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During the setup of helions for the FY2014 RHIC run it was discovered that the standard scheme for merging bunches on the AGS injection porch required an injection kicker pulse shorter than what was available. To overcome this difficulty, K. Zeno [1] proposed and developed an interesting and unusual alternative which uses RF harmonic numbers 12, 4, 2 (rather than the standard 8, 4, 2) to merge 8 helion bunches into 2. In this note we carry out simulations that illustrate how the alternative scheme works and how it compares with the standard scheme. This is done in Sections 13 and 14. A scheme in which 6 bunches are merged into 1 is simulated in Section 15. This may be useful if more helions per merged bunch are needed in future runs.

General formulae for the simulations are given in Sections 9 through 12.

For completeness, Sections 1 through 8 give a derivation of the turn-by-turn equations of longitudinal motion at constant magnetic field. The derivation is based on the work of MacLachlan [2]. The reader may wish to skip over these Sections and start with Section 9.

1 Synchronous Parameters

Let $2\pi R$ and ρ be the circumference and radius-of-curvature of the design orbit in a given ring, and let B and $2\pi R_s$ be the magnetic field and orbit circumference for the synchronous particle. We assume that B and R_s are given and calculate the other synchronous particle parameters in terms of these. The radius-of-curvature of the synchronous particle is

$$\rho_s = \rho(R_s/R)^{1/\alpha} \tag{1}$$

where

$$\alpha = \frac{1}{\gamma_t^2} \tag{2}$$

is the "momentum compaction" factor and γ_t is the transition gamma of the ring. The momentum of the synchronous particle is given by

$$cp_s = eQB\rho_s \tag{3}$$

where e is the proton charge and eQ is the charge of the particle. The energy is

$$E_s = \sqrt{(cp_s)^2 + m^2c^4} \tag{4}$$

where m is the mass of the particle. The synchronous β, γ and angular velocity are

$$\beta_s = cp_s/E_s, \quad \gamma_s = E_s/(mc^2), \quad \omega_s = c\beta_s/R_s. \tag{5}$$

The revolution period and frequency of the synchronous particle are

$$T_s = \frac{2\pi}{\omega_s}, \quad f_s = \frac{1}{T_s}. \tag{6}$$

We also define the phase slip factor

$$\eta_s = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}. \tag{7}$$

2 Time Equation

We consider a ring with a single RF gap. Let T_n^s and T_n be respectively the times at which the synchronous and asynchronous particles make their nth pass through the gap. The synchronous particle experiences no acceleration in the gap and revolves around the ring with constant angular velocity ω_s. Thus

$$T_{n+1}^s = T_n^s + \frac{2\pi}{\omega_s} \tag{8}$$

and taking $T_1^s = 0$ it follows that

$$T_{n+1}^s = nT_s. \tag{9}$$

Similarly, for the asynchronous particle we have

$$T_{n+1} = T_n + \frac{2\pi}{\omega_n} \tag{10}$$

where ω_n is the angular velocity just after the nth pass through the gap. Defining

$$t_{n+1} = T_{n+1} - T_{n+1}^s, \quad t_n = T_n - T_n^s \tag{11}$$

we then have

$$t_{n+1} = t_n + 2\pi\left(\frac{1}{\omega_n} - \frac{1}{\omega_s}\right) = t_n + \left(\frac{\omega_s - \omega_n}{\omega_n}\right) T_s. \tag{12}$$

3 Energy Equation

Now let E_n be the energy of the asynchronous particle just after its nth pass through the gap. Then we have

$$E_{n+1} = E_n + eQV(T_{n+1}), \tag{13}$$

where $V(T)$ is the voltage across the gap at time T. Since the synchronous particle undergoes no acceleration we must have

$$V(T_n^s) = 0 \tag{14}$$

for all n. We assume that

$$V(T + T_s/h) = V(T) \tag{15}$$

where h is a positive integer called the fundamental harmonic. It then follows that

$$V(T + nT_s) = V(T). \tag{16}$$

In terms of E_n, the other asynchronous parameters are

$$cp_n = \sqrt{E_n^2 - m^2c^4}, \quad \beta_n = cp_n/E_n \tag{17}$$

and

$$\rho_n = \frac{cp_n}{eQB}, \quad R_n = R(\rho_n/\rho)^\alpha, \quad \omega_n = c\beta_n/R_n. \tag{18}$$

Define

$$e_{n+1} = E_{n+1} - E_s, \quad e_n = E_n - E_s. \tag{19}$$

Then using

$$T_{n+1} = t_{n+1} + T_{n+1}^s = t_{n+1} + nT_s \tag{20}$$

we have

$$V(T_{n+1}) = V(t_{n+1}) \tag{21}$$

and

$$e_{n+1} = e_n + eQV(t_{n+1}). \tag{22}$$

This, together with

$$t_{n+1} = t_n + \left(\frac{\omega_s - \omega_n}{\omega_n}\right) T_s \tag{23}$$

gives the turn-by-turn longitudinal motion of the asynchronous particle.

4 Symplectic Map

The Jacobian matrix elements for the map from (t_n, e_n) to (t_{n+1}, e_{n+1}) are

$$\frac{\partial t_{n+1}}{\partial t_n} = 1, \quad \frac{\partial t_{n+1}}{\partial e_n} = 2\pi \frac{\partial(1/\omega_n)}{\partial e_n} \tag{24}$$

$$\frac{\partial e_{n+1}}{\partial t_n} = eQV', \quad \frac{\partial e_{n+1}}{\partial e_n} = 1 + 2\pi eQV' \frac{\partial(1/\omega_n)}{\partial e_n} \tag{25}$$

where V' is the derivative of V with respect to T at time T_{n+1}. Thus we have

$$\left(\frac{\partial t_{n+1}}{\partial t_n}\right)\left(\frac{\partial e_{n+1}}{\partial e_n}\right) - \left(\frac{\partial t_{n+1}}{\partial e_n}\right)\left(\frac{\partial e_{n+1}}{\partial t_n}\right) = 1 \tag{26}$$

and the map is symplectic. This guarantees that the map preserves area.

5 Approximate Time Equation

Writing

$$\frac{\omega_s}{\omega_n} = \left\{1 + \left(\frac{\omega_n - \omega_s}{\omega_s}\right)\right\}^{-1} \tag{27}$$

and expanding we have

$$\frac{\omega_s}{\omega_n} = 1 - \left(\frac{\omega_n - \omega_s}{\omega_s}\right) + \left(\frac{\omega_n - \omega_s}{\omega_s}\right)^2 - \left(\frac{\omega_n - \omega_s}{\omega_s}\right)^3 + \cdots. \tag{28}$$

To first order in $p_n - p_s$ and $E_n - E_s$ we also have

$$\left(\frac{\omega_n - \omega_s}{\omega_s}\right) = -\eta_s \left(\frac{p_n - p_s}{p_s}\right) = -\eta_s \left(\frac{E_n - E_s}{\beta_s^2 E_s}\right). \tag{29}$$

Thus to first order

$$\frac{\omega_s}{\omega_n} = 1 - \left(\frac{\omega_n - \omega_s}{\omega_s}\right) = 1 + \left(\frac{\eta_s}{\beta_s^2 E_s}\right) e_n \tag{30}$$

and (23) becomes

$$t_{n+1} = t_n + T_s \left(\frac{\eta_s}{\beta_s^2 E_s}\right) e_n. \tag{31}$$

This together with

$$e_{n+1} = e_n + eQV(t_{n+1}) \tag{32}$$

again produces a symplectic map from (t_n, e_n) to (t_{n+1}, e_{n+1}).

6 Phase and W Equations

Let us now introduce new variables

$$\phi_n = h\omega_s t_n, \quad W_n = \frac{e_n}{h\omega_s}. \tag{33}$$

Here ϕ_n is the phase that corresponds to time t_n, and W_n is defined so that the transformation from (t_n, e_n) to (ϕ_n, W_n) is symplectic. Note that ϕ_n varies from $-\pi$ to $+\pi$ as t_n varies from $-T_s/(2h)$ to $+T_s/(2h)$. In terms of the new variables equations (31) and (32) become

$$\phi_{n+1} = \phi_n + T_s \left(\frac{h^2\omega_s^2\eta_s}{\beta_s^2 E_s}\right) W_n \tag{34}$$

and

$$W_{n+1} = W_n + T_s \left(\frac{eQ}{2\pi h}\right) V\left(\frac{\phi_{n+1}}{h\omega_s}\right). \tag{35}$$

Defining

$$a = \left(\frac{h^2\omega_s^2\eta_s}{\beta_s^2 E_s}\right) \tag{36}$$

and

$$F(\phi_{n+1}) = \left(\frac{eQ}{2\pi h}\right) V\left(\frac{\phi_{n+1}}{h\omega_s}\right) \tag{37}$$

we then have

$$\phi_{n+1} = \phi_n + aT_s W_n \tag{38}$$

and

$$W_{n+1} = W_n + T_s F(\phi_{n+1}). \tag{39}$$

These equations again generate a sympletic map. Note that since $V(0) = 0$ and $V(T + T_s/h) = V(T_s)$ we have

$$F(0) = 0, \quad F(\phi + 2\pi) = F(\phi). \tag{40}$$

Since we are interested in the momentum deviation $p_n - p_s$, it is useful to have an expression for this in terms of W_n. Using (29) and (33) we find

$$\left(\frac{p_n - p_s}{p_s}\right) = \left(\frac{E_n - E_s}{\beta_s^2 E_s}\right) = \left(\frac{h\omega_s}{\beta_s^2 E_s}\right) W_n. \tag{41}$$

7 Fixed Points of the Motion

Fixed points of the motion are points (ϕ_f, W_f) that are unchanged by (38) and (39). For these points we must have

$$F(\phi_f) = 0, \quad W_f = 0. \tag{42}$$

Since $F(0) = 0$ we see that the point (0, 0) is a fixed point.

If a particle launched near a fixed point stays close to the fixed point, the fixed point is said to be stable. In the Appendix we show that below transition ($a < 0$) a fixed point is stable if

$$F'(\phi_f) > 0 \tag{43}$$

and unstable if

$$F'(\phi_f) < 0 \tag{44}$$

where the primes denote differentiation with respect to ϕ. Similarly above transition ($a > 0$) a fixed point will be stable if

$$F'(\phi_f) < 0 \tag{45}$$

and unstable if

$$F'(\phi_f) > 0. \tag{46}$$

8 RF Buckets

Consider the hamiltonian

$$H(\phi, W) = \frac{1}{2}\, aW^2 + U(\phi) \tag{47}$$

where

$$\frac{\partial U}{\partial \phi} = -F(\phi). \tag{48}$$

The equations of motion are

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} = aW \tag{49}$$

$$\frac{dW}{dt} = -\frac{\partial H}{\partial \phi} = F(\phi) \tag{50}$$

and we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{\partial U}{\partial t}. \tag{51}$$

The hamiltonian is therefore a constant of the motion if U has no explicit dependence on time. This constraint can be used to determine the extent of the regions of bounded motion generated by H.

