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SEMI-ANALYTICAL DESCRIPTION OF THE MODULATOR SECTION OF THE COHERENT ELECTRON COOLING *

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Abstract

We discuss the theoretical description of the modulator section of the coherent electron cooling (CeC) [1], the modern realization of the stochastic electron cooling, where the electron beam serves as a modulator and a kicker, i.e., it records the information about the hadron beam via electron density perturbations resulting from the shielding of the hadrons and then accelerates or decelerates hadrons by its electric field with respect to their velocities. To analyze the performance of the CeC shielding of a hadron in an electron beam should be computed with high precision. We propose a solution of this problem via Fourier and Laplace transforms for 1D, 2D and 3D plasmas. In some cases there are fully analytical solutions, which gave an opportunity to test semi-analytical ones involving numerical evaluations of the inverse integral transforms. Having its own practical value this solution will also serve as a testing ground for our general solution via numerical treatment of the integral equations applicable for the realistic case of the finite beam [3].

INTRODUCTION

The dynamics of the shielding of the charged particle in plasma is a longstanding problem, the last achievements for an infinite plasma are presented in [2] and the method for a finite plasma, the realistic model of an electron beam, is described in [3], the comprehensive list of references to the earlier works can be found in these articles. In this article we describe some new analytical and numerical results for an infinite plasma. We start with a general solution and then discuss various dimensions and equilibrium distributions.

GENERAL SOLUTION

We describe the dynamics of the shielding of the charged particle in an infinite isotropic electron plasma via Fourier and Laplace transforms. In the paper we use the following dimensionless variables:

$$\vec{\mathsf{x}} = \frac{\vec{x}}{r_{\rm D}}, \quad \vec{\mathsf{v}} = \frac{\vec{v}}{v_{\rm rms}}, \quad \mathsf{t} = \frac{t}{t_{\rm p}} \equiv t\omega_{\rm p}, \quad r_{\rm D} = \frac{v_{\rm rms}}{\omega_{\rm p}},$$

where

$$v_{\rm rms} = \sqrt{\frac{1}{\rho}\int v^2 f_0(\vec{v})d\vec{v}}, \quad \omega_{\rm p} = \sqrt{\frac{e^2\rho}{m_0\gamma\epsilon_0}}, \qquad (1)$$

and dimensionless equilibrium density $\mathsf{f}_0(\vec{\mathsf{v}})$:

$$f_0(\vec{v}) = \rho f_{\rm d}\mathsf{f}_0(\vec{\mathsf{v}}), \int f_0(\vec{v})d\vec{v} = \rho. \qquad (2)$$

For the unitary point charge moving along a straight line $\vec{\mathsf{y}}(\mathsf{t}) = \vec{\mathsf{x}}_0 + \vec{\mathsf{v}}_0\mathsf{t}$ we have for the induced electron density perturbation for any number of spacial dimensions d:

$$\mathsf{n}_1(\vec{\mathsf{x}},\mathsf{t}) = \mathsf{L}^{-1}\mathsf{F}^{-1}\left[\frac{\mathrm{e}^{-i\vec{\mathsf{k}}\cdot\vec{\mathsf{x}}_0}}{\left(\frac{f_{\rm d}^{-1}v_{\rm rms}^{-d}}{\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}}))}+1\right)\left(\mathsf{s}+i\vec{\mathsf{k}}\cdot\vec{\mathsf{v}}_0\right)}\right], \qquad (3)$$

where $\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}}))$ depends on equilibrium distribution:

$$\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \int_0^\infty \mathrm{e}^{-\mathsf{ts}}\mathsf{t}\int \mathsf{f}_0(\vec{\mathsf{v}})\,\mathrm{e}^{-i\vec{\mathsf{k}}\cdot\vec{\mathsf{v}}\mathsf{t}}d\vec{\mathsf{v}}d\mathsf{t},$$

$f_{\rm d}^{-1}v_{\rm rms}^{-d}$ is a dimensionless factor, and L^{-1}, F^{-1} are the inverse Laplace and Fourier transforms, respectively.

