BNL–49534
AD/RHIC–122
Informal Report

Symplectic Tracking Using Point Magnets in the Presence of a Longitudinal Magnetic Field

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September 1993

RHIC PROJECT

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Under Contract No. DE–AC02–76CH00016 with the
UNITED STATES DEPARTMENT OF ENERGY

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1. Introduction

In the absence of a longitudinal magnetic field, symplectic tracking can be achieved by replacing the magnets[1] by a series of point magnets and drift spaces. To treat the case when a longitudinal magnetic field is also present, this procedure is modified in this paper by replacing the drift space by a solenoidal drift, which is defined as the motion of a particle in a uniform longitudinal magnetic field. A symplectic integrator can be obtained by subdividing each magnet into pieces and replacing each magnet piece by point magnets, with only transverse fields, and solenoidal drift spaces. The reference orbit used here is made up of arcs of circles and straight lines[2] which join smoothly with each other. For this choice of reference orbit, the required results are obtained to track particles, which are the transfer functions, and the transfer time for the different elements. It is shown that these results provide a symplectic integrator, and they are exact in the sense that as the number of magnet pieces is increased, the particle motion will converge to the particle motion of the exact equations of motion.

2. The Approximate Lattice

In the absence of a longitudinal magnetic field, one procedure for symplectic integration is to replace each magnet in the given lattice by a series of point magnets and drift spaces. The equations of motion for the approximate lattice which has only point magnets and drifts can be integrated exactly, which gives a symplectic second order integrator[2] for the case where the longitudinal magnetic field, B_s, is absent.

For the case where a longitudinal magnetic field is present, $B_s \neq 0$, the following approximate lattice is proposed. Each magnet is broken into a number of pieces. A magnet piece of length h is replaced by point magnets, at each end of the piece with only transverse fields B_x, B_y, and a solenoidal drift which is defined as the particle motion in a uniform longitudinal magnetic field. The point magnets at the ends of the piece, kick the values of p_x and p_y at each end of the piece. In between the point magnets, the particle performs a solenoidal drift; the particle coordinates change as they would in a uniform longitudinal magnetic field.

It will be shown below that the above proposed approximate lattice for the case when $B_s \neq 0$ gives a symplectic integrator. This integrator is correct to first order in h, using the simplest procedure for specifying the longitudinal field in the solenoidal drift. More complicated procedures for specifying the longitudinal field may improve the accuracy. However first order in h accuracy may be sufficient as the effects due to the longitudinal fields are often small. As one increases the number of magnet pieces, decreasing h, the result obtained by integrating this approximate lattice will converge to the actual motion for the given lattice.

3. The Equations of Motion

The equations of motion for the transverse coordinates may be written as[3]

$$\begin{aligned}
\frac{dx}{ds} &= \frac{1+x/\rho}{p_s} p_x \\
\frac{dp_x}{ds} &= \frac{p_s}{\rho} + \frac{e}{c}\left[(1+x/\rho)\, B_y - \frac{p_y}{p_s}(1+x/\rho)\, B_s\right] \\
\frac{dy}{ds} &= \frac{1+x/\rho}{p_s} p_y \\
\frac{dp_y}{ds} &= \frac{e}{c}\left[\frac{p_x}{p_s}(1+x/\rho)\, B_s - (1+x/\rho)\, B_x\right] \\
p_s &= \left(p^2 - p_x^2 - p_y^2\right)^{1/2}
\end{aligned} \tag{3.1a}$$

x, y are the transverse coordinates in a coordinate system based on a reference orbit with the radius of curvature $\rho(s)$. As the longitudinal coordinates one can use t, the particle time of arrival at s, and E the particle energy. The longitudinal coordinates obey the

equations

$$\frac{dt}{ds} = \frac{1 + x/\rho}{p_s}\frac{p}{v}$$
$$\frac{dE}{ds} = e\left(1 + x/\rho\right)\mathcal{E}_s \tag{3.1b}$$

In Eq. (3.1) it has been assumed that the electric field has only the longitudinal component, $\mathcal{E}_s$. One can show that the equation for dt/ds is equivalent to

$$dt = \frac{d\ell}{v}$$
$$d\ell = \left[(1 + x/\rho)^2 + (dx/ds)^2 + (dy/ds)^2\right]^{1/2} ds \tag{3.1c}$$

where $d\ell$ is the path length over ds.

The equations of motion from Eqs. (3.1) may be derived from the hamiltonian[3]

$$H = -\left(1 + x/\rho\right)\left[E^2/c^2 - m^2c^2 + \left(\Pi_x - eA_x/c\right)^2 - \left(\Pi_y - eA_y/c\right)^2\right]^{1/2} \tag{3.2a}$$

where Π_x, Π_y, the coordinates canonical to x, y are

$$\Pi_x = p_x + eA_x/c$$
$$\Pi_y = p_y + eA_y/c \tag{3.2b}$$

The fields are related to the vector potential A_x, A_s, A_y by

$$B_x = \frac{1}{1 + x/\rho}\left[\frac{\partial}{\partial s}A_y - \frac{\partial}{\partial y}\left(\left(1 + x/\rho\right)A_s\right)\right]$$
$$B_s = \frac{\partial}{\partial y}A_x - \frac{\partial}{\partial x}A_y$$
$$B_y = \frac{1}{1 + x/\rho}\left[\frac{\partial}{\partial x}\left(\left(1 + x/\rho\right)A_s\right) - \frac{\partial}{\partial s}A_x\right] \tag{3.3}$$
$$\mathcal{E}_s = -\frac{1}{c}\frac{\partial A_s}{\partial t}$$

It then follows that transfer functions found by integrating Eqs. (3.1) exactly are symplectic transfer functions. The phrase transfer functions is used here to indicate the set of functions that relate the final coordinates to the initial coordinates.