The fixed points are the same as those for the motion generated by (38) and (39). They are given by (42). Let

$$U_{\phi\phi} = U''(\phi_f) \tag{52}$$

where the primes denote differentiation with respect to ϕ. Then below transition ($a < 0$) the fixed point will be stable if $U_{\phi\phi} < 0$ and unstable if $U_{\phi\phi} > 0$. Similarly above transition ($a > 0$) the fixed point will be stable if $U_{\phi\phi} > 0$ and unstable if $U_{\phi\phi} < 0$.

Let H_u be the value of H at an unstable fixed point $(\phi_u, 0)$. Then

$$H_u = U(\phi_u) \tag{53}$$

and the equation

$$H(\phi, W) = H_u \tag{54}$$

can be solved to obtain

$$W^2(\phi) = \frac{2}{a} \left\{ U(\phi_u) - U(\phi) \right\}. \tag{55}$$

The curve $W(\phi)$ is called the separatrix. Differentiating $W^2(\phi)$ with respect to ϕ we obtain

$$\frac{dW^2}{d\phi} = -\frac{2}{a} U'(\phi) \tag{56}$$

and

$$\frac{d^2W^2}{d^2\phi} = -\frac{2}{a} U''(\phi) \tag{57}$$

which shows that $W^2(\phi)$ reaches a local maximum at each stable fixed point. The area around a stable fixed point and bounded by the separatrix is an RF bucket. This is the extent of the region of bounded motion around the fixed point. The beam contained in the RF bucket is called a bunch. The height W_b of the bucket is given by

$$W_b^2 = \frac{2}{a} \left\{ U(\phi_u) - U(\phi_s) \right\} \tag{58}$$

where ϕ_s is the stable fixed point phase.

First-order symplectic integration [3] of (49) and (50) from time t to time $t + T_s$ yields

$$\phi(t + T_s) = \phi(t) + aT_s W(t) \tag{59}$$

and

$$W(t + T_s) = W(t) + T_s F(\phi(t + T_s)). \tag{60}$$

These are the same as (38) and (39). This suggests that the motion generated by the hamiltonian approximates that generated by (38) and (39). To the extent that this is true, the regions of bounded motion generated by H will approximate those generated by (38) and (39).

9 Capturing Unbunched Beam into RF Buckets

For simulations that start with bunched beam we want an initial beam distribution that consists of $\hat{h}$ bunches in $\hat{h}$ adjacent harmonic $\hat{h}h$ buckets. Here $\hat{h}h$ is the total number of RF buckets around the ring and $\hat{h}$ is the number of buckets in $1/h$ of the ring. To obtain the desired $\hat{h}$ bunches we start with a uniform distribution of unbunched beam consisting of an 80-by-80 array of particles which cover the region

$$-\phi_I \le \phi \le \phi_I, \quad -W_I \le W \le W_I \tag{61}$$

where

$$\phi_I = \pi \tag{62}$$

and

$$W_I = \frac{E_I}{h\omega_s} = \frac{E_I T_s}{2\pi h}. \tag{63}$$

Here E_I is the energy half-width of the unbunched beam. As defined by (33), the phase ϕ varies from $-\pi$ to $+\pi$ as the time deviation t varies from $-T_s/(2h)$ to $+T_s/(2h)$. The region defined by (61) therefore occupies $1/h$ of the ring as desired. The area of the region is

$$\epsilon = (2\phi_I)(2W_I) = 2E_I T_s/h. \tag{64}$$

This is the longitudinal emittance of the unbunched beam (in $1/h$ of the ring).

To capture the beam into $\hat{h}$ adjacent harmonic $\hat{h}h$ buckets we take

$$F(\phi) = -A \sin \hat{h}\phi \tag{65}$$

in (38) and (39). During capture A is raised slowly and quadratically from zero to A_C over time interval $0 \le T \le T_C$ according to

$$A(T) = A_C \left(\frac{T}{T_C}\right)^2 \tag{66}$$

where

$$A_C = \frac{eQV_C}{2\pi h} \tag{67}$$

and V_C is the RF harmonic $\hat{h}h$ voltage at time T_C. The time T_C is chosen to be long enough to ensure that the longitudinal emittance of the $\hat{h}$ bunches is as close to that of the initial distribution of unbunched beam as practical.

10 Formulae for the Standard 4 to 1 Merge

Here we divide the $\hat{h}h$ buckets into h groups of $\hat{h} = 4$ adjacent buckets and consider the merge of the 4 bunches in a given group into 1 bunch. If all of the $\hat{h}h$ buckets are occupied then the gap of time available for the injection kicker pulse is

$$T_G = 2\left(\frac{T_s}{\hat{h}h}\right) - T_W \tag{68}$$

where T_W is the bunch width and T_s is the revolution period.

For the merge we take

$$F(\phi) = A_1 \sin\phi - A_2 \sin 2\phi - A_4 \sin 4\phi \tag{69}$$

in (38) and (39). The parameters A_1, A_2 and A_4 are either zero or positive. They are varied slowly during the merging process. Note again that the phase ϕ varies from $-\pi$ to $+\pi$ as the time deviation t varies from $-T_s/(2h)$ to $+T_s/(2h)$.

In order to satisfy (48) we take

$$U(\phi) = A_1 \cos\phi - \frac{1}{2}A_2 \cos 2\phi - \frac{1}{4}A_4 \cos 4\phi \tag{70}$$

which gives

$$U''(\phi) = -A_1 \cos\phi + 2A_2 \cos 2\phi + 4A_4 \cos 4\phi. \tag{71}$$

This is positive at fixed point phases

$$\phi_u = \pm\pi \tag{72}$$

which shows that these are unstable fixed point phases below transition. We then have

$$U(\phi_u) = -A_1 - \frac{A_2}{2} - \frac{A_4}{4} \tag{73}$$

which can be used in (55) to obtain the separatrix.

Starting with 4 bunches in 4 adjacent harmonic $4h$ buckets, the merge is done in two steps. First the 4 bunches are merged into 2 and then the 2 are merged into 1. The merge of 4 bunches into 2 takes place over the time interval $0 \le T \le T_K$. During this time A_1 is held at zero, A_2 is raised linearly from zero to A_K, and A_4 is lowered linearly from A_C to zero. Thus

$$A_1(T) = 0 \tag{74}$$

and

$$A_2(T) = A_K \left(\frac{T}{T_K} \right), \quad A_4(T) = A_C \left(\frac{T_K - T}{T_K} \right). \tag{75}$$

Here

$$A_K = \frac{eQV_K}{2\pi h}, \quad A_C = \frac{eQV_C}{2\pi h} \tag{76}$$

where V_K is the harmonic $2h$ voltage at time T_K and V_C is the harmonic $4h$ voltage at time 0.

The merge of 2 bunches into 1 takes place over the time interval $T_K \le T \le T_L$. During this time A_4 is held at zero, A_1 is raised linearly from zero to A_L, and A_2 is lowered linearly from A_K to zero. Thus

$$A_1(T) = A_L \left(\frac{T - T_K}{T_L - T_K} \right), \quad A_2(T) = A_K \left(\frac{T_L - T}{T_L - T_K} \right) \tag{77}$$

and

$$A_4(T) = 0. \tag{78}$$

Here

$$A_L = \frac{eQV_L}{2\pi h} \tag{79}$$

where V_L is the harmonic h voltage at time T_L.

11 Formulae for the Alternative 4 to 1 Merge

In this case we divide the $\hat{h}h$ buckets into h groups of $\hat{h} = 6$ adjacent buckets and consider the merge of the 4 inner bunches of a given group into 1 bunch. The motivation here is that if the 2 outer buckets in each group are unoccupied, they provide time for the injection kicker pulse. The total gap of time available is

$$T_G = 4 \left(\frac{T_s}{\hat{h}h} \right) - T_W \tag{80}$$

where T_W is the bunch width and T_s is the revolution period. For harmonic $\hat{h}h = 12$ this turns out to be just long enough for the kicker pulse [4].

For this merge we take

$$F(\phi) = A_1 \sin\phi - A_2 \sin 2\phi - A_6 \sin 6\phi \tag{81}$$

in (38) and (39). The parameters A_1, A_2, and A_6 are either zero or positive. They are varied slowly during the merging process. Note again

that the phase ϕ varies from $-\pi$ to $+\pi$ as the time deviation t varies from $-T_s/(2h)$ to $+T_s/(2h)$.

In order to satisfy (48) we take

$$U(\phi) = A_1 \cos\phi - \frac{1}{2} A_2 \cos 2\phi - \frac{1}{6} A_6 \cos 6\phi \tag{82}$$

which gives

$$U''(\phi) = -A_1 \cos\phi + 2A_2 \cos 2\phi + 6A_6 \cos 6\phi. \tag{83}$$

This is positive at fixed point phases

$$\phi_u = \pm\pi \tag{84}$$

which shows that these are unstable fixed point phases below transition. We then have

$$U(\phi_u) = -A_1 - \frac{A_2}{2} - \frac{A_6}{6} \tag{85}$$

which can be used in (55) to obtain the separatrix.

At fixed point phase

$$\phi_f = 0 \tag{86}$$

we have

$$U''(\phi_f) = -A_1 + 2A_2 + 6A_6. \tag{87}$$

This gives an unstable fixed point (below transition) if

$$A_1 < 2A_2 + 6A_6. \tag{88}$$

In this case one has unstable fixed point phase

$$\phi_u = 0 \tag{89}$$

and

$$U(\phi_u) = A_1 - \frac{A_2}{2} - \frac{A_6}{6} \tag{90}$$

which can be used in (55) to obtain the associated separatrix.

For $0 \le T \le T_K$ we take

$$A_1(T) = \mathcal{F} A_L \left(\frac{T}{T_K}\right), \quad A_2(T) = A_K \left(\frac{T}{T_K}\right) \tag{91}$$

and

$$A_6(T) = A_C \left(\frac{T_K - T}{T_K}\right) \tag{92}$$

where

$$0 \le \mathcal{F} \le 1. \tag{93}$$

The parameter $\mathcal{F}$ is adjusted to keep the 4 inner bunches separated from the phase space of the 2 outer buckets, and to minimize the gross emittance of the final merged bunch.

For $T_K \le T \le T_L$ we take

$$A_6(T) = 0, \quad A_2(T) = A_K \left(\frac{T_L - T}{T_L - T_K} \right) \tag{94}$$

and

$$A_1(T) = \mathcal{F} A_L + (1 - \mathcal{F}) A_L \left(\frac{T - T_K}{T_L - T_K} \right). \tag{95}$$

Here

$$A_L = \frac{eQV_L}{2\pi h}, \quad A_K = \frac{eQV_K}{2\pi h}, \quad A_C = \frac{eQV_C}{2\pi h} \tag{96}$$

where V_L, V_K, and V_C are the harmonic h, $2h$, and $6h$ voltages at times T_L, T_K, and 0, respectively.

12 Formulae for a 6 to 1 Merge

In this case we divide the $\hat{h}h$ buckets into h groups of $\hat{h} = 6$ adjacent buckets and consider the merge of the 6 bunches in a given group into 1 bunch. The motivation here is to get more helions per merged bunch. If all of the $\hat{h}h$ buckets are occupied then the gap of time available for the injection kicker pulse is

$$T_G = 2 \left(\frac{T_s}{\hat{h}h} \right) - T_W \tag{97}$$

where T_W is the bunch width and T_s is the revolution period.

