



**BNL-94966-2011-CP**

***Calculating Point-Charge Wakefields from Finite  
Length Bunch Wake-potentials***

*B. Podobedov, G. Stupakov*

*Presented at the 2011 Particle Accelerator Conference (PAC'11)  
New York, NY  
March 28-April 1, 2011*

March 2011

**Photon Sciences Directorate**

**Brookhaven National Laboratory**

**U.S. Department of Energy  
DOE - Office of Science**

Notice: This manuscript has been authored by employees of Brookhaven Science Associates, LLC under Contract No. DE-AC02-98CH10886 with the U.S. Department of Energy. The publisher by accepting the manuscript for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for United States Government purposes.

This preprint is intended for publication in a journal or proceedings. Since changes may be made before publication, it may not be cited or reproduced without the author's permission.

## DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

# CALCULATING POINT-CHARGE WAKEFIELDS FROM FINITE LENGTH BUNCH WAKE-POTENTIALS\*

Boris Podobedov, Brookhaven National Laboratory, Upton, NY 11973  
Gennady Stupakov, SLAC National Accelerator Laboratory, Menlo Park, CA 94025.

## Abstract

Starting from analytical properties of high frequency geometric impedance we show how one can accurately calculate short bunch wake-potentials (and even point-charge wakefields) from time domain calculations performed with a much longer bunch. In many practical instances this drastically reduces the need for computer resources, speeds up the calculations, and improves their accuracy. To illustrate this method we give examples for 2D accelerator structures of various complexities.

## INTRODUCTION AND MAIN IDEA

Knowledge of wakefields, and in particular geometric wakefields, is critically important for studies of accelerator beam dynamics. While analytical solutions are known for a number of simple geometries, detailed wakefield calculations for realistic vacuum chamber components is typically done utilizing time domain EM solvers. These, by design, calculate the fields due to finite length bunches, and one is forced to use extremely fine mesh (small fraction of the bunch length) to compute wakes at small distances. This is where the wakes are usually dominated by singularities, so that a wake-potential due a bunch of rms length  $\sigma$  scales as  $W^\sigma(z) \propto \sigma^{-q}$ ,  $q > 0$  [1]. Utilizing fine meshes has obvious implications for computer memory requirements as well as calculation speed.

To get around the fine mesh requirement we propose to take care of the wake singularities by representing the wake-potential as a sum of two parts,

$$W^\sigma(z) = W_{\text{model}}^\sigma(z) + D^\sigma(z), \quad (1)$$

where  $W_{\text{model}}^\sigma(z)$  is the short-bunch asymptotic model (that includes all the singularities at  $\sigma \rightarrow 0$ ), and function  $D^\sigma(z)$  that remains finite for arbitrarily short bunches. Furthermore, we suggest that for bunches shorter than certain length  $\sigma_0$  (usually correlated with the smallest dimension of the structure) the  $\sigma$ -dependence of  $D^\sigma(z)$  is rather weak, so it can be dropped,

$$D^\sigma(z) \approx D(z), \quad 0 \leq \sigma \leq \sigma_0. \quad (2)$$

Thus, if we pick the initial bunch length to be  $\sigma_0$  (or shorter) and calculate the corresponding wake-potential  $W^{\sigma_0}(z)$  with an EM solver, we can then set

$$D(z) = W^{\sigma_0}(z) - W_{\text{model}}^{\sigma_0}(z). \quad (3)$$

For bunches, shorter than  $\sigma_0$ , we now have

$$W^\sigma(z) \approx W_{\text{model}}^\sigma(z) + D(z), \quad 0 \leq \sigma \leq \sigma_0. \quad (4)$$

In the limit of vanishing bunch length we can write for the point-charge wakefield

$$W^\delta(z) \approx W_{\text{model}}^\delta(z) + D(z). \quad (5)$$

For this method to be practical, we need to show that 1)  $W_{\text{model}}^\sigma(z)$  is easily identifiable, tractable, and it captures all singularities; 2) with some reasonable choice of  $\sigma_0$  Eq. (2) holds true. With the help of time domain EM code ECHO [2], we show in the rest of this paper, that both of these statements are correct for a large number of 2D accelerator structures of the collimator and cavity type. We also show that for these structures a short bunch wake can be accurately reconstructed from EM solver results for a much longer bunch.