SOLUTIONS FOR SOME DISTRIBUTIONS

We consider several distributions for 1D, 2D and 3D. $v_{\rm rms}$, $f_{\rm d}$, and $f_{\rm d}^{-1}v_{\rm rms}^{-d}$ can be computed via (1) and (2) for all distributions excepting the Cauchy. The solution (3) is valid for all these cases, we only need to compute $\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}}))$ and $f_{\rm d}^{-1}v_{\rm rms}^{-d}$. For the Kapchinskij-Vladimirskij (KV) distribution we have:

$$\text{1D: } \mathsf{f}_0(\vec{\mathsf{v}}) = \delta(\mathsf{v}^2-1), \qquad \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \frac{\mathsf{s}^2-\mathsf{k}^2}{(\mathsf{s}^2+\mathsf{k}^2)^2},$$

$$\text{2D: } \mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{\pi}\delta(\mathsf{v}^2-1), \qquad \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \frac{\mathsf{s}}{(\mathsf{s}^2+\mathsf{k}^2)^{\frac{3}{2}}},$$

$$\text{3D: } \mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{2\pi}\delta(\mathsf{v}^2-1), \qquad \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \frac{1}{2}\hat{I}(\mathsf{s},\vec{\mathsf{k}},1),$$

where

$$\hat{I}(\mathsf{s},\vec{\mathsf{k}},\mathsf{v}) = \frac{i\mathsf{k}_3\mathsf{v}\left(\mathcal{S}_- - \mathcal{S}_+\right) + s\left(\mathcal{S}_- + \mathcal{S}_+\right)}{(\mathsf{s}^2+\mathsf{k}^2\mathsf{v}^2)\mathcal{S}_-\mathcal{S}_+},$$

$$\mathcal{S}_\pm = \sqrt{\left(\mathsf{s} \pm i\frac{\mathsf{k}_3^2\mathsf{v}}{\mathsf{k}}\right)^2},$$

for the water-bag (WB):

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{2}\Theta(1-\mathsf{v}^2),\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \frac{1}{\mathsf{k}^2+\mathsf{s}^2},$$

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{\pi}\Theta\left(1-\mathsf{v}^2\right),\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \frac{2}{\mathsf{k}^2}\frac{\sqrt{\mathsf{k}^2+\mathsf{s}^2}-s}{\sqrt{\mathsf{k}^2+\mathsf{s}^2}},$$

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{3}{4\pi}\Theta(1-\mathsf{v}^2),\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \frac{3}{2}\int_0^1 \mathsf{v}^2\hat{I}(\mathsf{s},\vec{\mathsf{k}},\mathsf{v})d\mathsf{v},$$

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for 1D, 2D, and 3D respectively. For the Normal (Maxwell) distribution $\mathsf{f}_0(\vec{\mathsf{v}}) = \pi^{-\frac{d}{2}} \mathrm{e}^{-\mathsf{v}^2}$ we have:

$$\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{t}\mathsf{f}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{2}{\mathsf{k}^2}\left[1 - \sqrt{\pi}\mathrm{e}^{\frac{s^2}{\mathsf{k}^2}} \frac{s}{|\mathsf{k}|} \mathrm{Erfc}\frac{s}{|\mathsf{k}|}\right],$$

where $\mathrm{Erfc}(z)$ is a complementary error function,

$$v_{\mathrm{rms}} = \sqrt{\frac{dH_c}{2\beta}},\ f_{\mathrm{d}} = \left(\frac{\beta}{H_c}\right)^{\frac{d}{2}},\ f_{\mathrm{d}}^{-1} v_{\mathrm{rms}}^{-d} = (2/d)^{\frac{d}{2}},$$

And for the Cauchy distribution we have:

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{\Gamma(\frac{1+d}{2})}{\Gamma(\frac{1}{2})\pi^{\frac{d}{2}}} \frac{1}{(1+\mathsf{v}^2)^{\frac{1+d}{2}}},\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{t}\mathsf{f}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{1}{(\mathsf{s}+\mathsf{k})^2}.,$$

$$v_{\mathrm{rms}} = \sqrt{\frac{H_c}{\beta}}, \quad f_{\mathrm{d}} = \left(\frac{\beta}{H_c}\right)^{\frac{d}{2}}, \quad f_{\mathrm{d}}^{-1} v_{\mathrm{rms}}^{-d} = 1.$$

Then inverse integral transforms in (3) have to be computed. They can be rewritten as discrete Fourier transforms and then evaluated numerically using fast Fourier transform (FFT) algorithm. For 1D Cauchy distribution it is possible to compute this expression analytically.