For this merge we take

$$F(\phi) = A_1 \sin\phi - A_2 \sin 2\phi + A_3 \sin 3\phi - A_6 \sin 6\phi \tag{98}$$

in (38) and (39). The parameters A_1, A_2, A_3, and A_6 are either zero or positive. They are varied slowly during the merging process. Note again that the phase ϕ varies from $-\pi$ to $+\pi$ as the time deviation t varies from $-T_s/(2h)$ to $+T_s/(2h)$.

In order to satisfy (48) we take

$$U(\phi) = A_1 \cos\phi - \frac{1}{2}A_2 \cos 2\phi + \frac{1}{3}A_3 \cos 3\phi - \frac{1}{6}A_6 \cos 6\phi \tag{99}$$

which gives

$$U''(\phi) = -A_1 \cos\phi + 2A_2 \cos 2\phi - 3A_3 \cos 3\phi + 6A_6 \cos 6\phi. \tag{100}$$

This is positive at fixed point phases

$$\phi_u = \pm\pi \tag{101}$$

which shows that these are unstable fixed point phases below transition. We then have

$$U(\phi_u) = -A_1 - \frac{A_2}{2} - \frac{A_3}{3} - \frac{A_6}{6} \tag{102}$$

which can be used in (55) to obtain the separatrix.

For the case $A_6 = 0$ we have

$$F(\phi) = A_1 \sin\phi - A_2 \sin 2\phi + A_3 \sin 3\phi \tag{103}$$

$$U(\phi) = A_1 \cos\phi - \frac{1}{2}A_2 \cos 2\phi + \frac{1}{3}A_3 \cos 3\phi \tag{104}$$

and

$$U''(\phi) = -A_1 \cos\phi + 2A_2 \cos 2\phi - 3A_3 \cos 3\phi. \tag{105}$$

Using

$$\sin 2\phi = 2\cos\phi \sin\phi \tag{106}$$

and

$$\sin 3\phi = 3\sin\phi - 4\sin^3\phi \tag{107}$$

we then have

$$F(\phi) = \left\{A_1 - 2A_2 \cos\phi + A_3\left(4\cos^2\phi - 1\right)\right\}\sin\phi \tag{108}$$

which shows that in addition to the unstable fixed point phases (101) we have fixed point phases given by

$$A_1 - 2A_2 \cos\phi_f + A_3\left(4\cos^2\phi_f - 1\right) = 0. \tag{109}$$

Solving this for $\cos\phi_f$ gives

$$\cos\phi_f = \frac{A_2}{4A_3} \pm \frac{1}{4A_3}\sqrt{A_2^2 + 4A_3(A_3 - A_1)}. \tag{110}$$

To evaluate $U(\phi)$ and $U''(\phi)$ at ϕ_f we use

$$\cos 2\phi = 2\cos^2\phi - 1 \tag{111}$$

and

$$\cos 3\phi = 4\cos^3\phi - 3\cos\phi \tag{112}$$

which give

$$U(\phi) = A_1\cos\phi - \frac{A_2}{2}\left(2\cos^2\phi - 1\right) + \frac{A_3}{3}\left(4\cos^3\phi - 3\cos\phi\right) \tag{113}$$

and

$$U''(\phi) = -A_1\cos\phi + 2A_2\left(2\cos^2\phi - 1\right) - 3A_3\left(4\cos^3\phi - 3\cos\phi\right). \tag{114}$$

For the values of A_1, A_2, and A_3 considered here, $U''(\phi_f)$ is positive for the phase given by the upper sign in (110). This is then the phase for an unstable fixed point below transition. The lower sign gives the phase for a stable fixed point below transition. Thus using

$$\cos\phi_u = \frac{A_2}{4A_3} + \frac{1}{4A_3}\sqrt{A_2^2 + 4A_3(A_3 - A_1)} \tag{115}$$

in (113) gives $U(\phi_u)$ which can be used in (55) to obtain the associated separatrix.

Starting with 6 bunches in 6 adjacent harmonic $6h$ buckets, the merge is done in three steps. First the 6 bunches are merged into 3. This is a straightforward 2 to 1 merge. Next the 3 bunches are moved toward one another but are kept in separate buckets. Finally, as soon as the bunches touch one another, they are allowed to mix and merge into a single bunch. The 3 to 1 merge is based on the the work of Garoby [5, 6]. A similar scheme was developed by M. Blaskiewicz and J.M. Brennan for merging 24 bunches of gold ions into 4 bunches on the AGS injection porch [7].

The merge of 6 bunches into 3 takes place over the time interval $0 \le T \le T_K$. During this time A_1 and A_2 are held at zero, A_3 is raised linearly from zero to A_M, and A_6 is lowered linearly from A_C to zero. Thus we have

$$A_1(T) = 0, \quad A_2(T) = 0 \tag{116}$$

and

$$A_3(T) = A_M\left(\frac{T}{T_K}\right), \quad A_6(T) = A_C\left(\frac{T_K - T}{T_K}\right). \tag{117}$$

Here

$$A_M = \frac{eQV_M}{2\pi h}, \quad A_C = \frac{eQV_C}{2\pi h} \tag{118}$$

where V_M is the harmonic $3h$ voltage at time $T = T_K$ and V_C is the harmonic $6h$ voltage at time $T = 0$.

The first part of the merge of 3 bunches into 1 takes place over the time interval $T_K \le T \le T_{L1}$. During this time A_6 is held at zero, A_3 is lowered linearly from A_M to A_{M1}, and A_1 is raised linearly from zero to A_{L1}. The parameter A_2 is raised linearly from zero to A_{K1}. Thus we have

$$A_6(T) = 0 \tag{119}$$

$$A_3(T) = A_M \left(\frac{T_{L1} - T}{T_{L1} - T_K} \right) + A_{M1} \left(\frac{T - T_K}{T_{L1} - T_K} \right) \tag{120}$$

and

$$A_1(T) = A_{L1} \left(\frac{T - T_K}{T_{L1} - T_K} \right), \quad A_2(T) = A_{K1} \left(\frac{T - T_K}{T_{L1} - T_K} \right) \tag{121}$$

where

$$A_{K1} = \frac{eQV_{K1}}{2\pi h}, \quad A_{L1} = \frac{eQV_{L1}}{2\pi h}, \quad A_{M1} = \frac{eQV_{M1}}{2\pi h}. \tag{122}$$

Here V_{K1}, V_{L1}, and V_{M1} are the harmonic $2h$, h, and $3h$ voltages, respectively, at time T_{L1}.

The second part of the 3 to 1 merge takes place over the time interval $T_{L1} \le T \le T_{L2}$. During this time A_6 continues to be held at zero, A_3 is lowered linearly from A_{M1} to A_{M2}, and A_2 is lowered linearly from A_{K1} to A_{K2}. The parameter A_1 is held at A_{L1}. Thus

$$A_3(T) = A_{M1} \left(\frac{T_{L2} - T}{T_{L2} - T_{L1}} \right) + A_{M2} \left(\frac{T - T_{L1}}{T_{L2} - T_{L1}} \right) \tag{123}$$

$$A_2(T) = A_{K1} \left(\frac{T_{L2} - T}{T_{L2} - T_{L1}} \right) + A_{K2} \left(\frac{T - T_{L1}}{T_{L2} - T_{L1}} \right) \tag{124}$$

and

$$A_1(T) = A_{L1}, \quad A_6(T) = 0 \tag{125}$$

where

$$A_{K2} = \frac{eQV_{K2}}{2\pi h}, \quad A_{M2} = \frac{eQV_{M2}}{2\pi h}. \tag{126}$$

Here V_{K2} and V_{M2} are the harmonic $2h$ and $3h$ voltages, respectively, at time T_{L2}.

The final part of the 3 to 1 merge takes place over the time interval $T_{L2} \le T \le T_F$. During this time A_6 continues to be held at zero, A_3 is lowered linearly from A_{M2} to zero, A_2 is lowered linearly from A_{K2} to zero, and A_1 continues to be held at A_{L1}. Thus

$$A_3(T) = A_{M2}\left(\frac{T_F - T}{T_F - T_{L2}}\right), \quad A_2(T) = A_{K2}\left(\frac{T_F - T}{T_F - T_{L2}}\right) \tag{127}$$

and

$$A_1(T) = A_{L1}, \quad A_6(T) = 0. \tag{128}$$

13 Simulation of Standard 4 to 1 Merge

In this scheme we take harmonic $\hat{h}h = 8$ on the AGS injection porch and populate all 8 buckets with helion bunches from Booster. This gives $h = 2$ groups of $\hat{h} = 4$ adjacent bunches. The 4 bunches in each group are merged into 2 bunches by bringing on harmonic $4 = 2h$ voltage with RF cavity KL while reducing the harmonic $8 = 4h$ voltage to zero. The 2 bunches are then merged into 1 by bringing on harmonic $2 = 1h$ voltage with RF cavity L10 while reducing the harmonic $4 = 2h$ voltage to zero.

For the simulation we start with 4 adjacent bunches of helions. These are obtained from a uniform distribution of unbunched beam as described in Section 9. We use capture time $T_C = 400$ ms and harmonic 8 voltage $V_C = 40$ kV. The harmonic 8 frequency is $8f_s = 2.51$ MHz. The longitudinal emittance of the unbunched beam is taken to be 0.80 eV s per nucleon. **Figure 1** shows the resulting bunches in 4 buckets on the porch. One can see by inspection that the emittance of the 4 bunches is close to that of the initial distribution. This gives an emittance of 0.2 eV s per nucleon per bunch. Note also that taking bunch width $T_W = 90$ ns in (68) gives time gap

$$T_G = 707 \text{ ns} \tag{129}$$

which is too short (by some 300 ns) for the present injection kicker pulse. This means that in practice only 7 of the 8 buckets can be filled with bunches from Booster.

Figure 2 shows the RF voltages for the 4 to 1 merge. These are given by the formulae of Section 10 with h set equal to 2. Here the 4 to 2 merge

takes place over the time interval $0 \le T \le 80$ ms. The 2 to 1 merge takes place over the interval $80 \le T \le 160$ ms.

Figure 3 shows the "potential" $U(\phi_u) - U(\phi)$ at various times during the merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 10.

The merge from 4 to 2 bunches is shown in **Figures 4** and **5**. The merge from 2 to 1 bunches is shown in **Figures 6** and **7**. One can see by inspection that the emittance of the final merged bunch is close to that (0.80 eV s per nucleon) of the initial distribution of unbunched beam. The bunch is easily held in the harmonic 2 bucket with voltage $V_L = 20$ kV.

14 Simulation of Alternative 4 to 1 Merge

In this scheme we take harmonic $\hat{h}h = 12$ on the AGS injection porch. This gives $h = 2$ groups of $\hat{h} = 6$ adjacent buckets. The 4 inner buckets of each group are populated with helion bunches from Booster. The 2 outer buckets of each group are left empty to provide time for the injection kicker pulse. The 4 bunches in each group are merged into 2 bunches by bringing on harmonic $4 = 2h$ and $2 = 1h$ voltages with RF cavities KL and L10, respectively, while reducing the harmonic $12 = 6h$ voltage to zero. The 2 bunches are then merged into 1 by reducing the harmonic $4 = 2h$ voltage to zero.