## COLLIMATOR-LIKE GEOMETRY

For a collimator chamber that transitions from radius  $a$  to smaller radius  $b$  and back, the point-charge wakefield and the wake-potential due to a Gaussian bunch of rms length  $\sigma$ , are given, respectively, by

$$W_o^\delta(z) = k_o \delta(z), \quad (6)$$

$$W_o^\sigma(z) = k_o (2\pi)^{-1/2} \sigma^{-1} \exp\left(-\frac{z^2}{2\sigma^2}\right), \quad (7)$$

where  $Z_0$  is the free space impedance, and

$$k_o = -Z_0 c \text{Log}(a/b) / \pi. \quad (8)$$

In our sign convention  $W^\sigma(z) < 0$  corresponds to the energy loss of the particle with longitudinal coordinate  $z$ ;  $z < 0$  is the head of the bunch. Eq. (7) implies that the minimum of the wake-potential diverges for short bunches as  $W_o^\sigma(0) \propto \sigma^{-1}$ .

Eqs. (6-8) were derived a long time ago by various authors. Recently it was shown [3] that this example is a particular case of the so-called optical regime, where the wakes of all collimator-like structures (including 3D) have the same  $z$ -dependence as is shown above. Furthermore, in this asymptotic regime one can find exact impedances and wakefields of very complicated collimators (which amounts to calculating the value of geometry-dependent coefficient  $k_o$ , for which the recipe was given in [3]). This makes us believe that what we illustrate below for axially symmetric case is applicable to

3D. Meanwhile through the rest of this section we will assume that  $W_{\text{model}}^{\sigma}(z)$  is given by Eqs. (7-8).

We investigated in detail a family of collimator-like geometries; their radial dependence along the collimator is shown in Fig. 2 (inset). In the short bunch limit all of them have the same asymptotic wake given by Eqs.(7-8).

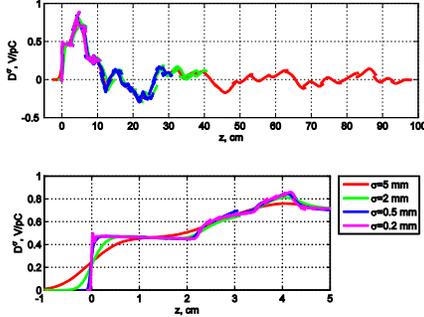


Figure 1: Function  $D^{\sigma}(z)$  for 3-step collimator.

First, in Fig. 1, we present the calculations for 3-step collimator (geometry shown in Fig. 2 inset in red dash). Specifically, we plot the function  $D^{\sigma}(z)$  defined in Eq. (1), i.e. the wake-potential (calculated with ECHO) with the optical model subtracted. Clearly, the plots in Fig. 1 do not show any singular behaviour as  $\sigma$  gets shorter. Furthermore, for the range of the bunch lengths shown, the traces basically overlap for most values of  $z$ , which confirms that  $D^{\sigma}(z)$  is weakly dependent on  $\sigma$ , i.e. Eq. (2) holds true. However, at certain ‘‘corner’’ points ( $z \approx 2.2$  cm, 4.2 cm, etc.),  $D^{\sigma}(z)$  has discontinuous derivatives, so strictly-speaking  $\sigma \rightarrow 0$  is needed to exactly match this behaviour. As long as we are not interested in the derivatives of the wake at these points, then  $D^{\sigma}(z)$  calculated with a reasonably short bunch (i.e.  $\sigma < 2$  mm) clearly provides a very good accuracy near these locations as well.

Fig. 2 shows  $D^{0.5\text{mm}}(z)$  for the set of collimators. For  $z < 1$  cm (which happens to be the smallest radial step among this set) all  $D^{0.5\text{mm}}(z)$  curves converge to the same linear trend already discussed above. Thus, for fixed  $a$  and  $b$ , the short-range  $D^{\sigma}(z)$  is geometry-independent. This observation is important, since this simple short range behaviour (straight line) can be accurately calculated with a fairly long bunch, i.e.  $\sigma_0 = 0.5$  mm. Furthermore,  $\sigma_0$  can be predicted in advance for most geometries; it simply needs to be short compared to the characteristic dimensions of the structure.

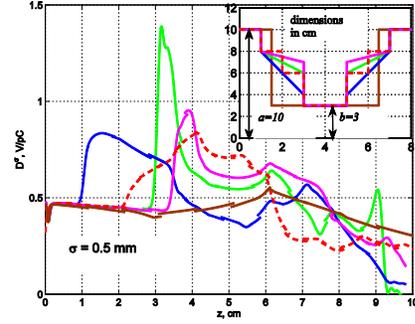


Figure 2: Functions  $D^{\sigma}(z)$  for a family of collimators.