For the case when $B_x = B_y = 0$, one can find an equation for dp_s/ds,

$$\frac{dp_s}{ds} = -\frac{p_x}{\rho} \tag{3.4}$$

which follows from

$$\frac{dp_s}{ds} = -\frac{p_x}{p_s}\frac{dp_x}{ds} - \frac{p_y}{p_s}\frac{dp_y}{ds}$$

and using Eq. (3.1a) for dp_x/ds and dp_y/ds.

In the region of the lattice outside the rf cavities where the particle velocity is constant it is convenient to use the coordinates q_x, q_y instead of p_x, p_y

$$\begin{aligned} q_x &= p_x/p, \qquad q_y = p_y/p \\ q_s &= \left(1 - q_x^2 - q_y^2\right)^{1/2} = p_s/p \end{aligned} \tag{3.5}$$

For large accelerators $q_x \simeq dx/ds$, $q_y \simeq dy/ds$. Eqs. (3.1) can then be written as

$$\begin{aligned} \frac{dx}{ds} &= \frac{1 + x/\rho}{q_s} q_x \\ \frac{dq_x}{ds} &= \frac{q_s}{\rho} + \frac{1}{B\rho}\left[(1 + x/\rho)\, B_y - \frac{q_y}{q_s}(1 + x/\rho)\, B_s\right] \\ \frac{dy}{ds} &= \frac{1 + x/\rho}{q_s} q_y \\ \frac{dq_y}{ds} &= \frac{1}{B\rho}\left[\frac{q_x}{q_s}(1 + x/\rho)\, B_s - (1 + x/\rho)\, B_x\right] \\ \frac{dq_s}{ds} &= -\frac{q_x}{\rho} \qquad \text{when } B_x = B_y = 0 \\ B\rho &= pc/e \end{aligned} \tag{3.6}$$

4. Transfer Functions when $B_s = 0$

This case was treated in Ref. 2. The results are summarized here. All the results given in this paper use a reference orbit made up of smoothly joining circular arcs and straight lines.

For the point magnets at each end of the magnet pieces of length h

$$\begin{aligned} x_2 &= x_1, \qquad y_2 = y_1 \\ q_{x_2} &= q_{x_1} + \frac{1}{B\rho}\frac{h}{2}(1 + x_1/\rho)\frac{\sin\theta/2}{\theta/2} B_y\left(x_1 s_1 y_1\right) \\ q_{y_2} &= q_{y_1} - \frac{1}{B\rho}\frac{h}{2}(1 + x_1/\rho)\frac{\sin\theta/2}{\theta/2} B_x\left(x_1 s_1 y_1\right) \\ \theta &= h/\rho \end{aligned} \tag{4.1}$$

$x_1 y_1 s_1\ q_{x_1} q_{y_1}$ are the coordinates just before the point magnet. The $\sin(\theta/2)/(\theta/2)$ is included so that in the case where the dipoles are uniform field dipoles, the central closed orbit in the dipoles is the chord that connects the end points on the reference orbit in that magnet piece. The $\sin(\theta/2)/(\theta/2)$ factor changes the transfer function by terms of order h^3. Thus the transfer function is the same up to terms of order h^2, with or without this factor.

For the drift space between the point magnets the transfer functions are given by

$$\begin{aligned}
q_{x_2} &= q_{x_1}\cos\theta + q_{s_1}\sin\theta \\
q_{s_2} &= -q_{x_1}\sin\theta + q_{s_1}\cos\theta \\
x_2 &= x_1 + (1 + x_1/\rho)\, 2\rho\sin\theta/2 \frac{q_x(\theta/2)}{q_{s_2}} \\
q_x(\theta/2) &= q_{x_1}\cos\theta/2 + q_{s_1}\sin\theta/2 \\
y_2 &= y_1 + q_{y_1} L_{12} \\
L_{12} &= (1 + x_1/\rho)\,\rho\sin\theta/q_{s_2} \\
q_{y_2} &= q_{y_1} \\
\theta &= h/\rho
\end{aligned} \tag{4.2}$$

L_{12} is the path length between s_1 and s_2.

In regions of the lattice where $1/\rho = 0$, Eqs. (4.2) become

$$\begin{aligned}
q_{x_2} &= q_{x_1}, \quad q_{y_2} = q_{y_1}, \quad q_{s_2} = q_{s_1} \\
x_2 &= x_1 + q_{x_1} L_{12}, \quad y_2 = y_1 + q_{y_1} L_{12} \\
L_{12} &= (s_2 - s_1)/q_{s_1}
\end{aligned} \tag{4.3}$$

5. Transfer Function when $B_s \neq 0$ and $1/\rho = 0$

When a longitudinal field is present, $B_s \neq 0$, then for the approximate lattice each magnet is broken in pieces of length h. Each magnet piece is replaced by point magnets at each end having only transverse fields B_x, B_y, and a solenoidal drift between the point magnets. The solenoidal drift is the motion of a particle in a uniform longitudinal field.

The effect of the point magnets at the ends of each piece are given by the transfer functions, Eqs. (4.1)

$$\begin{aligned} x_2 &= x_1, \qquad y_2 = y_1 \\ q_{x_2} &= q_{x_1} + \frac{1}{B\rho}\frac{h}{2}\hat{B}_y \\ q_{y_2} &= q_{y_1} - \frac{1}{B\rho}\frac{h}{2}\hat{B}_x \end{aligned} \tag{5.1}$$

The strength of the fields $\hat{B}_y$, $\hat{B}_x$ to be used in the point magnet transfer functions will be given below, and will be specified by the requirement that the transfer functions be symplectic.