For the simulation we start with 6 adjacent bunches of helions. These are obtained from a uniform distribution of unbunched beam as described in Section 9. We use capture time $T_C = 200$ ms and harmonic 12 voltage $V_C = 47$ kV. The harmonic 12 frequency is $12f_s = 3.765$ MHz. The longitudinal emittance of the unbunched beam is taken to be 1.2 eV s per nucleon. **Figure 8** shows the resulting bunches in 6 buckets on the porch. One can see by inspection that the emittance of the 6 bunches is close to that of the initial distribution. This gives an emittance of 0.2 eV s per nucleon per bunch. Note also that taking bunch width $T_W = 90$ ns in (80) gives time gap

$$T_G = 972 \text{ ns} \tag{130}$$

which is just long enough for the present injection kicker pulse [4].

Figure 9 shows the RF voltages for the 4 to 1 merge. These are given by the formulae of Section 11 with h set equal to 2. Here the parameter $\mathcal{F}$ (defined in Section 11) has been set equal to 0.95. This setting keeps the inner 4 bunches separated from the outer 2 during the merging process and

minimizes the gross emittance of the final merged bunch.

Figure 10 shows the "potential" $U(\phi_u) - U(\phi)$ at various times during the merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 11.

The merge from 4 to 2 bunches is shown in **Figures 11** and **12**. Here we see that the brown separatrix keeps the inner 4 bunches separated from the outer 2. The two outer bunches (which are not present in practice) form a halo around the inner 4.

The merge from 2 to 1 bunches is shown in **Figures 13** and **14**. One can see by inspection that the emittance of the final merged bunch is close to that (0.80 eV s per nucleon) of the inner 4 bunches of the initial distribution of bunched beam. The bunch is easily held in the harmonic 2 bucket with voltage $V_L = 22$ kV.

Comparing **Figures 7** and **14** one sees that alternative 4 to 1 merge produces a merged bunch of the same size and quality as the standard 4 to 1 merge.

15 Simulation of 6 to 1 Merge

In this scheme we again take harmonic $\hat{h}h = 12$ on the AGS injection porch. This gives $h = 2$ groups of $\hat{h} = 6$ adjacent buckets. The 6 buckets of each group are populated with helion bunches from Booster. The 6 bunches in each group are merged into 3 bunches by bringing on harmonic $6 = 3h$ while reducing the harmonic $12 = 6h$ voltage to zero. The 3 bunches are then merged into 1 by manipulating harmonics $6 = 3h$, $4 = 2h$, and $2 = 1h$ as described in Section 12. Harmonics 4 and 2 are provided by the KL and L10 cavities respectively.

For the simulation we start with 6 adjacent bunches of helions. These are obtained from a uniform distribution of unbunched beam as described in Section 9. We use capture time $T_C = 200$ ms and harmonic 12 voltage $V_C = 47$ kV. The harmonic 12 frequency is $12f_s = 3.765$ MHz. The longitudinal emittance of the unbunched beam is taken to be 1.2 eV s per nucleon. **Figure 15** shows the resulting bunches in 6 buckets on the porch. One can see by inspection that the emittance of the 6 bunches is close to that of the initial distribution. This gives an emittance of 0.2 eV s per nucleon per bunch. Note also that taking bunch width $T_W = 90$ ns in (97) gives time gap

$$T_G = 441 \text{ ns}. \tag{131}$$

This is too short for the present injection kicker pulse, but still allows for 6 of the 12 buckets to be filled.

Figures 16 and **17** show the RF voltages for the 6 to 1 merge. These are given by the formulae of Section 12 with h set equal to 2. Here the 6 to 3 merge takes place over the time interval $0 \leq T \leq 30$ ms. The 3 to 1 merge takes place over the interval $30 \leq T \leq 120$ ms.

Figures 18 and **19** show the "potential" $U(\phi_u) - U(\phi)$ at various times during the merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 12.

The merge from 6 to 3 bunches is shown in **Figures 20** and **21**.

Figures 22 and **23** show the 3 bunches one sixth and one third of the way through the 3 to 1 merge. In both figures the bunches are kept separated by the brown separatrix. In **Figure 23** the bunches are nearly touching but are still in separate buckets.

In **Figure 24** the bunches are halfway through the 3 to 1 merge. Here the bunches have started to mix with one another.

In **Figure 25** the bunches are two thirds of the way through the 3 to 1 merge. The bunches continue to mix and merge.

The completion of the 3 to 1 merge is shown in **Figure 26**. The magnified view in **Figure 27** shows a tightly merged bunch. This is the final result of the 6 to 1 merge.

Note that if the 2 outer buckets in **Figure 15** are left empty, then the 6 to 1 merge becomes another alternative scheme for merging 4 bunches into 1. In this case, however, the phase space of the 2 outer buckets mixes with the 4 inner bunches and dilutes the gross emittance. One ends up with the bunch shown in **Figure 28**. (This was obtained by simply turning off the black and brown points in **Figure 26**.) The gross emittance of the bunch is 6/4 times larger than that obtained with the alternative 4 to 1 scheme of Section 14.

16 Appendix

For motion near a fixed point (ϕ_f, 0) we have

$$F(\phi) = -k(\phi - \phi_f) \tag{132}$$

where

$$k = -F'(\phi_f). \tag{133}$$

Equations (38) and (39) then become

$$\phi_{n+1} - \phi_f = (\phi_n - \phi_f) + aT_s W_n \tag{134}$$

and

$$W_{n+1} = W_n - kT_s(\phi_n - \phi_f) - akT_s^2 W_n. \tag{135}$$

Defining

$$\Phi = \phi - \phi_f \tag{136}$$

and

$$\mathbf{X}_{n+1} = \begin{pmatrix} \Phi_{n+1} \\ W_{n+1} \end{pmatrix}, \quad \mathbf{X}_n = \begin{pmatrix} \Phi_n \\ W_n \end{pmatrix} \tag{137}$$

we then have

$$\mathbf{X}_{n+1} = \mathbf{M}\mathbf{X}_n \tag{138}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & aT_s \\ -kT_s & 1 - akT_s^2 \end{pmatrix}. \tag{139}$$

The matrix $\mathbf{M}$ has unit determinant and its trace is

$$\mathcal{T} = 2 - akT_s^2. \tag{140}$$

It can be expressed in Courant-Snyder [8] form

$$\mathbf{M} = \begin{pmatrix} C + \mathcal{A}S & \mathcal{B}S \\ -\mathcal{G}S & C - \mathcal{A}S \end{pmatrix} \tag{141}$$

where

$$C = 1 - \frac{1}{2}akT_s^2, \quad S^2 = 1 - C^2 \tag{142}$$

$$\mathcal{B}S = aT_s, \quad 2\mathcal{A}S = akT_s^2, \quad \mathcal{G}S = kT_s \tag{143}$$

and

$$\mathcal{B}\mathcal{G} - \mathcal{A}^2 = 1. \tag{144}$$

16.1 Conserved Quadratic Form

Defining

$$\mathbf{E} = \begin{pmatrix} \mathcal{B} & -\mathcal{A} \\ -\mathcal{A} & \mathcal{G} \end{pmatrix}, \quad \mathbf{E}^{-1} = \begin{pmatrix} \mathcal{G} & \mathcal{A} \\ \mathcal{A} & \mathcal{B} \end{pmatrix} \tag{145}$$

and

$$\mathbf{T} = \mathbf{M}^T \mathbf{E}^{-1} \mathbf{M} \tag{146}$$

where the superscript T denotes the transpose of the matrix, we have

$$\mathbf{T} = \begin{pmatrix} C + \mathcal{A}S & -\mathcal{G}S \\ \mathcal{B}S & C - \mathcal{A}S \end{pmatrix} \begin{pmatrix} \mathcal{G} & \mathcal{A} \\ \mathcal{A} & \mathcal{B} \end{pmatrix} \begin{pmatrix} C + \mathcal{A}S & \mathcal{B}S \\ -\mathcal{G}S & C - \mathcal{A}S \end{pmatrix} \tag{147}$$

$$\mathbf{T} = \begin{pmatrix} C + \mathcal{A}S & -\mathcal{G}S \\ \mathcal{B}S & C - \mathcal{A}S \end{pmatrix} \begin{pmatrix} \mathcal{G}C & S + \mathcal{A}C \\ \mathcal{A}C - S & \mathcal{B}C \end{pmatrix} \tag{148}$$

and

$$\mathbf{T} = \begin{pmatrix} \mathcal{G} & \mathcal{A} \\ \mathcal{A} & \mathcal{B} \end{pmatrix}. \tag{149}$$

Thus

$$\mathbf{M}^T \mathbf{E}^{-1} \mathbf{M} = \mathbf{E}^{-1} \tag{150}$$

and

$$\mathbf{X}_{n+1}^T \mathbf{E}^{-1} \mathbf{X}_{n+1} = \mathbf{X}_n^T \mathbf{M}^T \, \mathbf{E}^{-1} \, \mathbf{M} \mathbf{X}_n = \mathbf{X}_n^T \mathbf{E}^{-1} \mathbf{X}_n. \tag{151}$$

Here we have quadratic forms

$$\mathbf{X}_{n+1}^T \mathbf{E}^{-1} \mathbf{X}_{n+1} = \mathcal{G}\Phi_{n+1}^2 + 2\mathcal{A}\Phi_{n+1} W_{n+1} + \mathcal{B} W_{n+1}^2 \tag{152}$$

and

$$\mathbf{X}_n^T \mathbf{E}^{-1} \mathbf{X}_n = \mathcal{G}\Phi_n^2 + 2\mathcal{A}\Phi_n W_n + \mathcal{B} W_n^2 \tag{153}$$

and we can write

$$G(\Phi_{n+1}, W_{n+1}) = G(\Phi_n, W_n) \tag{154}$$

where

$$G(\Phi, W) = \mathcal{G}\Phi^2 + 2\mathcal{A}\Phi W + \mathcal{B} W^2. \tag{155}$$

This shows that $G(\Phi, W)$ is a conserved quantity for the turn-by-turn motion generated by (134) and (135).

16.2 Stable Fixed Point

We assume that $k < 0$ below transition and $k > 0$ above transition. Then since $a < 0$ below transition and $a > 0$ above transition, we have

$$ak > 0. \tag{156}$$

We assume further that $|k|$ is small enough to ensure that

$$|\mathcal{T}| < 2. \tag{157}$$

It then follows that $|C| < 1$ and $S^2 > 0$. The parameters S, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{G}$ are then all real numbers. The sign of S is chosen so that $\mathcal{B} > 0$. We then have

$$G(\Phi, W) = \mathcal{E} \tag{158}$$

where $\mathcal{E}$ is a real number. With the help of (144) we have

$$\Phi^2 + (\mathcal{A}\Phi + \mathcal{B}W)^2 = \mathcal{E}\mathcal{B} \tag{159}$$

and

$$W^2 + (\mathcal{G}\Phi + \mathcal{A}W)^2 = \mathcal{E}\mathcal{G}. \tag{160}$$

The maximum possible Φ is then

$$\Phi_M = \sqrt{\mathcal{E}\mathcal{B}} \tag{161}$$

and the corresponding W is

$$W(\Phi_M) = -\frac{\mathcal{A}}{\mathcal{B}}\,\Phi_M. \tag{162}$$

Similarly the maximum possible W is

$$W_M = \sqrt{\mathcal{E}\mathcal{G}} \tag{163}$$

and the corresponding Φ is

$$\Phi(W_M) = -\frac{\mathcal{A}}{\mathcal{G}}\,W_M. \tag{164}$$

Here

$$-\frac{\mathcal{A}}{\mathcal{B}} = -\frac{1}{2}kT_s, \quad -\frac{\mathcal{A}}{\mathcal{G}} = -\frac{1}{2}aT_s. \tag{165}$$

This analysis shows that under the conditions assumed, a particle launched close to the fixed point will stay close to the fixed point. The fixed point is then said to be stable.

