rotation in phase space by an angle μ, followed by a single sextupole kick with strength ϵ. Note that on the left, only x and p appear, whereas on the right,

we have different powers. By extending the phase space point to include higher order monomials and truncating at a fixed order, we may construct a square matrix representation. The other elements are given by finding higher order monomials of x and p. For example, with the Henon map, up to 4th order, we find

$$
\begin{aligned}
x &= x_0 \cos \mu + p_0 \sin \mu + \epsilon x_0^2 \sin \mu \\
p &= -x_0 \sin \mu + p_0 \cos \mu + \epsilon x_0^2 \cos \mu \\
x^2 &= x_0^2 \cos^2 \mu + 2 x_0 p_0 \cos \mu \sin \mu \\
&\quad + p_0^2 \sin^2 \mu + \epsilon x_0^3 \cos \mu \sin \mu + \epsilon x_0^2 p_0 \sin^2 \mu + \epsilon^2 x_0^4 \cos^2 \mu \\
xp &= -x_0^2 \cos \mu \sin \mu + x_0 p_0 (\cos^2 \mu - \sin^2 \mu) \\
&\quad + \epsilon x_0^3 (\cos^2 \mu - \sin^2 \mu) + 2 \epsilon x_0^2 p_0 + \epsilon^2 x_0^4 \cos \mu \sin \mu
\end{aligned}
$$

$$\ldots$$

$$
\begin{aligned}
p^4 &= x_0^4 \cos^4 \mu + 4 x_0^3 p_0 \cos^3 \mu \sin \mu + \\
&\quad + 6 x_0^2 p_0^2 \cos^2 \mu \sin^2 \mu + 4 x_0 p_0^3 \cos \mu \sin^3 \mu + p_0^4 \cos^4 \mu
\end{aligned}
$$

We may now write this in the form

$$X = M X_0 \qquad (2)$$

where to 4th order, we define the 14×1 monomial array

$$\tilde{X}_0 = (\, x_0 \quad p_0 \quad x_0^2 \quad x_0 p_0 \quad p_0^2 \quad \ldots \quad p_0^4 \,) \qquad (3)$$

The tilde represents matrix transposition. We refer to X_0 as an extended phase space vector. In the more general case, M in (2) could represent a map for a full turn, part of a turn, or simply a transformation of variables. Here it represents a one turn map. We point out a benefit of this formulation is that map composition reduces simply to matrix multiplication.

By extending the phase space vector, we have linearized the evolution equations. In the Lie Algebra formulation, maps act on phase space functions. If we represent the coefficient of the polynomial functions as row vectors, (e.g. (0 0 k 0 1) represents $kx^2 + p^2$) then a Lie map may be represented as a square matrix acting to the left. Thus, in this matrix picture, both the Lie map and the transfer map may be represented by the same object.[4]

ANALYSIS OF SQUARE MATRIX MAP USING LINEAR ALGEBRA

We would like to extract the spectral structure from the matrix M. The first inclination may be to try to diagonalize and find the eigenvalues. It turns out, however, that

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[2]We note that this matrix description has been mentioned in the literature,(see e.g. [3], eq. 9.36 and following) though not pursued.

[3]Non-integrability via tune diffusion is part of the numerical method known as Frequency Map Analysis [8].

[4]In case one represents functions as column vectors, then the Lie map and transfer map matrix are transposes of each other. That these objects are transposes (or adjoints) has been pointed out by Forest [7].

for symplectic non-linear maps M is not diagonalizable. All square matrices may be transformed into Jordan form, however. When transforming the non-linear map matrix, we find that the Jordan form always has the same structure, dependent only on the eigenvalues of the linear map. The dependence on the specific form of the non-linearity is hidden in the structure of the transformation. We will actually find it more convenient to do a Jordan decomposition on $\tilde{M}$ as will become clearer later. Doing so, we find

$$\tilde{N} = U\tilde{M}U^{-1} \tag{4}$$

$$= \begin{pmatrix} \tilde{N}_0 & & & & & \\ & \tilde{N}_1 & & & & \\ & & \tilde{N}_{-1} & & & \\ & & & \tilde{N}_2 & & \\ & & & & \tilde{N}_{-2} & \\ & & & & & \ddots \end{pmatrix} \tag{5}$$

$$\tilde{N}_k = e^{ik\mu}I + \tau^\dagger \tag{6}$$

with $k = \pm 1, \pm 2, \ldots$. I is the identity matrix and $\tau^\dagger$ the matrix with 1's just above the diagonal:

$$\tau^\dagger = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \tau = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7}$$

We have written out τ and $\tau^\dagger$ explicitly for a 4×4 case. We will use the symbols interchangably regardless of their dimension. The different blocks will have different dimensions l_k dependent on the truncation order. The 0 block $N_0 = I$ gives invariants.