For the solenoidal drift from s_1 to s_2, $B_x = B_y = 0$ and assuming a region where $1/\rho = 0$, then the equations of motion become from Eqs. (3.6)

$$\begin{aligned} \frac{dx}{ds} &= \frac{q_x}{q_s}, \qquad \frac{dy}{ds} = \frac{q_y}{q_s} \\ \frac{dq_x}{ds} &= -\frac{B_s}{B\rho}\frac{1}{q_s}, \qquad \frac{dq_y}{ds} = \frac{B_s}{B\rho}\frac{1}{q_s}q_x \\ \frac{dq_s}{ds} &= 0. \end{aligned} \tag{5.2}$$

In Eq. (5.1) B_s is constant from s_1 to s_2 and parallel to the reference orbit which is a straight line when $1/\rho = 0$. The simplest assumption is to put B_s equal to the value of B_s at beginning of the piece at $x_1 s_1 y_1$. This will result in a integrator which is correct only to first order in h. This may be acceptable as in large accelerators, the longitudinal orbit effects are small compared to the orbit effects due to the transverse fields. More complicated and more accurate ways of specifying B_s for the solenoidal drift may be constructed.

One sees from Eq. (5.2) that for the solenoidal drift when $1/\rho = 0$, q_s is constant and q_x, q_y rotates through the angle $\alpha = -B_s L_{12}/B\rho$,

$$\begin{aligned} q_{x_2} &= q_{x_1}\cos\alpha + q_{y_1}\sin\alpha \\ q_{y_2} &= -q_{x_1}\sin\alpha + q_{y_1}\cos\alpha \\ \alpha &= -B_s L_{12}/B\rho, \quad L_{12} = (s_2 - s_1)/q_{s_1} \\ q_{s_2} &= q_{s_1} \end{aligned} \tag{5.2a}$$

L_{12} is the path length in going from s_1 to s_2,

$$L_{12} = \int ds \frac{1 + x/\rho}{q_s} = \frac{s_2 - s_1}{q_{s_1}} \tag{5.3}$$

The results for x_2, y_2 can be found from the invariants

$$\begin{aligned} q_x + \frac{B_s}{B\rho}\frac{1}{q_s} y &= \text{ constant} \\ q_y - \frac{B_s}{B\rho}\frac{1}{q_s} x &= \text{ constant} \end{aligned} \tag{5.4}$$

which follows from Eq. (5.2). Thus one finds

$$\begin{aligned} x_2 &= x_1 + L_{12}\left[q_{x_1}\frac{\sin\alpha}{\alpha} + q_{y_1}\frac{1-\cos\alpha}{\alpha}\right] \\ y_2 &= y_1 + L_{12}\left[-q_{x_1}\left(\frac{1-\cos\alpha}{\alpha}\right) + q_{y_1}\frac{\sin\alpha}{\alpha}\right] \\ L_{12} &= (s_2 - s_1)/q_{s_1} \\ \alpha &= -B_s L_{12}/B\rho \end{aligned} \tag{5.5}$$

The transfer functions for the solenoidal drift Eqs. (5.2) and Eqs. (5.5) can be rewritten to make more obvious the contribution due to the solenoidal field B_s

$$\begin{aligned} x_2 &= x_1 + q_{x_1}L_{12} + g_1 \\ y_2 &= y_1 + q_{y_1}L_{12} + g_3 \\ q_{x_2} &= q_{x_1} + g_2 \\ q_{y_2} &= q_{y_1} + g_4 \\ q_{s_2} &= q_{s_1} \\ g_1 &= L_{12}\left[q_{x_1}\frac{\sin\alpha - \alpha}{\alpha} + q_{y_1}\frac{1-\cos\alpha}{\alpha}\right] \\ g_2 &= q_{x_1}(\cos\alpha - 1) + q_{y_1}\sin\alpha \\ g_3 &= L_{12}\left[-q_{x_1}\frac{1-\cos\alpha}{\alpha} + q_{y_1}\frac{\sin\alpha - \alpha}{\alpha}\right] \\ g_4 &= -q_{x_1}\sin\alpha + q_{y_1}(\cos\alpha - 1) \\ \alpha &= -B_s L_{12}/B\rho, \quad B_s = B_s(x_1 s_1 y_1) \\ L_{12} &= (s_2 - s_1)/q_{s_1} \end{aligned} \tag{5.6}$$

Eqs. (5.6) separate the effects of the solenoidal field and the effects due to a simple drift as given by Eq. (4.3). Since the longitudinal effects are often small, α is small and one can evaluate the $g_i(\alpha)$ by expanding $\cos\alpha$ and $\sin\alpha$ in powers of α. One has to keep enough powers of α to achieve an accuracy of about one point in 10^{14} in the transfer function, as this is the accuracy of computers often used in long term tracking. An accuracy of one point in 10^{14} may be achievable by keeping terms up to order α^2 or α^3 for large accelerators.

We will now treat the question of whether the transfer functions for a magnet piece given by Eq. (5.6), the solenoidal drift, and Eq. (5.1), the point magnets, are symplectic. The transfer functions are symplectic if they are shown to be the exact solution of the equations of motion which can be derived from a hamiltonian. A hamiltonian will exist if the magnetic fields for the approximate lattice can be described by a vector potential, $B = \text{curl } A$, for then Eq. (3.2) will give the hamiltonian.