Note that (158) defines an ellipse with area $\pi\mathcal{E}$ and Courant-Snyder parameters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$. Since we have assumed that $k < 0$ below transition we also have $\mathcal{A} < 0$. The ellipse is therefore rotated clockwise with respect to the Φ and W axes.

Note also that defining

$$\mathcal{H}(\Phi, W) = \frac{S}{2T_s}\, G(\Phi, W) \tag{166}$$

and using the identities

$$\frac{S\mathcal{B}}{2T_s} = \frac{a}{2}, \quad \frac{2\mathcal{A}S}{2T_s} = \frac{1}{2}akT_s, \quad \frac{S\mathcal{G}}{2T_s} = \frac{k}{2} \tag{167}$$

we have

$$\mathcal{H}(\Phi, W) = \frac{1}{2}aW^2 + \frac{1}{2}k\Phi^2 + \frac{1}{2}akT_s\Phi W. \tag{168}$$

This is a conserved quantity for the motion generated by (134) and (135). It shows that the hamiltonian

$$H = \frac{1}{2}aW^2 + \frac{1}{2}k\Phi^2 \tag{169}$$

is not conserved. In fact we have

$$H(\Phi, W) = \mathcal{H}(\Phi, W) - \frac{1}{2}akT_s\Phi W \tag{170}$$

which we can write as

$$H(\Phi, W) = \frac{S\mathcal{E}}{2T_s} - \frac{1}{2}akT_s\Phi W \tag{171}$$

where

$$\mathcal{E} = G(\Phi, W). \tag{172}$$

The first term on the right hand side of (171) is constant but the second term varies with Φ and W.

16.3 Explicit Turn-by-Turn Motion

Now let

$$\psi = \arcsin S \tag{173}$$

where S is given by (142). Then

$$C = \cos\psi, \quad S = \sin\psi \tag{174}$$

and following Courant and Snyder [8] we can write

$$\mathbf{M} = \mathbf{I}\cos\psi + \mathbf{J}\sin\psi \tag{175}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{G} & -\mathcal{A} \end{pmatrix}. \tag{176}$$

Here

$$\mathbf{J}^2 = -\mathbf{I} \tag{177}$$

and it follows that

$$\{\mathbf{I}\cos\psi + \mathbf{J}\sin\psi\}\{\mathbf{I}\cos\chi + \mathbf{J}\sin\chi\} = \mathbf{I}\cos(\psi+\chi) + \mathbf{J}\sin(\psi+\chi). \tag{178}$$

Thus we have

$$\mathbf{M}^n = \mathbf{I}\cos n\psi + \mathbf{J}\sin n\psi \tag{179}$$

and

$$\Phi_n = (C_n + \mathcal{A}S_n)\Phi_0 + \mathcal{B}S_n W_0 \tag{180}$$

$$W_n = -\mathcal{G}S_n\Phi_0 + (C_n - \mathcal{A}S_n)W_0 \tag{181}$$

where

$$C_n = \cos n\psi, \quad S_n = \sin n\psi \tag{182}$$

and Φ_0 and W_0 are initial values of ϕ and W. Collecting terms we have

$$\Phi_n = \Phi_0 C_n + (\mathcal{A}\Phi_0 + \mathcal{B}W_0)S_n \tag{183}$$

$$W_n = W_0 C_n - (\mathcal{G}\Phi_0 + \mathcal{A}W_0)S_n \tag{184}$$

and therefore

$$\Phi_n = \sqrt{\mathcal{E}\mathcal{B}}\,\{C_n\cos\xi - S_n\sin\xi\} = \sqrt{\mathcal{E}\mathcal{B}}\cos(n\psi+\xi) \tag{185}$$

$$W_n = \sqrt{\mathcal{E}\mathcal{G}}\,\{C_n\sin\chi + S_n\cos\chi\} = \sqrt{\mathcal{E}\mathcal{G}}\sin(n\psi+\chi) \tag{186}$$

where

$$\mathcal{E}\mathcal{B} = \Phi_0^2 + (\mathcal{A}\Phi_0 + \mathcal{B}W_0)^2 \tag{187}$$

$$\mathcal{E}\mathcal{G} = W_0^2 + (\mathcal{G}\Phi_0 + \mathcal{A}W_0)^2 \tag{188}$$

and

$$\sqrt{\mathcal{E}\mathcal{B}}\cos\xi = \Phi_0, \quad \sqrt{\mathcal{E}\mathcal{B}}\sin\xi = -(\mathcal{A}\Phi_0 + \mathcal{B}W_0) \tag{189}$$

$$\sqrt{\mathcal{E}\mathcal{G}}\sin\chi = W_0, \quad \sqrt{\mathcal{E}\mathcal{G}}\cos\chi = -(\mathcal{G}\Phi_0 + \mathcal{A}W_0). \tag{190}$$

These equations give explicitly the turn-by-turn motion near the stable fixed point.

16.4 Synchrotron Oscillation Tune and Period

The synchrotron tune is defined to be

$$Q_s = \frac{|\psi|}{2\pi}. \tag{191}$$

The synchrotron oscillation period is then

$$P_s = \frac{T_s}{Q_s}. \tag{192}$$

For sufficiently small akT_s^2 we have

$$\psi = -T_s\sqrt{ak} \tag{193}$$

which gives

$$Q_s = \frac{T_s}{2\pi}\sqrt{ak}, \quad P_s = \frac{2\pi}{\sqrt{ak}}. \tag{194}$$

For the case

$$F(\phi) = \left(\frac{eQV_G}{2\pi h}\right)\sin\phi \tag{195}$$

we have stable fixed point phase $\phi_s = 0$ (below transition) and

$$ak = -\frac{eQV_G}{2\pi h}\left(\frac{h^2\omega_s^2\eta_s}{\beta_s^2 E_s}\right) = -\frac{eQV_G}{2\pi h}\left(\frac{h^2c^2\eta_s}{R_s^2 E_s}\right). \tag{196}$$

16.5 Unstable Fixed Point

For all of the above work we have assumed that $k < 0$ below transition and $k > 0$ above transition. If instead we assume that $k > 0$ below transition and $k < 0$ above transition then

$$ak < 0 \tag{197}$$

and we see from (140) that the trace of $\mathbf{M}$ is greater than 2. One then finds that a particle launched close to the fixed point will not stay close to the fixed point. The fixed point is then said to be unstable.

We again have conserved quadratic form (159) which we can write as

$$\Phi^2 - (i\mathcal{A}\Phi + i\mathcal{B}W)^2 = \mathcal{E}\mathcal{B}. \tag{198}$$

But now $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ are pure imaginary numbers which means that $i\mathcal{A}$, $i\mathcal{B}$, and $\mathcal{E}\mathcal{B}$ are all real. Equation (198) therefore defines a hyperbola. The asymptotes are given by

$$i\mathcal{A}\Phi + i\mathcal{B}W = \pm\Phi \tag{199}$$

which we can write as

$$W = \frac{-i\mathcal{A} \pm 1}{i\mathcal{B}}\,\Phi. \tag{200}$$

For a particle on either of the asymptotes we have $\mathcal{E} = 0$. Since $\mathcal{E}$ is a conserved quantity, the particle stays on the asymptote.

Again following Courant and Snyder [8] we can write

$$\mathbf{M} = C\mathbf{I} + S\mathbf{J} \tag{201}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{G} & -\mathcal{A} \end{pmatrix} \tag{202}$$

and

$$\mathbf{J}^2 = -\mathbf{I}. \tag{203}$$

Here C and S are given by (142) and since $C > 1$, we can find real ψ such that

$$C = \cosh\psi, \quad S = i\sinh\psi. \tag{204}$$

We then have

$$\mathbf{M} = \mathbf{I}\cosh\psi + i\mathbf{J}\sinh\psi \tag{205}$$

and

$$\begin{aligned} &\{\mathbf{I}\cosh\psi + i\mathbf{J}\sinh\psi\}\,\{\mathbf{I}\cosh\chi + i\mathbf{J}\sinh\chi\} \\ &\qquad = \mathbf{I}\cosh(\psi+\chi) + i\mathbf{J}\sinh(\psi+\chi). \end{aligned} \tag{206}$$

It then follows that

$$\mathbf{M}^n = \mathbf{I}\cosh n\psi + i\mathbf{J}\sinh n\psi \tag{207}$$

and

$$\mathbf{M}^n = \begin{pmatrix} \cosh n\psi + i\mathcal{A}\sinh n\psi & i\mathcal{B}\sinh n\psi \\ -i\mathcal{G}\sinh n\psi & \cosh n\psi - i\mathcal{A}\sinh n\psi \end{pmatrix}. \tag{208}$$

This gives the turn-by-turn motion near the unstable fixed point.

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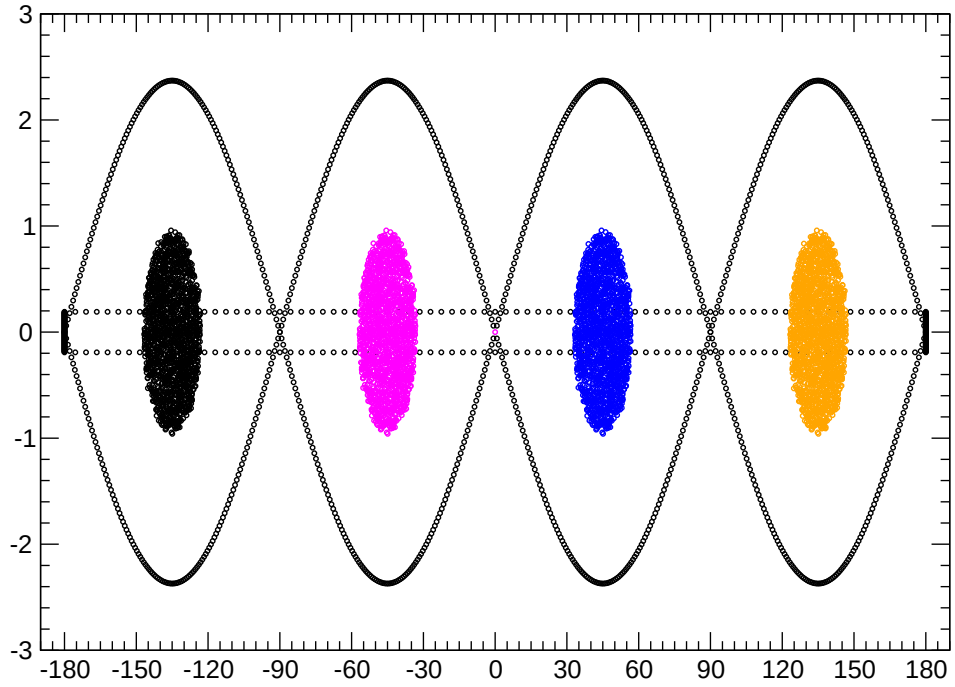