EXACT SOLUTION FOR 1D CAUCHY

The inverse Laplace and Fourier transforms in (3) can be evaluated exactly for 1D Cauchy distribution giving the following solution:

$$\begin{aligned}\mathsf{n}_1\left(\vec{\mathsf{x}},\mathsf{t}\right) = \frac{1}{4\pi}\frac{1}{\mathsf{v}_0 - i}\big(\mathrm{e}^{-\mathcal{A}_+}\left(\mathrm{Ei}(\mathcal{A}_+) - \mathrm{Ei}(\mathcal{B}_+)\right) + \\ + \mathrm{e}^{\mathcal{A}_+}\left(\mathrm{E}_1(\mathcal{A}_+) - \mathrm{E}_1(\mathcal{B}_+)\right)\big) + \\ + \frac{1}{4\pi}\frac{1}{\mathsf{v}_0 + i}\big(\mathrm{e}^{-\mathcal{A}_-}\left(\mathrm{Ei}(\mathcal{A}_-) - \mathrm{Ei}(\mathcal{B}_-)\right) + \\ + \mathrm{e}^{\mathcal{A}_-}\left(\mathrm{E}_1(\mathcal{A}_-) - \mathrm{E}_1(\mathcal{B}_-)\right)\big), \quad (4)\end{aligned}$$

$$\mathcal{A}_\pm = \frac{\mathsf{t}\mathsf{v}_0 - \mathsf{x} + \mathsf{x}_0}{1 \pm i\mathsf{v}_0}, \quad \mathcal{B}_\pm = \frac{\mathsf{x}_0 - \mathsf{x} \pm i\mathsf{t}}{1 \pm i\mathsf{v}_0},$$

and $\mathrm{E}_1(z)$ and $\mathrm{Ei}(z)$ are the exponential integral functions, which can be computed via series expansions.

NUMERICAL RESULTS

Here we show numerical results for $\mathsf{n}_1\left(\vec{\mathsf{x}},\mathsf{t}\right)$ for some distributions obtained using the program we developed. The program works for any number of spacial dimensions and for any equilibrium distribution, its detailed description and full analysis of the results will appear in our next publications. On the Fig. 1 we compare the numerical results with the exact ones for 1D Cauchy distribution for wide ranges of time and spacial coordinate. q is an FFT parameter defining number of data points via $N = 2^q$. We found that for all ranges for $q = 10$ FFT values stabilize and further increasing of q doesn't change them. On the Fig 2 we show our numerical results for all 1D distributions we considered. The dynamics of the perturbations for 2D KV, WB and Cauchy distributions is depicted on the Fig 3, where we also show results for 3D Cauchy distribution.

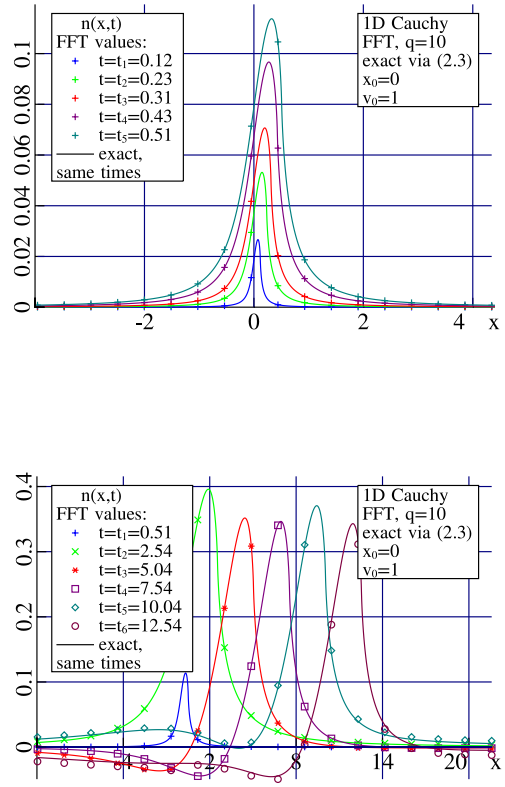

Figure 1: The exact and FFT values for 1D Cauchy case.

CONCLUDING REMARKS

In this short paper we presented our new results for the dynamics of the shielding of the charged particle in an isotropic infinite electron plasma. Our numerical results are in the perfect agreement with the theoretical ones for the exactly solvable case.

Being useful for practical purposes on its own, the program we developed can serve as a reliable testing ground for PIC simulations and for the general method capable to deal with realistic finite electron beam based on numerical solution of the integral equation. This method was described in our IPAC'12 contribution [3] and now is being developed.