It is instructive to reconsider the case where the longitudinal field is absent, $B_s = 0$. In this case, it is often assumed that the fields are described by just the longitudinal component of the vector potential

$$\begin{aligned} B_x &= -\frac{\partial}{\partial y} A_s \\ B_y &= \frac{1}{1 + x/\rho} \frac{\partial}{\partial x} \left[(1 + x/\rho) A_s \right] \end{aligned} \tag{5.7}$$

This is achieved in large accelerators where usually it is the integrated fields that are measured in each magnet. The integrated fields are defined by

$$\begin{aligned} \overline{B}_x(x,y) &= \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} ds B_x(xsy) \\ \overline{B}_y(x,y) &= \frac{1}{s_2 - s_1} \int_{s_1}^{s_L} ds B_y(xsy) \end{aligned} \tag{5.8}$$

s_1 to s_2 is the entire length of the magnet along the reference orbit. It can then be shown that for the usual acceleration magnet, $\overline{B}_x, \overline{B}_y$ satisfy $\text{div}\overline{B} = 0$ and can be derived from a vector potential $\overline{A}_s(x,y)$ according to Eq. (5.7), also $\overline{B}_s = 0$. One may note that the actual field in each magnet is replaced by $\overline{B}_x, \overline{B}_y$ distributed uniformly in s along the magnet. For the approximate lattice using point magnets, the vector potential for each magnet piece is given by $\hat{A}_s$

$$\hat{A}_s = \frac{1}{2} h \overline{A}_s \left[\delta(s - s_1) + \delta(s - s_2) \right] \tag{5.9}$$

This vector potential for the approximate lattice using point magnets, shows the transfer functions found from the corresponding hamiltonian and equations of motion are symplectic. Also $\hat{B}_x, \hat{B}_y$ the fields used in the transfer functions for the point magnets are the integrated fields,

$$\hat{B}_x = \overline{B}_x, \quad \hat{B}_y = \overline{B}_y \tag{5.10}$$

Now let us consider the case where the longitudinal field is present, $B_s \neq 0$. In order to see the effects of the longitudinal field, it is not sufficient to measure just the integrated field. One has to measure, to some degree, the magnetic field all along the magnet, B_x, B_s, B_y which can be derived from a vector potential A_x, A_s, A_y according to Eq. (3.3). For the approximate lattice, which in this case uses point magnets separated by solenoidal drifts, the vector potential for each magnet may be chosen as

$$\begin{aligned}
\hat{A}_s &= A_s(xsy)\frac{h}{2}\left[\delta(s-s_1)+\delta(s-s_2)\right] \\
\hat{A}_x &= \frac{1}{2}B_s(x_1s_1y_1)\,y + A_x(x_1sy_1)\frac{h}{2}\left[\delta(s-s_1)+\delta(s-s_2)\right] \\
\hat{A}_y &= \frac{1}{2}B_s(x_1s_1y_1)\,x + A_y(x_1sy_1)\frac{h}{2}\left[\delta(s-s_1)+\delta(s-s_2)\right]
\end{aligned} \tag{5.11}$$

$\hat{A}_x, \hat{A}_y$ include the vector potential for a uniform longitudinal field, $B_s(x_1s_1y_1)$, that does not depend on s.

One may note in $\hat{A}_x$ and $\hat{A}_y$, the first term proportional to B_s does not contribute to $\hat{B}_x$ and $\hat{B}_y$ as s is fixed at $s = s_1$, the second term proportional to A_x or A_y does not contribute to $\hat{B}_s$ as x, y are fixed at x_1 and y_1.

This vector potential for the approximate lattice describes a lattice where each magnet piece is represented by point magnets at each end of the piece separated by a drift in a uniform longitudinal field, $B_s(x_1s_1y_1)$. $\hat{B}_x, \hat{B}_y$ the field used in the point magnet transfer functions are then given by

$$\begin{aligned}
\hat{B}_x &= -\frac{\partial}{\partial y}A_s(xsy) + \frac{\partial}{\partial s}A_y(x_1sy_1) \\
\hat{B}_y &= \frac{\partial}{\partial x}A_s(xsy) - \frac{\partial}{\partial s}A_x(x_1s_1y_1)
\end{aligned} \tag{5.12}$$

One sees that at $s = s_1$ $\hat{B}_x, \hat{B}_y$ are just the fields B_x, B_y at $s = s_1$. However, at $s = s_2$ $\hat{B}_x, \hat{B}_y$ differ from the fields B_x, B_y at $s = s_2$ by a term which is of order h. With this

choice of $\hat{B}_x, \hat{B}_y$ and with $\hat{B}_s(x_1 s_1 y_1)$, Eqs. (5.1) and (5.6) are a symplectic integrator, and it will be seen in Section 8 that it is correct to first order in h.

6. Transfer Functions when $B_s \neq 0$ and $1/\rho \neq 0$

In this paper the reference orbit is assumed to be made up of smoothly joining circular arcs and straight lines. At the locations of the dipoles, the reference orbit will have a radius of curvature $1/\rho \neq 0$. This allows us to construct a reference orbit which has a continuous slope and is fairly close to the central closed orbit of the accelerator. It also allows the calculation of the tune and other linear parameters by multiplying the transfer matrices of each magnet piece.

As in the case where $1/\rho = 0$, Section 5, for the approximate lattice each magnet is broken into pieces of length h along the reference orbit and each magnet piece is replaced by point magnets at each end having only transverse fields and a solenoidal drift in a uniform longitudinal field between the point magnets.

The effect of the point magnets at the ends of each magnet piece are given by the transfer functions

$$\begin{aligned} x_2 &= x_1 \quad , \quad y_2 = y_1 \\ q_{x_2} &= q_{x_1} + \frac{1}{B\rho}\frac{\sin\theta/2}{\theta/2}\frac{h}{2}\hat{B}_y \\ q_{y_2} &= q_{y_1} - \frac{1}{B\rho}\frac{\sin\theta/2}{\theta/2}\frac{h}{2}\hat{B}_x \\ \theta &= h/\rho \end{aligned} \tag{6.1}$$

The strength of the $\hat{B}_x, \hat{B}_y$ will be chosen below.