Figure 1: Four bunches of helions in harmonic 8 buckets on the AGS injection porch. The harmonic 8 frequency is $8f_s = 2.51$ MHz. The RF voltage is $V_C = 40$ kV. The black rectangle is the border of the uniform distribution used to make the bunches. The longitudinal emittance of the distribution is 0.80 eV s per nucleon. One can see by inspection that the emittance of the 4 bunches is close to that of the initial distribution. This gives an emittance of 0.2 eV s per nucleon per bunch. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

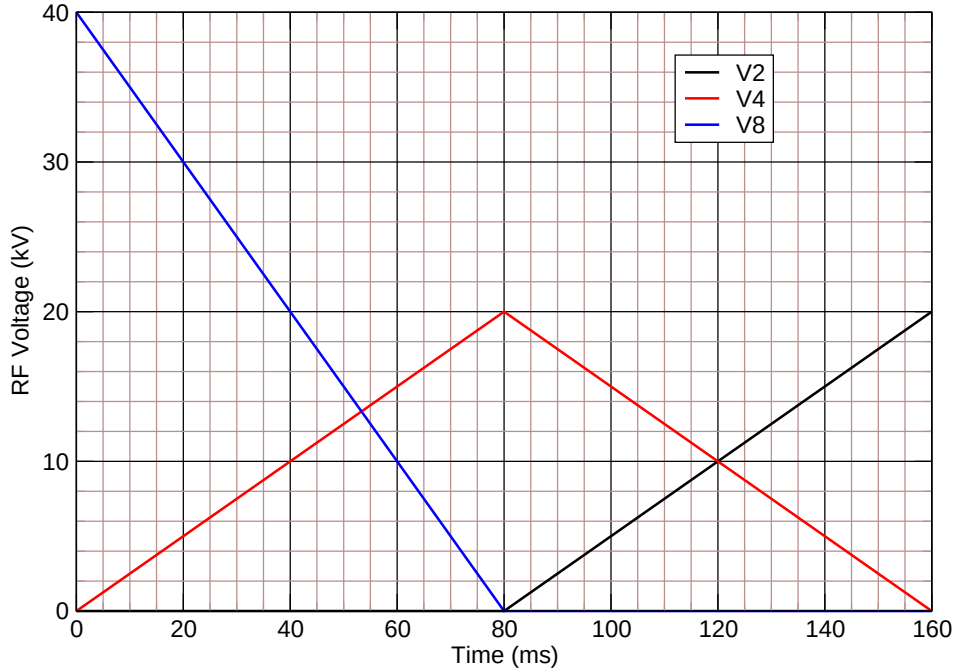

Figure 2: RF voltages for the standard 4 to 1 merge. Here the black, red, and blue curves are the harmonic $2 = 1h$, $4 = 2h$, and $8 = 4h$ voltages respectively. These are given by the formulae of Section 10 with h set equal to 2. Here the harmonic 8, 4, and 2 voltages are $V_C = 40$ kV, $V_K = 20$ kV, and $V_L = 20$ kV respectively. The times T_K and T_L are 80 and 160 ms respectively.

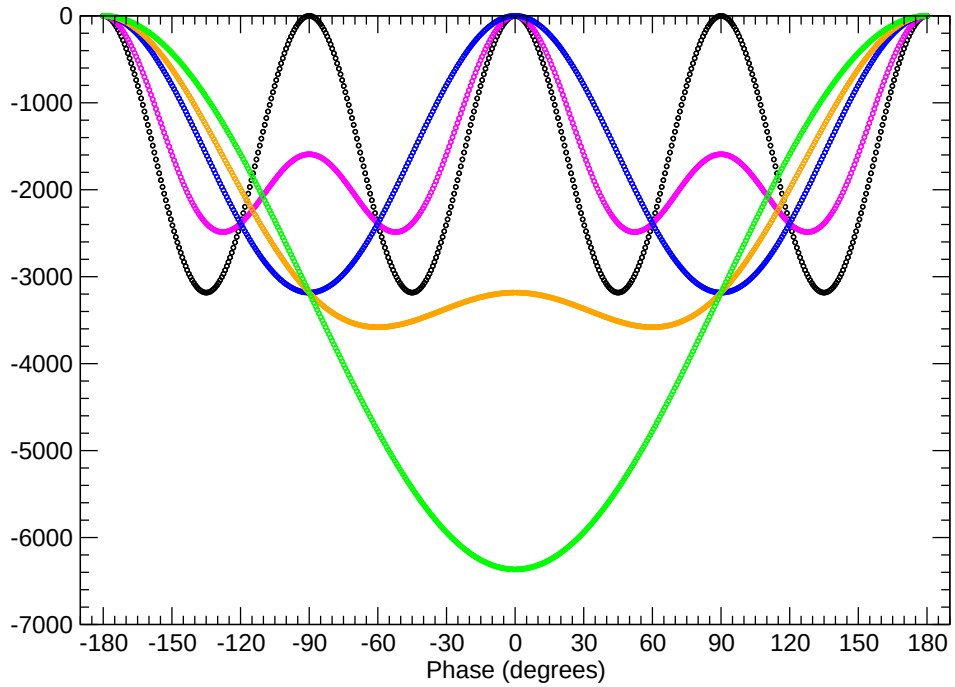

Figure 3: Plots of the "potential" $U(\phi_u) - U(\phi)$ at various times during the standard 4 to 1 merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 10. The black, magenta, blue, orange, and green curves are $U(\phi_u) - U(\phi)$ at times 0, 40, 80, 120, and 160 ms respectively. The local minima of the curves correspond to stable fixed points. The local maxima correspond to unstable fixed points.

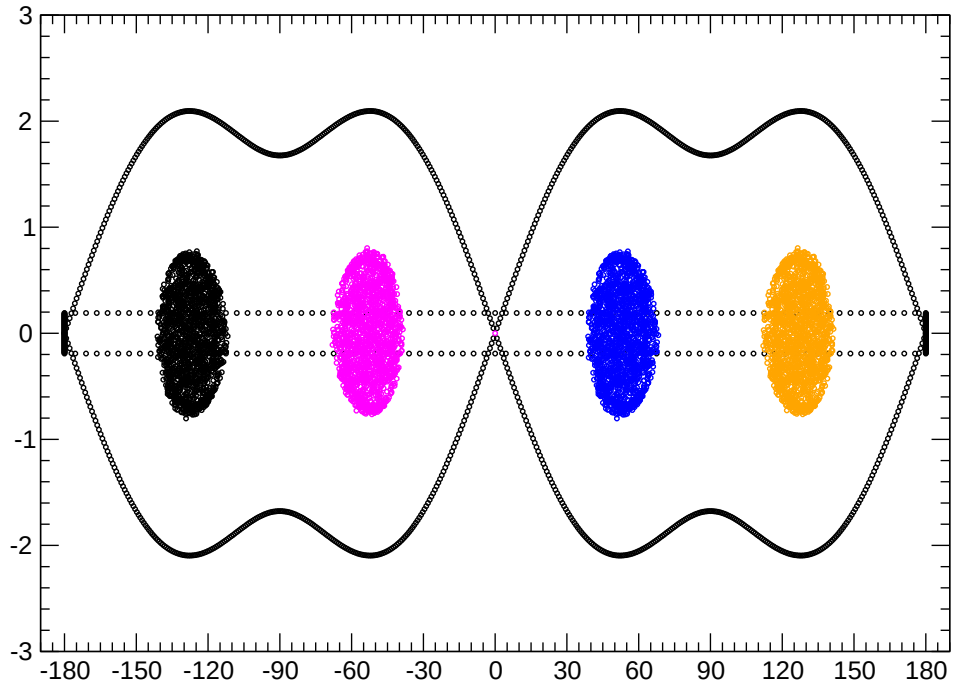

Figure 4: Halfway through the 4 to 2 merge of the bunches in **Figure 1**. This is at time 40 ms in **Figure 2** and corresponds to the magenta curve in **Figure 3**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

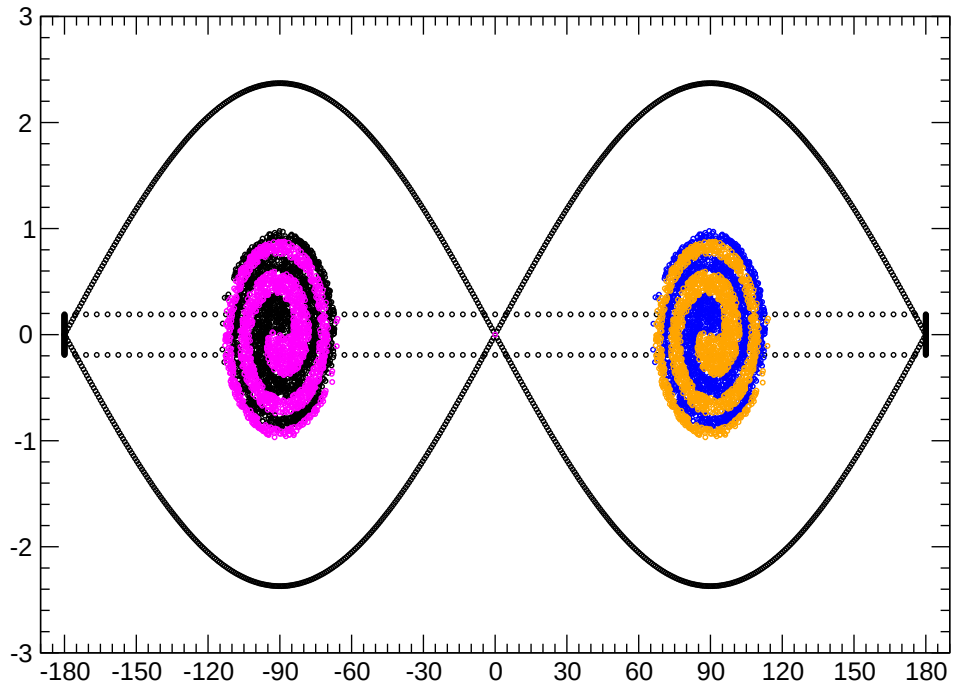

Figure 5: Completion of the 4 to 2 merge. This is at time 80 ms in **Figure 2** and corresponds to the blue curve in **Figure 3**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