The situation becomes more complicated for slowly tapered collimators. However, recently we used a similar approach [4] (although in the frequency domain) to reconstruct accurate short bunch wake-potentials of a slowly tapered collimator from band-limited impedance.

## CAVITY-LIKE GEOMETRY

For cavity-like structures the singular parts of the wakes are given by the diffraction model [1],

$$W_d^{\delta}(z) = k_d z^{-1/2} \quad (z > 0), \quad (9)$$

$$W_d^{\sigma}(z) = k_d \sigma^{-1/2} f(z/\sigma), \quad (10)$$

$$\text{where } k_d = -Z_0 c \pi^{-2} a^{-1} \sqrt{g/2}, \quad (11)$$

$$f(s) = e^{-s^{2/4}} \sqrt{|s|} (\mathbf{I}_{-1/4}(\frac{s^2}{4}) + \text{sign}(s) \mathbf{I}_{1/4}(\frac{s^2}{4})),$$

and  $\mathbf{I}_{\pm 1/4}(\frac{s^2}{4})$  are the Bessel functions. The minimum of the wake-potential occurs at  $z \approx 0.76\sigma$ , and it diverges  $\propto \sigma^{-1/2}$  for short bunches. Below we assume that  $W_{\text{model}}^{\sigma}(z)$  is given by  $W_d^{\sigma}(z)$ , defined by Eq. (10).

We first consider a simple  $g=1$  cm long cylindrical cavity of rectangular profile; the cavity and outer pipe radii are  $b=5$  cm,  $a=1$  cm (Fig.4 inset, solid brown). ECHO-calculated wake-potentials are shown in Fig 3.

The short-range wake-potentials clearly conform to the diffraction model, i.e. their minima scale  $\propto \sigma^{-1/2}$ . However, similarly scaling peaks occur at multiples of  $2g$ ; particles at these locations gain energy. Subtracting the diffraction model from the calculated wake-potential results in a function that, in the limit of  $\sigma \rightarrow 0$ , is still singular at these points (see Fig. 4, ‘‘Rectangular’’ traces).

For this simple geometry, one could expand the diffraction model to analytically describe the wake-potential at these locations [5], so that one can then eliminate all singularities by subtracting this expanded model. For now we omit this exercise, but rather point out that singularities at multiples of  $2g$  are due to reflections of the diffracted wave at the cavity sidewalls. These singularities disappear for realistic cavities with smooth corners and/or non-parallel sidewalls.

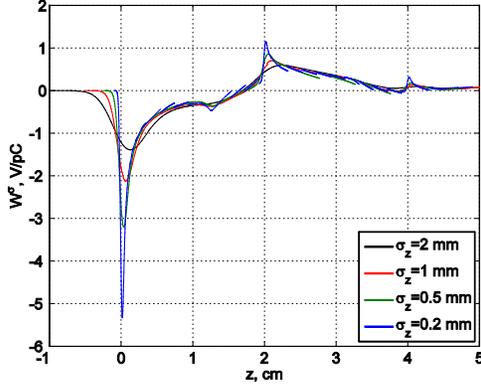


Figure 3: Wake-potentials for rectangular profile cavity.

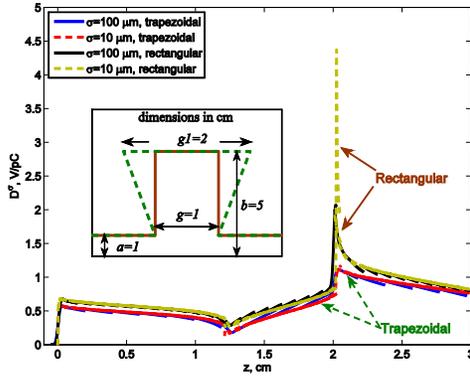


Figure 4:  $D^\sigma(z)$  for rectangular and trapezoidal cavities.

For example, if the inner corners of this cavity are replaced with 90 degree arc segments of radius  $r=1$  mm, then, after reaching the value of  $\sim 1.3$  V/pC at  $\sigma \approx 200\mu\text{m} \approx r/2\pi$ , the  $W^\sigma(z \approx 2\text{cm})$  peak stops growing with decreasing  $\sigma$ .

Another example is a cavity with non-parallel sidewalls, like the “trapezoidal” one shown in Fig. 4, inset. Its wakes at multiples of  $2g$  are not singular, as is clear from Fig. 4.  $D^{100\mu\text{m}}(z)$  and  $D^{10\mu\text{m}}(z)$  are almost exactly equal. In a separate set of calculations, we confirmed that these singularities are absent for other values of  $g_1 \neq g$ , all the way down to  $g_1=0$ .