RAISING THE MAP TO THE NTH POWER

In order to find the motion after n turns, we need to compute M^n. We replace (2) with $X = M^n X_0$. With (4) we find $M^n = \tilde{U}^{-1}N^n\tilde{U}$. Defining $\Psi_0 = \tilde{U}X_0$, we find the new value of the Ψ_0 vector to be $\Psi = N^n\Psi_0 = e^{n\ln N}\Psi_0$. We perform the last step in order to extract the frequency content. Expanding the logarithm in the kth block, using the transpose of (6) we get $\ln(N_k) = ik\mu I + e^{-ik\mu}\tau - \frac{1}{2}(e^{-ik\mu}\tau)^2 + \ldots$ This matrix is in lower triangular form, since powers of τ just move the ones further below the diagonal. The series will be automatically truncated at the order l_k. Now, we find a further transformation $\tilde{V}$ to bring each of the $\ln N_k$ into Jordan form. Let V_k be the block of V corresponding to N_k. Doing this, we find

$$\ln(N_k) = \tilde{V}_k^{-1}(ik\mu I + \tau)\tilde{V}_k \tag{8}$$

Now, if we define the vector

$$\Phi_{k0} = \tilde{V}_k\tilde{U}_k X_0 \tag{9}$$

Where U_k is the matrix that projects X onto the N_k subspace. [5] Then the new vector after n turns is

$$\Phi_k = e^{n(i\mu+\tau)}\Phi_{k0} \tag{10}$$

EIGENVECTORS OF JORDAN BLOCKS, LOWERING OPERATOR, AND COHERENT STATES

We now focus on Φ_1, the $k = 1$ projection of Φ, corresponding to $e^{i\mu}$. We drop the subscript and just refer to the initial and final vectors in the subspace as Φ_0 and Φ. We write

$$\tilde{\Phi}_0 = \begin{pmatrix} \phi_l & \ldots & \phi_2 & \phi_1 & \phi_0 \end{pmatrix} \tag{11}$$

with l the dimension of the subspace, (determined by the truncation order). The evolution equation (10) can be written $\Phi = e^{in\mu}e^{n\tau}\Phi_0$. We note that if Φ_0 were an eigenvector of τ, then (10) would be a simple eigenvector equation.

Let $|j\rangle$ represent a vector with a one in the jth slot from the bottom and all other elements 0. These form a basis for the subspace. The matrices τ and $\tau^\dagger$ act on these states as raising and lowering operators respectively:

$$\tau\,|j\rangle = |j-1\rangle \text{ for } 0 < n \leq l, \ \ \tau\,|0\rangle = 0 \tag{12}$$

$$\tau^\dagger\,|j\rangle = |j+1\rangle \text{ for } 0 \leq n < l, \ \ \tau^\dagger\,|l\rangle = 0 \tag{13}$$

Now, we define

$$|\alpha\rangle = \sum_{j=0}^{l} \alpha^j\,|j\rangle. \tag{14}$$

Consider for the moment the limit $l \to \infty$. We then find

$$\tau\,|\alpha\rangle = \alpha\,|\alpha\rangle. \tag{15}$$

In other words, $|\alpha\rangle$ is the eigenvector of τ with eigenvalue α. This is the definition of a coherent state, analogous to minimum uncertainty wave-packets in quantum mechanics. For finite l, $|\alpha\rangle$ is only an approximate eigenvector of τ. However, the failure is at the highest power in the coordinates, so we expect this to be a good approximation. From now on, we neglect this distinction, treating $|\alpha\rangle$ as an eigenvector of τ even for finite l.

Now, suppose that:

$$\Phi_0 = \phi_0\,|\alpha\rangle \tag{16}$$

Then

$$\Phi = e^{n(i\mu+\tau)}\phi_0\,|\alpha\rangle = e^{n(i\mu+\alpha)}\Phi_0 \tag{17}$$

So that $\mu + \text{Im}(\alpha)$ is the frequency associated with this block. From this, we interpret $\text{Im}(\alpha)$ to be the tune shift at initial phase space position (x_0, p_0). If $\text{Re}(\alpha) > 0$, then the particle is unstable.

An integrable system has a well-defined tune for any initial phase space position. Thus we make an identification between coherent states for Φ_0 and integrable systems.

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[5]We remark that U_1V_1 is a 35×4 matrix for 7th order truncation. The four columns form an eigenbasis for the $k = 1$ subspace of $\tilde{M}$. There are $l_k - 1$ degrees of freedom to choose this basis which can be fixed by choosing the generalized eigenvectors to be orthogonal to the ground state, which is defined below.