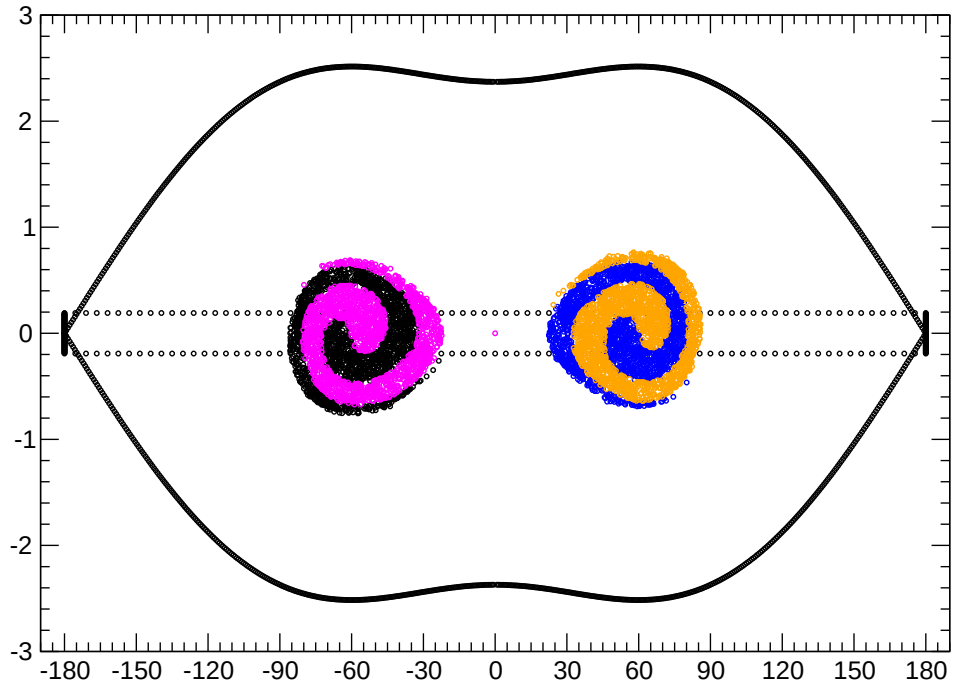

Figure 6: Halfway through the 2 to 1 merge of the bunches in **Figure 5**. This is at time 120 ms in **Figure 2** and corresponds to the orange curve in **Figure 3**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

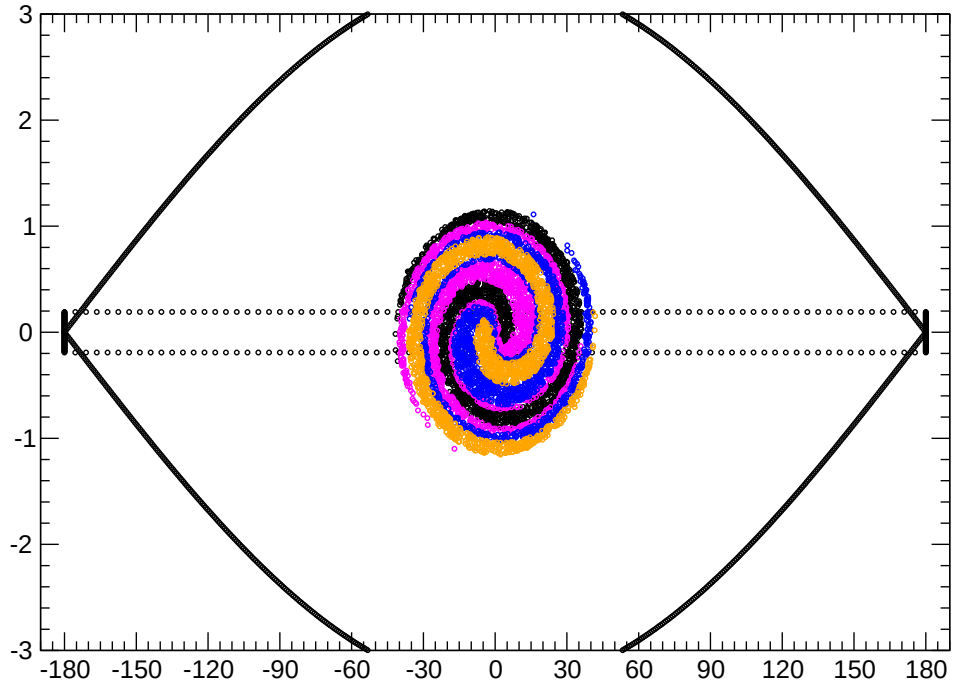

Figure 7: Completion of the 2 to 1 merge. This is at time 160 ms in **Figure 2** and corresponds to the green curve in **Figure 3**. One can see by inspection (of this and the previous figures) that the emittance of the bunch is close to that (0.80 eV s per nucleon) of the initial distribution of unbunched beam. The bunch is easily held in the harmonic 2 bucket. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

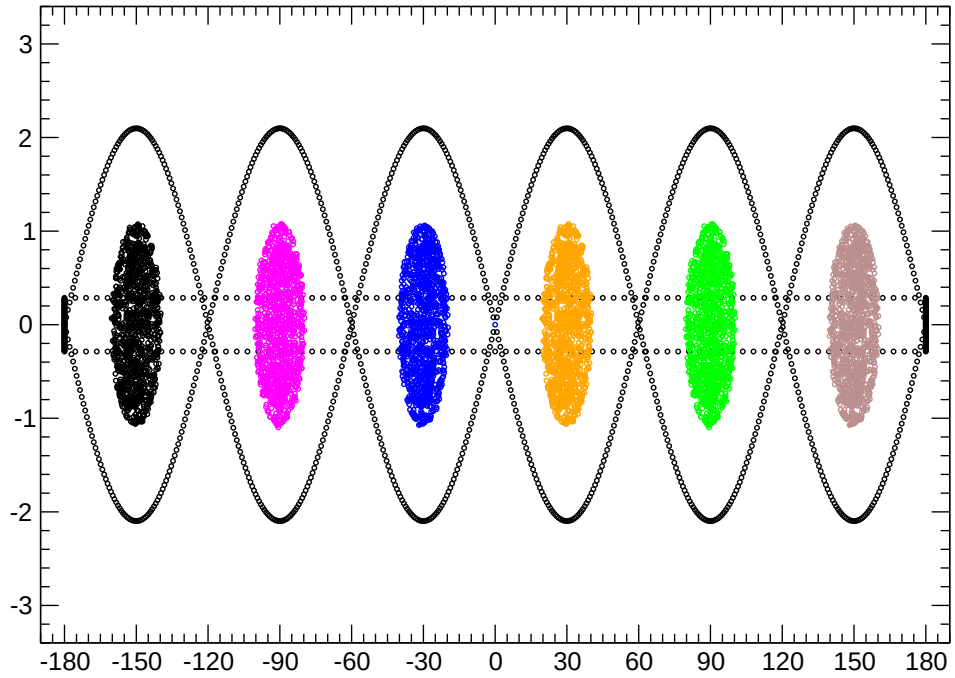

Figure 8: Six bunches of helions in harmonic 12 buckets on the AGS injection porch. The harmonic 12 frequency is $12f_s = 3.765$ MHz. The voltage is $V_C = 47$ kV. The black rectangle is the border of the uniform distribution used to make the bunches. The longitudinal emittance of the distribution is 1.20 eV s per nucleon. One can see by inspection that the emittance of the 6 bunches is close to that of the initial distribution. This gives an emittance of 0.2 eV s per nucleon per bunch. In practice the two outer buckets are left unoccupied to provide time for the injection kicker pulse. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

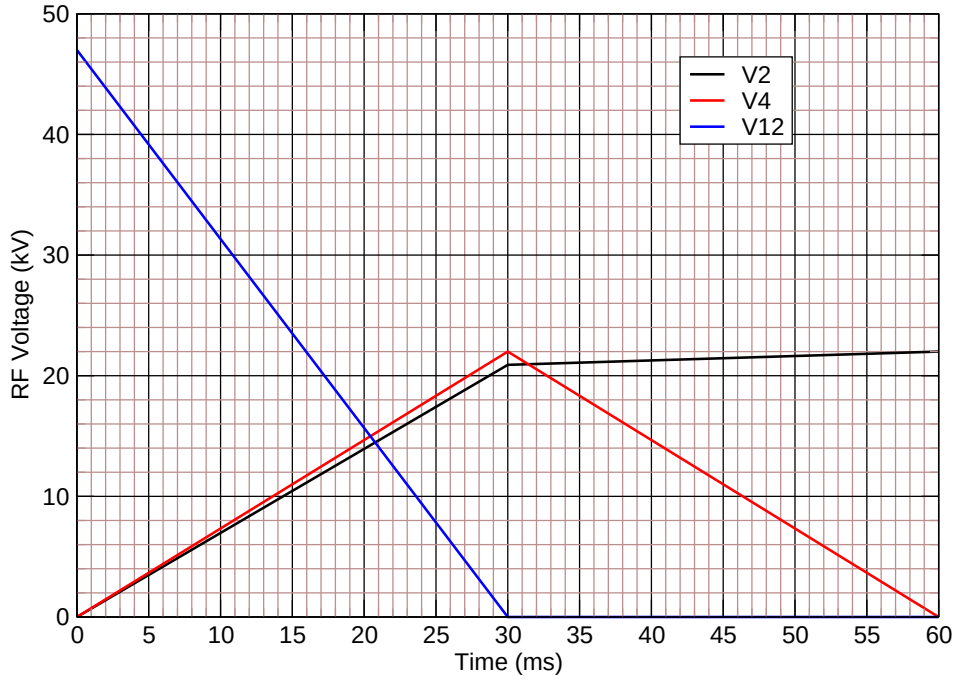

Figure 9: RF voltages for the alternative 4 to 1 merge. Here the black, red, and blue curves are the harmonic $2 = 1h$, $4 = 2h$, and $12 = 6h$ voltages respectively. These are given by the formulae of Section 11 with h set equal to 2. Here the harmonic 12, 4, and 2 voltages are $V_C = 47$ kV, $V_K = 22$ kV, and $V_L = 22$ kV respectively. The times T_K and T_L are 30 and 60 ms respectively. The parameter $\mathcal{F}$ has been set equal to 0.95. This setting keeps the inner 4 bunches separated from the outer 2 during the merging process and minimizes the gross emittance of the final merged bunch.

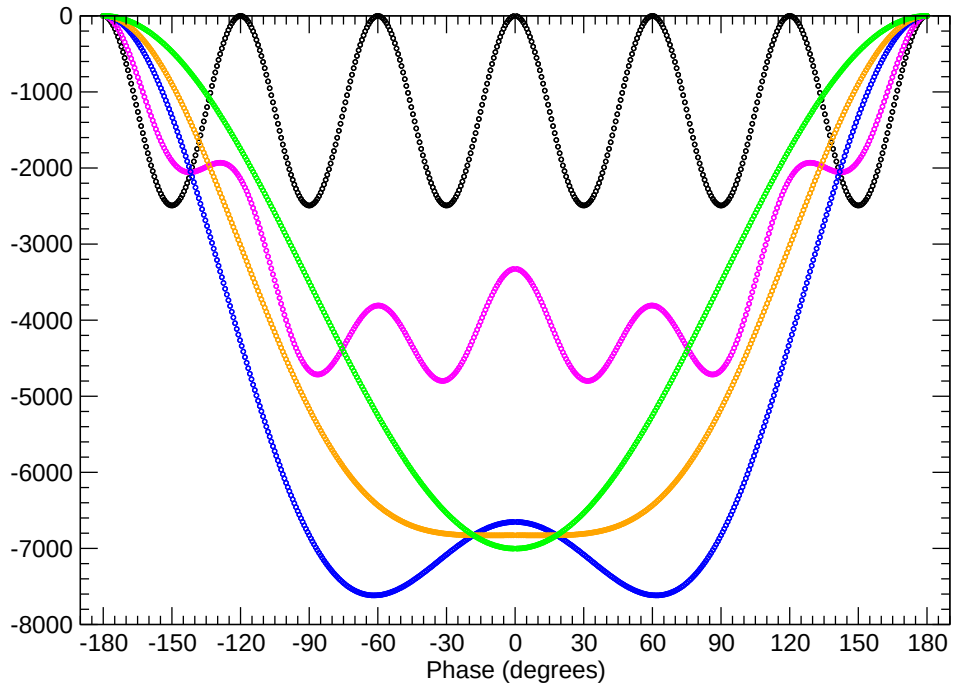

Figure 10: Plots of the "potential" $U(\phi_u) - U(\phi)$ at various times during the alternate 4 to 1 merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 11. The black, magenta, blue, orange, and green curves are $U(\phi_u) - U(\phi)$ at times 0, 15, 30, 45, and 60 ms respectively. The local minima of the curves correspond to stable fixed points. The local maxima correspond to unstable fixed points.