For the solenoidal drift between the point magnets, one has to define it so that the B_s for the solenoidal drift approaches the actual B_s as h goes to zero, and for which the equations of motion are exactly solvable. One possible procedure for defining the solenoidal drift is to have the longitudinal field be uniform in a Cartesian coordinate system based on the chord or straight line that joins the end prints of the magnet piece which are on the reference orbit. In this cartesian CS, the coordinates will be labeled $\overline{x}, \overline{s}, \overline{y}, q_{\overline{x}}, q_{\overline{y}}$. The field along $\overline{s}$, $B_{\overline{s}}$ is defined to be uniform in the solenoidal drift and

$$B_{\overline{s}} = B_s(x_1 s_1 y_1)\cos\theta/2 \tag{6.2}$$

The $\cos\theta/2$ factor in Eq. (6.2) may be omitted as it differs from 1 by term of order h^2. It may be useful in cases where the magnet pieces are relatively large. In the cartesian CS, the motion of the particle in the solenoidal drift is given by Eq. (5.6). Thus

$$\begin{aligned}
\overline{x}_2 &= \overline{x}_1 + q_{\overline{x}1} L_{12} + g_1 \\
L_{12} &= \left(\overline{s}_2 - \overline{s}_1\right)/q_{\overline{s}1} \\
\alpha &= -B_{\overline{s}} L_{12}/B\rho \\
g_1 &= L_{12}\left[q_{\overline{x}1}\frac{\sin\alpha - \alpha}{\alpha} + q_{\overline{y}1}\frac{1-\cos\alpha}{\alpha}\right]
\end{aligned} \tag{6.3}$$

$\overline{x}, \overline{s}, \overline{y}$ and $q_{\overline{x}}, q_{\overline{y}}, q_{\overline{s}}$ are the coordinates in the cartesian CS based on the chord of the reference orbit for the magnet piece. To convert Eq. (6.3) into relationship between x_1 and x_2, we note the following relationships

$$\begin{aligned}
\overline{x}_1 &= x_1\cos\theta/2 \quad , \quad \overline{x}_2 = x_2\cos\theta/2 \\
\overline{s}_1 &= -x_1\sin\theta/2 \quad , \quad \overline{s}_2 = 2\rho\sin\theta/2 + x_2\sin\theta/2 \\
\overline{s}_2 - \overline{s}_1 &= 2\rho\sin\theta/2\left[1 + \frac{1}{2\rho}\left(x_1 + x_2\right)\right] \\
q_{\overline{s}1} &= -q_{x1}\sin\theta/2 + q_{s1}\cos\theta/2 \\
q_{\overline{x}1} &= q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2 \\
\overline{y}_1 &= y_1 \quad , \quad q_{\overline{y}1} = q_{y1} \\
-\overline{q}_{x1}\sin\theta/2 + \overline{q}_{s1}\cos\theta/2 &= -q_{x1}\sin\theta + q_{s1}\cos\theta \\
\theta &= h/\rho
\end{aligned} \tag{6.4}$$

One can now relate x_2 to x_1 using Eq. (6.3) to give

$$\begin{aligned}
x_2 &= x_2^{(0)} + h_1\left(x_2\right) \\
x_2^{(1)} &= x_1 + \frac{2\rho\sin\theta/2\left(1 + x_1/\rho\right)\left(q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2\right)}{-q_{x1}\sin\theta + q_{s1}\cos\theta} \\
h_1\left(x_2\right) &= \frac{L_{12}\overline{q}_{s_1}}{-q_{x1}\sin\theta + q_{s1}\cos\theta} g_1 \\
L_{12} &= \frac{1}{q_{\overline{s}1}}\left[\left(x_2 + x_1\right)\sin\frac{\theta}{2} + 2\rho\sin\theta/2\right] \\
\alpha &= -B_{\overline{s}} L_{12}/B\rho \quad , \quad \theta = h/\rho
\end{aligned} \tag{6.5}$$

Eq. (6.5) is an implicit relationship between x_2 and x_1. h_1 depends on x_2 through L_{12} and α which both depend on x_2. $x_2^{(0)}$ is the result for x_2 when $B_s = 0$ [see Eq. (4.2)].

Eq. (6.4) cannot be solved analytically for x_2. It can be solved by iteration assuming that the term proportional to B_s, $h_1(x_2)$, is small. Because of the usual smallness of the longitudinal effects, a few iterations may give a result for x_2 which is accurate to 1 part in 10^{14}, which is roughly what is required for long term tracking.

The iteration may be done as follows:

$$\begin{aligned}
x_2^{(0)} &= x_2^{(0)} \quad , \quad L_{12}^{(0)} = L_{12}\left(x_2^{(0)}\right) \\
x_2^{(1)} &= x_2^{(0)} + h_1\left(L_{12}^{(0)}\right) \quad , \quad L_{12}^{(1)} \equiv L_{12}\left(x_2^{(1)}\right) \\
x_2^{(2)} &= x_2^{(0)} + h_1\left(L_{12}^{(1)}\right) \quad , \quad L_{12}^{(2)} = L_{12}\left(x_2^{(2)}\right) \\
x_2^{(3)} &= x_2^{(0)} + h_1\left(L_{12}^{(2)}\right) \quad , \quad L_{12}^{(3)} = L_{12}\left(x_2^{(3)}\right)
\end{aligned} \tag{6.6}$$

In doing the iteration, g_1 may be computed by expanding $\cos\alpha$, $\sin\alpha$ to get

$$\begin{aligned}
g_1 &= \sum_{n=1}^{\infty} g_n \alpha^n \\
\alpha &= -B_{\bar{s}} L_{12}/B\rho
\end{aligned} \tag{6.7a}$$

$$\begin{aligned}
g_1 &= q_{y1}\frac{1}{2} \quad , \quad g_3 = q_{y1}(-1)\frac{1}{4!}\cdots \\
g_2 &= q_{\bar{x}1}\left(\frac{-1}{3!}\right) \quad , \quad g_4 = q_{\bar{x}1}\frac{1}{5!}\cdots \\
g_n &= -q_{y1}\frac{(-1)^{\frac{n+1}{2}}}{(n+1)!} \qquad n \text{ odd} \\
g_n &= q_{\bar{x}1}(-1)^{n/2}\frac{1}{(n+1)!} \qquad n \text{ even}
\end{aligned} \tag{6.7b}$$