DEVIATION FROM COHERENCE, OR LACK OF INTEGRABILITY

When the system is not integrable, the ratio $\alpha_j \equiv \frac{\phi_{j+1}}{\phi_j}$ varies with j.[6] But away from chaotic regions, the variation is presumably small. $\phi_0(n)$ may be expanded as

$$\phi_0(n) = \phi_0 e^{(i\mu+\alpha_0)n} \qquad (18)$$
$$\times e^{\frac{1}{2}(\alpha_1\alpha_0-\alpha_0^2)n^2+\frac{1}{6}(2\alpha_0^3-3\alpha_0^2\alpha_1+\alpha_0\alpha_1\alpha_2)n^3+\dots}$$

When all α_j are equal, it is clear that the second factor equals one. Thus, this factor represents the deviation from coherence or integrability. In particular, $\frac{1}{2}\text{Im}(\alpha_0(\alpha_1-\alpha_0))$ gives the first order variation in the tune with turn number, or a tune diffusion.

RESULTS FOR HENON MAP

We carry out these calculations for the Henon map. We rescale[7] the coordinates such that $\epsilon = 1$. We choose as an example, $\mu = 2\pi(3-\sqrt{5})/2$. In Figure 1, we have plotted $\text{Im}(\alpha_1)$ and $\text{Im}(\alpha_0)$ (lines: 7th order, dots: 8th order) along the $x_0 = 0$ line in phase space. We note that these begin to differ beyond $|p_0| > 0.5$, representing a deviation from integrability. We see good agreement to the analytical tune shift with amplitude formula: $\Delta\mu = -(1/16)(\cot[3\mu/2] + 3\cot[\mu/2])(x_0^2 + p_0^2)$. In Figure 2, we show the contours of $|\phi_0|$ (calculated to 6th order) along with the tracking results near the boundary of stability. Note that the orbit does not fall on a single curve which is an indication of chaos in this region. Comparison to Figure 1 shows the deviation from coherence overlaps with this boundary.

CONCLUSIONS AND FUTURE WORK

We have found that our analysis of the non-linear mapping matrix for the Henon map successfully reproduces both the correct phase space structure and tune shift with amplitude. In addition, deviation from integrability can be seen. We expect that maps extracted from realistic storage ring models may also be efficiently analyzed. Extension to higher dimension is relatively straight forward. We have confined our analysis to a single block of the N matrix. This is valid off resonance. Near resonance, multiple eigenvalues become close to each other and one should consider multiple blocks to understand the dynamics. This is a topic of on-going investigation.

ACKNOWLEDGEMENTS

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[6]We remark that we expect that the transformation U to Jordan form does not represent a coordinate transformation for the non-integrable case, whereas for the integrable case, it is a coordinate transformation.

[7]See [4] for more details.
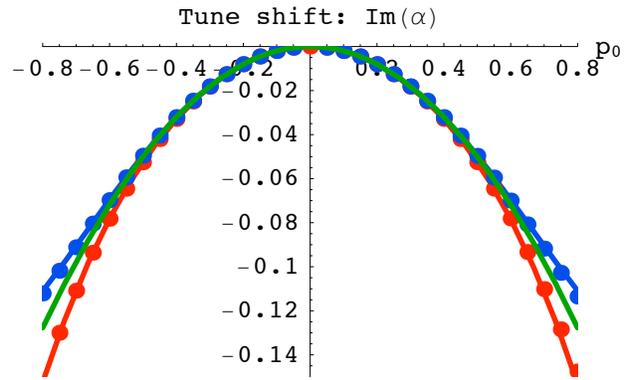
Figure 1: $\text{Im}(\alpha_1)$ (blue), $\text{Im}(\alpha_0)$ (red), and analytical tune shift formula (green). vs. p_0 for $x_0 = 0$.
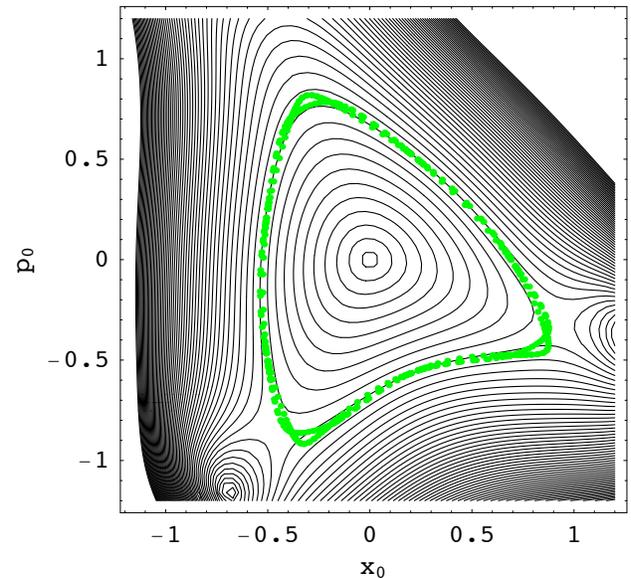
Figure 2: Invariant contours from $|\phi_0|$ for Henon map compared to tracking results (green).

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