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Abstract

This focus of this note is to determine what is the limitation on the amplification of the density modulation in FEL-type process caused by saturation. This derivation is applicable to any directional instability in the electron beam with a resonant wave-number $k_o \equiv 2\pi / \lambda_o$.

It is well known fact that e-beam instabilities, including that in FEL, are described by a set of self-consistent Maxwell and Vlasov equation. In classical limit, Maxwell equations are completely linear. The later is not true about the Vlasov equation. Hence, the later is responsible for the saturation (if any).

It is also a well established fact that Vlasov equation can be linearized when the density modulation is significantly smaller compared with initial beam density. In other words, the Vlasov equation becomes nonlinear (which can cause a saturation) when the density modulation becomes comparable with the initial beam density: $|\delta n| \sim n_o$.

When $|\delta n / n_o| << 1$ we can use linearized Vlassov equation, which then can be represented by a Green function. Here, for compactness, I consider a directional instability, where the response of the system on a perturbation can be described by one-dimensional Green function. This is usually true for an FEL, where diffraction makes transverse dependences smooth compared with the fast longitudinal oscillations.

Here I also consider a response of the system to be much shorter compared with the electron bunch length, e.g. the e-beam density could be considered locally constant.

General case

In a linear approximation, the reaction of an FEL with constant electron beam density n_o on a local perturbation

$$\delta n = \delta(z - z_o)$$

can be approximately described by a 1D Green function

$$n(\tau) = n_o + \delta(z - z_o) + G_\tau(z - z_o), \tag{1}$$

with obvious $\int G_\tau(z)dz = 0$.

Let's consider a case of typical random start of SASE FEL with initial distribution of electrons as

$$n_o(0, z) = \sum_{i=1}^{N} \delta(z - z_i). \tag{2}$$

where N is the number of electrons in the beam. Than the linear amplified FEL response reads:

$$n_o(\tau, z) = +\sum_{i=1}^{N} \delta(z - z_i) + \sum_{i=1}^{N} G_\tau(z - z_i). \tag{3}$$

In the presence of density modulation imprinted by ions with charge Z and the effective density X=x*Z, x~1[1]

$$n_o(\tau,z)=+\sum_{i=1}^{N_e}\delta(z-z_i)+\sum_{i=1}^{N_e}G_\tau(z-z_i)+X\sum_{j=1}^{N_i}G_\tau(z-z_j)\,. \qquad (4)$$

Let's calculate bunching factor, corresponding to the relative local density modulation by picking an arbitrary wavelet:

$$b(\tau)=\frac{\int_o^{\lambda_o} n(\tau,z)e^{ik_o z}\,dz}{\int_o^{\lambda_o} n(\tau,z)\,dz}=\frac{\sum_{i,z_i\in\{0,\lambda_o\}} e^{ik_o z_i}+g(z_i)\sum_{i=1}^{N_e}e^{ik_o z_i}+Xg(z_j)\sum_{j=1}^{N_i}e^{ik_o z_j}}{M};$$

$$\int_o^{\lambda_o} G_\tau(z-z_i)e^{ik_o z}\,dz=e^{ik_o z_i}\int_{-z_i}^{\lambda_o-z_i} G_\tau(z)e^{ik_o z}\,dz=e^{ik_o z_i}g(z_i) \qquad (5)$$

$$g(z_i)=\int_{-z_i}^{\lambda_o-z_i} G_\tau(z)e^{ik_o z}\,dz;$$

where M is number of electron in a wavelet $\{0,\lambda_o\equiv 2\pi/k_o\}$.