In computing $x_2^{(1)}$, one case uses the g_n for $n \leq 1$; for $x_2^{(2)}$ use the g_n for $n \leq 3$, etc. For large accelerators, it may be sufficient to iterate up to $x_2^{(2)}$. Having found x_2 and L_{12} for the solenoid drift, one can now proceed to find q_{x2}, y_2, q_{s2}. In the cartesian CS, these are

given by Eq. (5.6)

$$
\begin{aligned}
q_{\overline{x}2} &= q_{\overline{x}1} + g_2 \\
\overline{y}_2 &= \overline{y}_1 + q_{\overline{y}1} L_{12} + g_3 \\
q_{\overline{y}2} &= q_{\overline{y}1} + g_4 \\
q_{\overline{s}2} &= q_{\overline{s}1}
\end{aligned}
\tag{6.8}
$$

$q_{\overline{x}2}$, $\overline{y}_2$, $q_{\overline{y}2}$, $q_{\overline{s}_3}$ are related to q_{x2}, y_2, q_{y2} q_{s2} by

$$
\begin{gathered}
\begin{pmatrix} q_x \\ q_s \end{pmatrix}_2 = R\left(\frac{\theta}{2}\right) \begin{pmatrix} q_{\overline{x}} \\ q_{\overline{s}} \end{pmatrix}_2 \\
\overline{y}_2 = y_2 \quad , \quad q_{\overline{y}2} = q_{y2}
\end{gathered}
\tag{6.9a}
$$

where $R(\theta)$ is the rotation matrix

$$
R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\tag{6.9b}
$$

Now using Eq. (6.4) which can be written as

$$
\begin{pmatrix} q_{\overline{x}} \\ q_{\overline{s}} \end{pmatrix}_1 = R(\theta/2) \begin{pmatrix} q_x \\ q_s \end{pmatrix}_1 \quad ,
\tag{6.10}
$$

one finds

$$
\begin{aligned}
\begin{pmatrix} q_x \\ q_s \end{pmatrix}_2 &= R(\theta) \begin{pmatrix} q_x \\ q_s \end{pmatrix}_1 + R(\theta/2) \begin{pmatrix} g_2 \\ 0 \end{pmatrix} \\
q_{x2} &= q_{x1} \cos\theta + q_{s1} \sin\theta + g_2 \cos\theta/2 \\
q_{s2} &= -q_{x1} \sin\theta + q_{s1} \cos\theta - g_2 \sin\theta/2 \\
g_2 &= q_{\overline{x}1} (\cos\alpha - 1) + q_{\overline{y}1} \sin\alpha \\
\alpha &= -B_{\overline{s}} L_{12}/B\rho
\end{aligned}
\tag{6.11}
$$

The results for y_2, q_{y2} follow directly from Eq. (5.6)

$$
\begin{aligned}
y_2 &= y_1 + q_{y1} L_{12} + g_3 \\
q_{y2} &= q_{y1} + g_4 \\
g_3 &= L_{12} \left[-q_{\overline{x}1} \frac{1 - \cos\alpha}{\alpha} + q_{\overline{y}1} \frac{\sin\alpha - \alpha}{\alpha} \right] \\
g_4 &= -q_{\overline{x}1} \sin\alpha + q_{\overline{y}1} (\cos\alpha - 1)
\end{aligned}
\tag{6.12}
$$

Using the result found for L_{12} from Eq. (6.8), one can find q_{x2}, q_{s2}, y_2, q_{y2} using Eqs. (6.11) and (6.12).

Let us now treat the question of whether the above transfer functions are symplectic. The vector potential for the actual magnetic field is given by A_x, A_s, A_y. The vector potential for the approximate lattice may be written as

$$\hat{\mathbf{A}} = \hat{\mathbf{A}}_{PM} + \hat{\mathbf{A}}_{SOL} \tag{6.13a}$$

$\hat{\mathbf{A}}_{PM}$ is the part of the vector potential that describes the point magnets given by its components in the reference orbit CS

$$\begin{aligned}
\hat{A}_{PM_s} &= \frac{h}{2} A_s\left(xsy\right)\left[\delta\left(s-s_1\right)+\delta\left(s-s_2\right)\right]\frac{\sin\theta/2}{\theta/2} \\
\hat{A}_{PM_x} &= \frac{h}{2} A_x\left(x_1 s\ y_1\right)\left[\delta\left(s-s_1\right)+\delta\left(s-s_2\right)\right]\frac{\sin\theta/2}{\theta/2} \\
\hat{A}_{PM_y} &= \frac{h}{2} A_y\left(x_1 s\ y_1\right)\left[\delta\left(s-s_1\right)+\delta\left(s-s_2\right)\right]\frac{\sin\theta/2}{\theta/2}
\end{aligned} \tag{6.13b}$$

$\hat{\mathbf{A}}_{SOL}$ is the part of the vector potential that describes the solenoidal drift and is given by its components in the cartesian CS

$$\begin{aligned}
\hat{A}_{SOL\overline{x}} &= \frac{1}{2} B_{\overline{s}}\left(x_1 s_1 y_1\right) y \\
\hat{A}_{SOL\overline{y}} &= -\frac{1}{2} B_{\overline{s}}\left(x_1 s_1 y_1\right) x \\
B_{\overline{s}} &= \cos\theta/2\ \ B_s\left(x_1 s_1 y_1\right)
\end{aligned} \tag{6.13c}$$

The $\hat{B}_x$, $\hat{B}_y$ to be used in the point magnets transfer function, Eq. (6.1) are given by

$$\begin{aligned}
\hat{B}_x &= \left[\frac{\partial}{\partial s} A_y\left(x_1 s\ y_1\right) - \frac{\partial}{\partial y}\left(\left(1+x/\rho\right) A_s\left(xsy\right)\right)\right]\frac{1}{1+x/\rho} \\
\hat{B}_y &= \left[\frac{\partial}{\partial x}\left(\left(1+x\rho\right) A_s\left(xsy\right)\right) - \frac{\partial}{\partial s} A_x\left(x_1 s\ y_1\right)\right]\frac{1}{1+x/\rho}
\end{aligned} \tag{6.14}$$

As was found in Section 5, for the $1/\rho = 0$ case, the $\hat{B}_x, \hat{B}_y$ differ from the actual fields at the point magnets at $s = s_2$ by a term which is of order h.