With regards to the short-range behaviour of  $D^\sigma(z)$  for either cavity, we observe that it is qualitatively similar to that discussed earlier for the collimator case. A positive, almost linear and featureless trend continues to  $z \sim 1$  cm, the distance easily predictable from cavity dimensions.

Since for trapezoidal cavity  $D^\sigma(z)$  is non-singular everywhere, and it is  $\sigma$ -independent for  $\sigma < \sim 1$  mm,  $D^{100\mu\text{m}}(z)$  clearly allows for an accurate calculation of the wake-potential of shorter bunches.

As a final and more practical example we calculate a very short bunch wake for 9-cell TESLA structure.

Principle dimensions are 115.4 mm period, 70 mm iris diameter, and 1.036 m total length [6].

In this case the asymptotic wake model is unknown, although we expect  $\sigma$ -scaling similar to the diffraction model. To obtain an accurate  $W_{\text{model}}^\sigma(z)$ , we pick reasonably short  $\sigma_1$  and  $\sigma_2$  and calculate  $W^{\sigma_1}(z)$  and  $W^{\sigma_2}(z)$  with ECHO. Then we assume the equation

$$W^\sigma(z) = \kappa W_d^\sigma(z) + D(z), \quad (12)$$

to hold exactly for  $\sigma=\sigma_1$  and  $\sigma=\sigma_2$ , with constant  $\kappa$ .

Solving for  $\kappa$  by direct elimination of  $D(z)$  results in  $z$ -varying  $\kappa$ . However, near  $z=0$  (the region important for modelling short-bunch wakes) this dependence is almost flat. Thus we set  $\kappa = (W^{\sigma_1} - W^{\sigma_2})(W_d^{\sigma_1} - W_d^{\sigma_2})^{-1}|_{z=0}$ .

Now  $D(z)$  follows directly from Eq. (12), and wake-potential of a shorter bunch is calculated from Eq. (4).

A wake-potential for  $3\mu\text{m}$  rms bunch reconstructed in this manner from  $\sigma_1=100\mu\text{m}$  and  $\sigma_2=30\mu\text{m}$  ECHO calculations is plotted in Fig. 5. It shows perfect agreement with the direct ECHO calculation for  $\sigma=3\mu\text{m}$  bunch. A similar agreement was also found for the long-range wake-potentials.

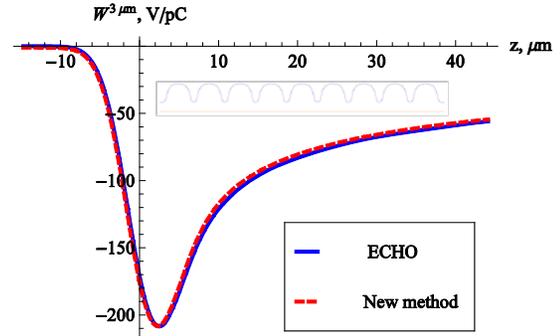


Figure 5: TESLA cavity wakes for  $3\mu\text{m}$  bunch calculated directly and from  $\sigma_1=100\mu\text{m}$  and  $\sigma_2=30\mu\text{m}$  calculations.

## CONCLUSIONS

We describe preliminary results of a new method that allows us to accurately obtain longitudinal wakefields of short bunches by adding a long-bunch result from an EM solver and a singular analytical wake model. In the future this work will be generalized to 3D geometries as well. Similarly, the method should be equally applicable to the calculations of transverse wakefields.

Periodic structures with a significant number of periods ( $N \geq a^2 / \sigma L$ , where  $L$  is the period length) have not been considered so far. They have asymptotic wakefields that differ from the examples described above. We believe this method is applicable to such geometries as well, as long as correct asymptotic solutions are used.

We acknowledge the support of US DOE under contracts DE-AC02-98CH10886 and DE-AC02-76SF00515.

## REFERENCES

- [1] A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, (Wiley, NY, 1993).
- [2] I. Zagorodnov, T. Weiland, PRSTAB 8, 042001 (2005).
- [3] G. Stupakov, K. Bane, I. Zagorodnov, PRSTAB 10, 054401 (2007).
- [4] G. Stupakov and B. Podobedov, PRSTAB, 13, 104401 (2010).
- [5] B. Podobedov and G. Stupakov, in preparation.
- [6] TESLA TDR, [www.desy.de](http://www.desy.de), (2001).