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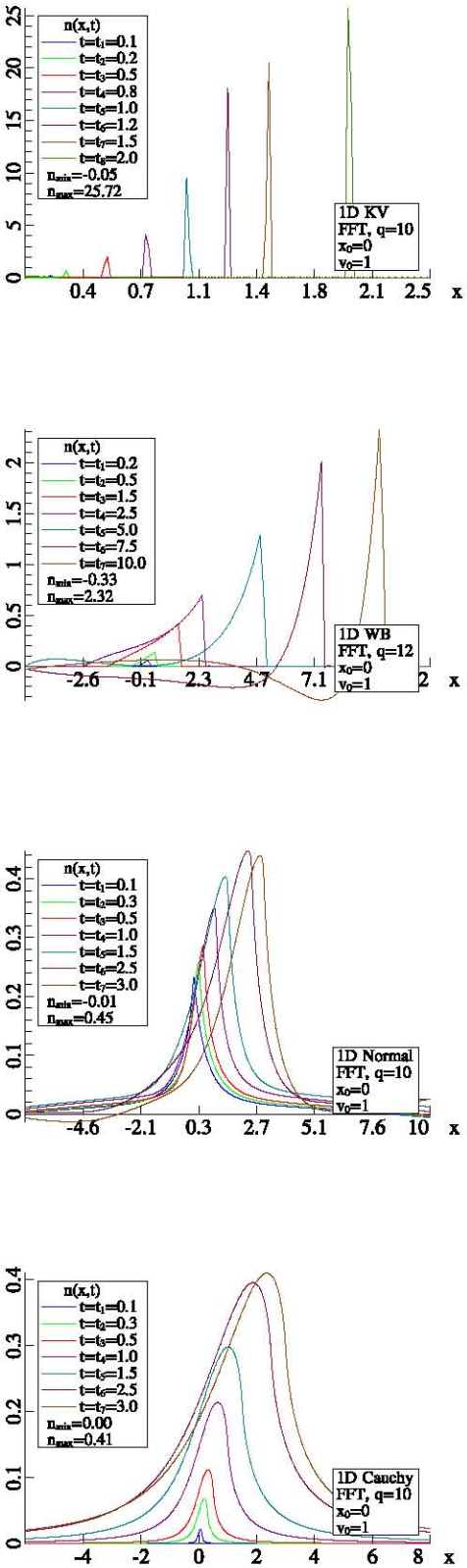

Figure 2: KV, WB, Normal, and Cauchy distributions in 1D.

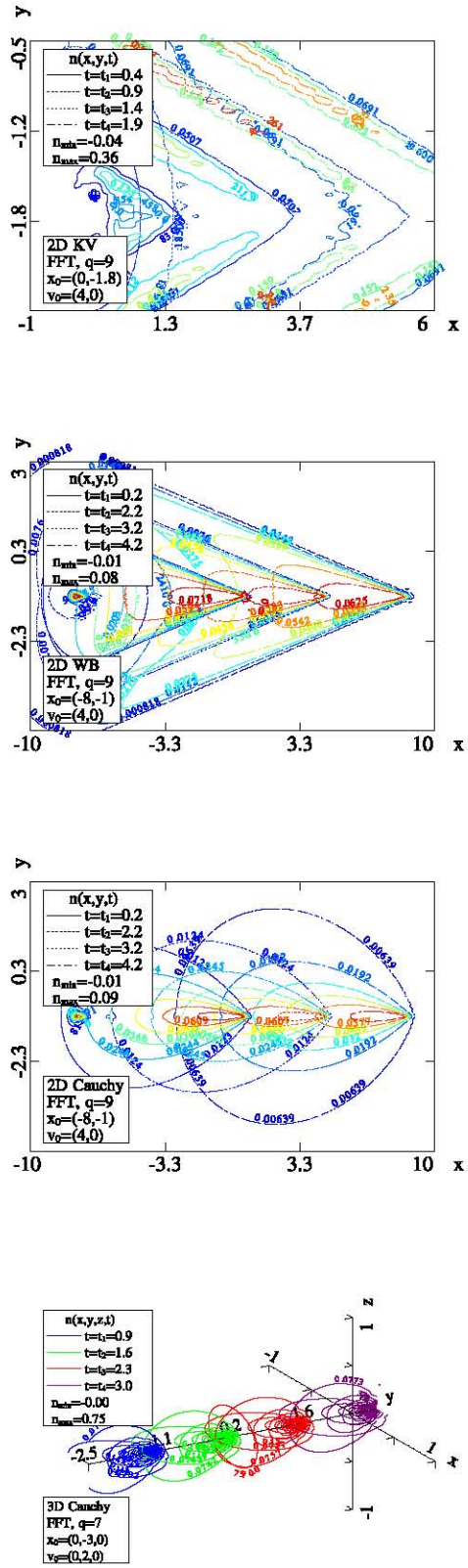

Figure 3: KV and WB in 2D and Cauchy in 2D and 3D.