7. Accuracy of the Integrator

It will be shown below that the integrator proposed above, using point magnets and solenoidal drifts, for each magnet piece has an accuracy up to terms of order h, where h

is the length of magnet piece. The error term is of order h^2. This is a contrast to the case where the longitudinal field is absent, $B_s = 0$, where it was shown[2] that the integrator, using point magnets and drifts, has an accuracy of order h^2, with an error term of order h^3. However, the error term of order h^2, when $B_s \neq 0$, is proportional to the longitudinal field and if the longitudinal effects may be considered small, the error term of order h^2 may also be correspondingly small.

The treatment given here uses the results found in Ref. 2 for the $B_s = 0$ case. The equations of motion are written as

$$\begin{aligned} \frac{dx_i}{d\lambda} &= f_i(x_i) \quad i = 1,6 \\ \frac{ds}{d\lambda} &= 1 \end{aligned} \tag{7.1}$$

The f_i do not depend on λ and a Taylor series result for the x_i at the end of the magnet piece, x_{i2} is

$$x_{i2} = x_{i1} + h\ f_{i1} + \sum_{y=1,6} \frac{\partial f_i}{\partial x_j} f_j \frac{h^2}{2} + \dots \tag{7.2}$$

The x_i found at the end of a magnet piece with the integrator, using point magnets and solenoidal drifts, has to be compared with the exact result Eq. (7.2) to determine the accuracy of the integrator. Following the procedure used in Ref. 2, we write

$$f_i = g_i + K_i \tag{7.3a}$$

where the g_i is the part of f_i that describes a solenoidal drift. g_i contains the field free term in the f_i and the field dependent terms corresponding to a uniform B_s in the local cartesian CS. The K_i are written as

$$K_i = \hat{K}_i + \Delta K_i. \tag{7.3b}$$

The $\hat{K}_i$ contains the field dependent terms corresponding to the fields $B_x = \hat{B}_x$, $B_y = \hat{B}_y$, where $\hat{B}_x, \hat{B}_y$ are the field used in the transfer functions for the point magnets. One can then see that the ΔK_i, evaluate at $s = s_1$, are terms of order h.

For the point magnets at s_1, the x_i are changed by

$$x_{i2} = x_{i1} + \frac{1}{2} h \hat{K}_i \tag{7.4}$$

Repeating the calculations done in Ref. 2, one finds for the final x_i using point magnets and a solenoidal drift,

$$x_{i2} = x_{i1} + h\hat{f}_{i1} + \frac{h^2}{2}\sum_y \left[\frac{\partial \hat{f}_i}{\partial x_j}\hat{f}_j - \frac{\partial \hat{K}_i}{\partial x_j}\frac{K_j}{2}\right]_{x_{i1}} \tag{7.5}$$
$$\hat{f}_{i1} = g_{i1} + \hat{K}_{i1} = f_{i1} - \Delta K_{i1}$$

Comparing Eq. (7.5) with Eq. (7.2) shows there is an error term of order h^2, which is proportional to the longitudinal field, B_s.

8. Summary of the Transfer Functions

It may be useful to summarize in one place the results for the transfer functions for the symplectic integrator proposed above. These are the results that might be used in writing a symplectic tracking program when longitudinal fields are present.

For the approximate lattice which is used to generate the integrator, it is assumed that each magnet is broken up into a number of pieces. Each piece is represented in the approximate lattice by point magnets at the ends of the piece and a solenoidal drift between the point magnets. A solenoidal drift is the motion of a particle in a uniform longitudinal field.

The results are given using a reference orbit made up of circular arcs and straight lines which join smoothly. Thus there are regions of the lattice where the reference orbit has a radius of curvalue $1/\rho = 0$, usually at the drifts and quadrupoles, and there are regions where $1/\rho =$ constant, usually at the dipoles. The results can also be used if one chooses a reference orbit which always uses the local cartesian CS, based on the chord that joins the end points of each magnet piece on the reference orbit.

8.1 Transfer Functions for the Point Magnets

In the approximate lattice, each magnet piece is represented by point magnets at the ends of the pieces and a solenoidal drift between the ends. The magnet piece goes from $s = s_1$ to $s = s_2$ and has a length along the reference orbit of $h = s_2 - s_1$. For the point

magnets, the transfer functions are

$$
\begin{aligned}
x_2 &= x_1 \quad , \quad y_2 = y_1 \\
q_{x2} &= q_{x1} + \frac{1}{B\rho}\frac{\sin\theta/2}{\theta/2}\frac{h}{2}\hat{B}_y \\
q_{y2} &= q_{y1} - \frac{1}{B\rho}\frac{\sin\theta/2}{\theta/2}\frac{h}{2}\hat{B}_x \\
\theta &= h/\rho - (s_2 - s_1)/\rho
\end{aligned}
\tag{8.1a}
$$

The fields $\hat{B}_x, \hat{B}_y$ may differ from the actual fields at the point magnet by terms of order h and are given by

$$
\begin{aligned}
\hat{B}_x &= \left[\frac{\partial}{\partial s}A_y(x_1 s\ y_1) - \frac{\partial}{\partial y}((1+x/\rho)A_s(xsy))\right]\frac{1}{1+x/\rho} \\
\hat{B}_y &= \left[\frac{\partial}{\partial x}((1+x/\rho)A_s(xsy)) - \frac{\partial}{\partial s}A_x(x_1 s_1 y_1)\right]\frac{1}{1+x/\rho}
\end{aligned}
\tag{8.1b}
$$

$\hat{B}_x = B_x$, $\hat{B}_y = B_y$ at $s = s_1$ but not at $s = s_2$.