Let's calculate a RMS value of the bunching factor, assuming absence of correlation between electrons and hadrons, i.e. assuming a random Poisson distribution of their initial phases:

$$\left\langle\left|Mb(\tau)\right|^2\right\rangle=\left|\sum_{i,z_i\in\{0,\lambda_o\}} e^{ik_o z_i}+g(z_i)\sum_{i=1}^{N_e}e^{ik_o z_i}+X(z_j)g(z_j)\sum_{i=1}^{N_i}e^{ik_o z_i}+\right|^2$$

$$\left|\sum_{i,z_i\in\{0,\lambda_o\}} e^{ik_o z_i}+g(z_i)\sum_{i=1}^{N}e^{ik_o z_i}\right|^2=\sum_{i,z_i\in\{0,\lambda_o\}}\left(1+2\operatorname{Re}g(z_i)\right)+\sum_{i=1}^{N_e}\left|g(z_i)\right|^2+X^2\sum_{j}^{N_i}\left|g(z_j)\right|^2 \,. \qquad (6)$$

In other words, the

$$\left\langle\left|Mb(\tau)\right|^2\right\rangle=M+2\sum_{i,z_i\in\{0,\lambda_o\}}\operatorname{Re}g(z_i)+\sum_{i=1}^{N_e}\left|g(z_i)\right|^2+\sum_{j}^{N_i}X^2(z_j)\left|g(z_j)\right|^2$$

$$\left\langle\left|Mb(\tau)\right|^2\right\rangle=M\left(1+2\cdot\left\langle\operatorname{Re}g(z)\right\rangle_{z\in\{0,\lambda_o\}}\right)+\int_{-\infty}^{\infty}\Lambda_e(z)\left|g(z)\right|^2dz+\int_{-\infty}^{\infty}\Lambda_I(z)X^2(z)\left|g(z)\right|^2dz \qquad (7)$$

where Λ_e and Λ_i are linear density of electrons and ions:

[1] To be exact, we should use a convolution of the induced density modulation in the modulator with Green function:

$$X\sum_{j=1}^{N_i}G_\tau(z-z_j)\to\sum_{j=1}^{N_i}G_{\tau h}(z-z_j);\quad G_{\tau h}(z-z_j)=\int G_{\tau h}(z-\zeta)\rho_h(\zeta-z_j)d\zeta\,.$$

where ρ_h is a density modulation induced by the hadron in the CeC modulator [1]. For cases of interest, the details of ρ_h are not important, when its duration is much shorter than that of the amplifier.

$$\Lambda_e(z)=\left\langle\frac{\sum\limits_{z_l\in\{z-\Delta z,z+\Delta z\}}1}{2\Delta z}\right\rangle_{\Delta z\to 0};\Lambda_I(z)=\left\langle\frac{\sum\limits_{z_j\in\{z-\Delta z,z+\Delta z\}}1}{2\Delta z}\right\rangle_{\Delta z\to 0}.$$

For simplicity let's look at the contentious beam with fixed density:

$$\Lambda_e=\frac{M_e}{\lambda_o};\Lambda_I=\frac{M_I}{\lambda_o}.$$

Then

$$\left\langle\left|Mb(\tau)\right|^2\right\rangle=M(1+2\cdot\left\langle\operatorname{Re}g(z)\right\rangle_{z\in\{0,\lambda_o\}})+\Lambda_e\int_{-\infty}^{\infty}\left|g(z)\right|^2dz+X^2\cdot\Lambda_I\int_{-\infty}^{\infty}\left|g(z)\right|^2dz \quad (8)$$

Causality in FEL means that the $\left|g(z)\right|_{z\in(0,\lambda_o)}\to 0$. Thus, the modulation is determined by the effective correlation length, which is defined as [1]:

$$\int_{-\infty}^{\infty}\left|g(z)\right|^2dz=g_{\max}^2N_c\lambda_o$$

Then, assuming that the FEL is saturated at b < 1;

$$1+g_{\max}^2N_c\left(1+X^2\cdot\frac{M_I}{M_e}\right)\le M_e$$

gives an estimate for the maximum attainable gain for an initial δ-function:

$$g_{\max}\le\sqrt{\frac{M}{N_c\left(1+X^2\cdot\dfrac{M_I}{M_e}\right)}}$$

To be exact, for am FEL operating at peak current of I_p and wavelength λ_o,

$$M_e=\frac{I_{pe}\lambda_o}{ec}=2.08\cdot10^4\cdot I_p[A]\lambda_o[\mu m]$$

This gives a following limitation on the maximum attainable gain:

$$g_{\max}\le144\cdot\sqrt{\frac{I_{pe}[A]\cdot\lambda_o[\mu m]}{N_c\left(1+\dfrac{X^2}{Z}\cdot\dfrac{I_{pI}}{I_{pe}}\right)}}$$