8.2 Transfer Functions for the Solenoidal Drift

For the solenoidal drift between the point magnets, the transfer function for x is given by

$$
\begin{aligned}
x_2 &= x_2^{(0)} + h_1 \\
x_2^{(0)} &= x_1 + \frac{2\rho\sin\theta/2\,(1+x_1/\rho)\,(q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2)}{-q_{x1}\sin\theta + q_{s1}\cos\theta} \\
h_1 &= \frac{L_{12}q_{\bar{s}1}}{-q_{x1}\sin\theta + q_{s1}\cos\theta}g_1 \\
g_1 &= q_{\bar{x}1}\frac{\sin\alpha - \alpha}{\alpha} + q_{y1}\frac{1-\cos\alpha}{\alpha} \\
\alpha &= -B_{\bar{s}}L_{12}/B\rho \quad , \quad B_{\bar{s}} = B_s(x_1 s_1 y_1)\cos\theta/2 \\
L_{12} &= \frac{1}{q_{\bar{s}1}}\left[1 + (x_1 + x_2)/2\rho\right]2\rho\sin\theta/2 \\
q_{\bar{x}1} &= q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2 \\
q_{\bar{s}1} &= -q_{x1}\sin\theta/2 + q_{s1}\cos\theta/2
\end{aligned}
\tag{8.2}
$$

$x_2^{(0)}$ is the transfer function when $B_s = 0$, h_1 vanishes when $B_s = 0$, and L_{12} is the path length between s_1 and s_2. h_1 depends on x_2 through L_{12} and α, and Eq. (8.2) is

an implicit equation for x_2 which can be solved by iteration, assuming that h_1 can be considered small. This gives the iteration result

$$\begin{aligned}
x_2^{(0)} &= x_2^{(0)} \quad , \quad L_{12}^{(0)} = L_{12}\left(x_2^{(0)}\right) \\
x_2^{(1)} &= x_2^{(0)} + h_1\left(L_{12}^{(0)}\right) \quad , \quad L_{12}^{(1)} = L_{12}\left(x_2^{(1)}\right) \\
x_2^{(2)} &= x_2^{(0)} + h_1\left(L_{12}^{(1)}\right) \quad , \quad L_{12}^{(2)} = L_{12}\left(x_2^{(2)}\right) \\
x_2^{(3)} &= x_2^{(0)} + h_1\left(L_{12}^{(2)}\right) \quad , \quad L_{12}^{(3)} = L_{12}\left(x_2^{(3)}\right)
\end{aligned} \tag{8.3}$$

Long term tracking is often done with an accuracy of 1 part in 10^{14} in the transfer functions. For large accelerators, where the longitudinal effects are small, the 1 part in 10^{14} accuracy may be achieved after a few iterations.

In doing the iteration indicated by Eq. (8.3), g_1 can be expanded in powers of α, keeping only up to the power of α as the order of the iteration. Thus

$$\begin{aligned}
g_1 &= \sum_{n=1}^{\infty} g_{1n}\alpha^n \\
g_{1n} &= -q_{y1}\frac{(-1)^{\frac{n+1}{2}}}{(n+1)!} \qquad n \text{ odd} \\
g_{1n} &= q_{\overline{x}1}\,(-1)^{n/2}\,\frac{1}{(n+1)!} \qquad n \text{ even}
\end{aligned} \tag{8.4}$$

Having found x_2 and L_{12} by solving Eq. (8.2) one can then find q_{x2}, y_2, q_{y2} using

$$\begin{aligned}
q_{x2} &= q_{x1}\cos\theta + q_{s1}\sin\theta + g_2\cos\theta/2 \\
q_{s2} &= -q_{x1}\sin\theta + q_{s1}\cos\theta - g_2\sin\theta/2 \\
y_2 &= y_1 + q_{y1}L_{12} + g_3 \\
q_{y2} &= q_{y1} + g_4 \\
g_2 &= q_{\overline{x}1}\,(\cos\alpha - 1) + q_{y1}\sin\alpha \\
g_3 &= L_{12}\left[-q_{\overline{x}1}\frac{1-\cos\alpha}{\alpha} + q_{y1}\frac{\sin\alpha - \alpha}{\alpha}\right] \\
g_4 &= -q_{\overline{x}1}\sin\alpha + q_{y1}\,(\cos\alpha - 1)
\end{aligned} \tag{8.5}$$

8.3 Transfer Functions when $1/\rho = 0$

In this case no iteration is required as L_{12} does not depend on x_2. The transfer functions for the solenoidal drift

$$
\begin{aligned}
x_2 &= x_1 + q_{x1} L_{12} + g_1 \\
q_{x2} &= q_{x1} + g_2 \\
y_2 &= y_1 + q_{y1} L_{12} + g_3 \\
q_{y2} &= q_{y1} + g_4 \\
q_{s2} &= q_{s1} \\
\alpha &= -B_s L_{12} / B\rho \quad , \quad B_s = B_s\left(x_1 s_1 y_1\right) \\
L_{12} &= \left(s_2 - s_1\right) / q_{s1}
\end{aligned}
\tag{8.6}
$$

The transfer functions for the point magnets when $1/\rho = 0$ are given by Eqs. (8.1) if one puts $\sin(\theta/2)/(\theta/2) = 1$.

The problem of tracking symplectically when longitudinal fields are present was treated in Ref. 4 for the case of hard edge fringe fields.

References

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