Since from plasma oscillations in a modulator [1], $X\le 2Z$

$$g_{\max} \sim 144 \cdot \sqrt{\frac{I_{pe}[A] \cdot \lambda_o[\mu m]}{N_c \left(1 + Z^* \frac{I_{\beta I}}{I_{pe}}\right)}}; \; Z^* = X^2 / Z \tag{9}$$

and

$$g_{\max} \sim \sqrt{\frac{I_{pe} \lambda_o}{ec \cdot N_c \left(1 + Z^* \frac{I_{\beta I}}{I_{pe}}\right)}} \tag{10}$$

To move further we can express the FEL wavelength through other parameters:

$$\lambda_o = \frac{\lambda_w \left(1 + a_w^2\right)}{2\gamma^2} \tag{11}$$

transferring it to

$$g_{\max} \sim \sqrt{\frac{I_{pe} \lambda_w \left(1 + a_w^2\right)}{2\gamma^2 ec \cdot N_c \left(1 + Z^* \frac{I_{\beta I}}{I_{pe}}\right)}} \tag{12}$$

This result does not depend neither on the type of FEL nor on the properties of the electron beam (assuming it is suitable for FEL) and fully applicable for 3D treatment. At the same time there is no analytical expression for N_c for an arbitrary 3D FEL.

1D FEL case

Hence, to move further, we need to make an approximation for N_c using a simple 1D theory. Here the expression derived in [2] :

$$N_c \propto \sqrt{\frac{N_w}{9\sqrt{3}\rho}} \tag{13}$$

giving

$$g_{\max} \sim \sqrt{\frac{I_{pe} \lambda_w \left(1 + a_w^2\right)}{2\gamma^2 ec \cdot \sqrt{\frac{N_w}{9\sqrt{3}\rho}} \left(1 + Z^* \frac{I_{\beta I}}{I_{pe}}\right)}} \tag{14}$$

Using 1D definition

$$L_G = \frac{1}{2\sqrt{3} k_w \rho} \tag{15}$$

and

$$N_w \equiv \frac{L_w}{\lambda_w} = Ln_{FEL}\frac{L_G \cdot}{\lambda_w} = Ln_{FEL} \cdot \frac{1}{4\pi\sqrt{3}\rho} \tag{16}$$

$$Ln_{FEL} \sim 18-20$$

where Ln_{FEL} number of the gain lengths needed for FEL to saturate.

$$g_{\max} \sim \sqrt{\frac{3\sqrt{3\pi} \cdot I_{pe}\lambda_w\left(1+a_w^2\right)\rho}{\gamma^2 ec \cdot \sqrt{Ln_{FEL}}\left(1+Z^* \frac{I_{pI}}{I_{pe}}\right)}} \tag{17}$$

Finally, in 1D case we shell conclude that maximum gain attainable for CeC is:

$$g_{\max} \sim \sqrt{\frac{3\sqrt{3\pi} \cdot I_{pe}\lambda_w\left(1+a_w^2\right)}{2\gamma^2 ec \cdot \sqrt{Ln_{FEL}}\left(1+Z^* \frac{I_{pI}}{I_{pe}}\right)}\sqrt[3]{\frac{2\lambda_w I_e e}{\gamma mc^3 k_o S}\frac{a_w^2}{1+a_w^2}}}$$

References

[1] V.N. Litvinenko, Ya.S. Derbenev, Physical Review Letters **102**, 114801 (2009)
[2] S. Krinsky, AIP Conference Proceedings **648**, 23 (2002)