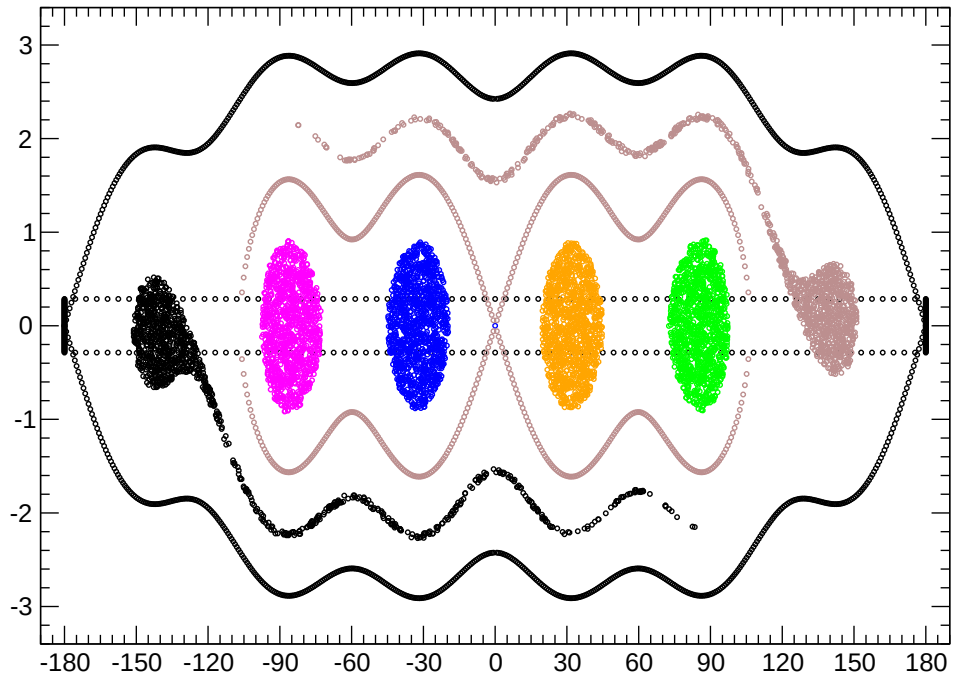

Figure 11: Halfway through the 4 to 2 merge of the bunches in **Figure 8**. This is at time 15 ms in **Figure 9** and corresponds to the magenta curve in **Figure 10**. Note that the brown separatrix keeps the inner 4 bunches separated from the outer 2. The two outer bunches (which are not present in practice) have started to form a halo around the inner 4. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

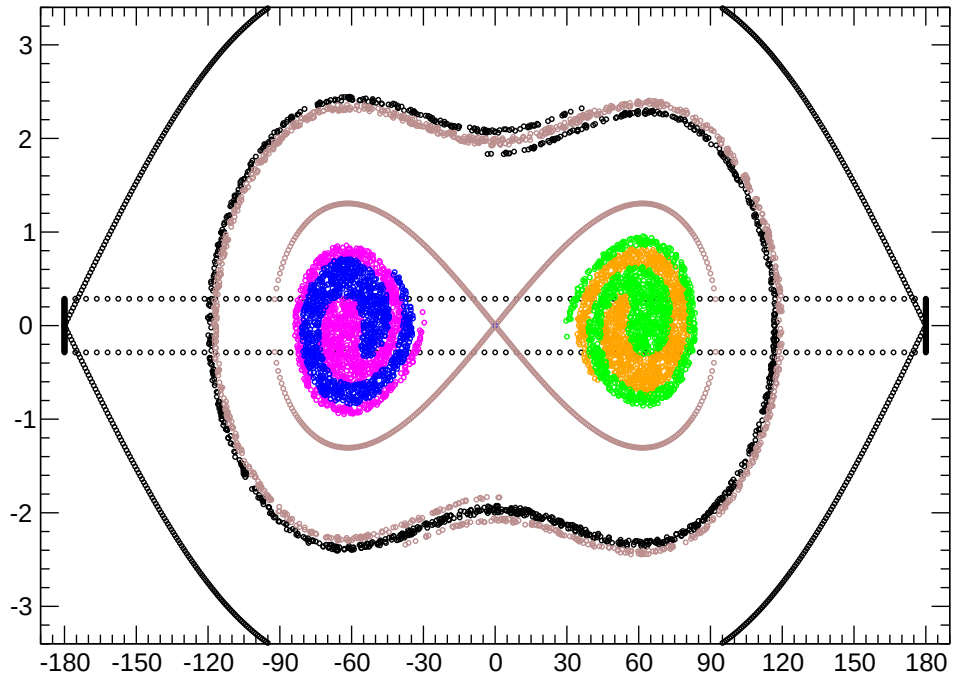

Figure 12: Completion of the 4 to 2 merge. This is at time 30 ms in **Figure 9** and corresponds to the blue curve in **Figure 10**. Note again that the brown separatrix keeps the inner 4 bunches separated from the outer 2. The two outer bunches (which are not present in practice) have formed a halo around the inner 4. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

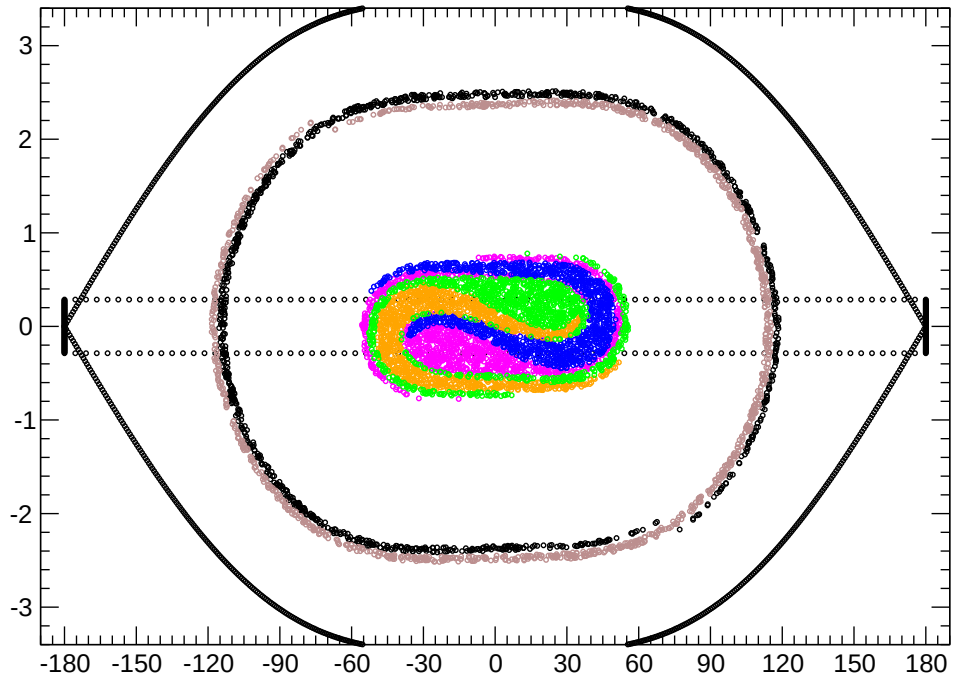

Figure 13: Halfway through the 2 to 1 merge of the bunches in **Figure 12**. This is at time 45 ms in **Figure 9** and corresponds to the orange curve in **Figure 10**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

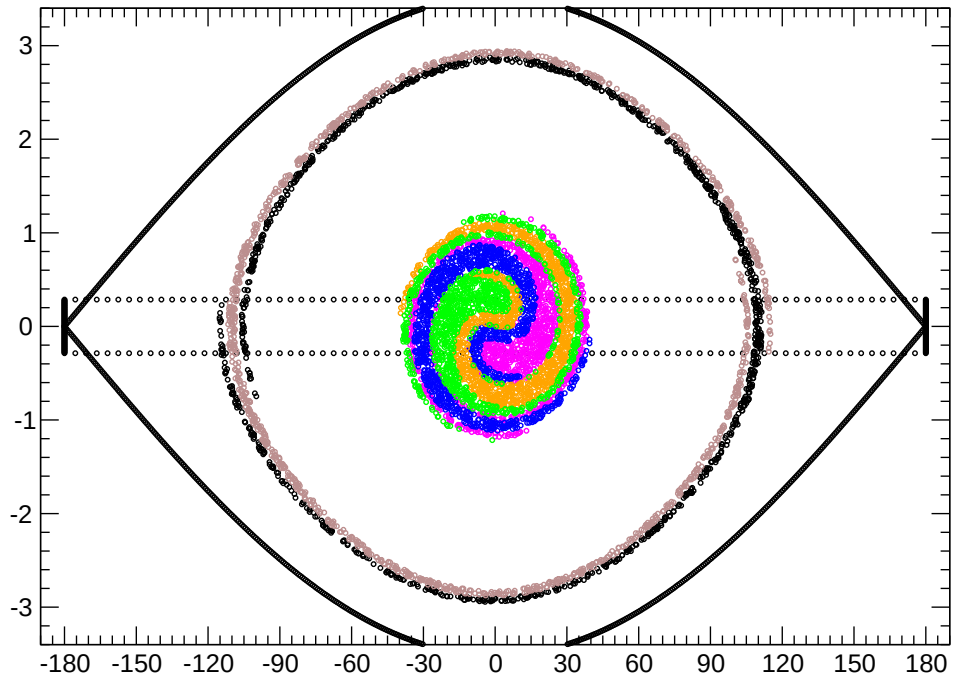

Figure 14: Completion of the 2 to 1 merge. This is at time 60 ms in **Figure 9** and corresponds to the green curve in **Figure 10**. One can see by inspection (of this and the previous figures) that the emittance of the merged bunch is close to that (0.80 eV s per nucleon) of the 4 inner bunches of the initial distribution of bunched beam. The bunch is easily held in the harmonic 2 bucket. (The halo, formed by the 2 outer bunches of the initial distribution of bunched beam, is not present in practice.) The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s. Comparing with **Figure 7** one sees that alternative 4 to 1 merge produces a merged bunch of the same size and quality as the standard 4 to 1 merge.

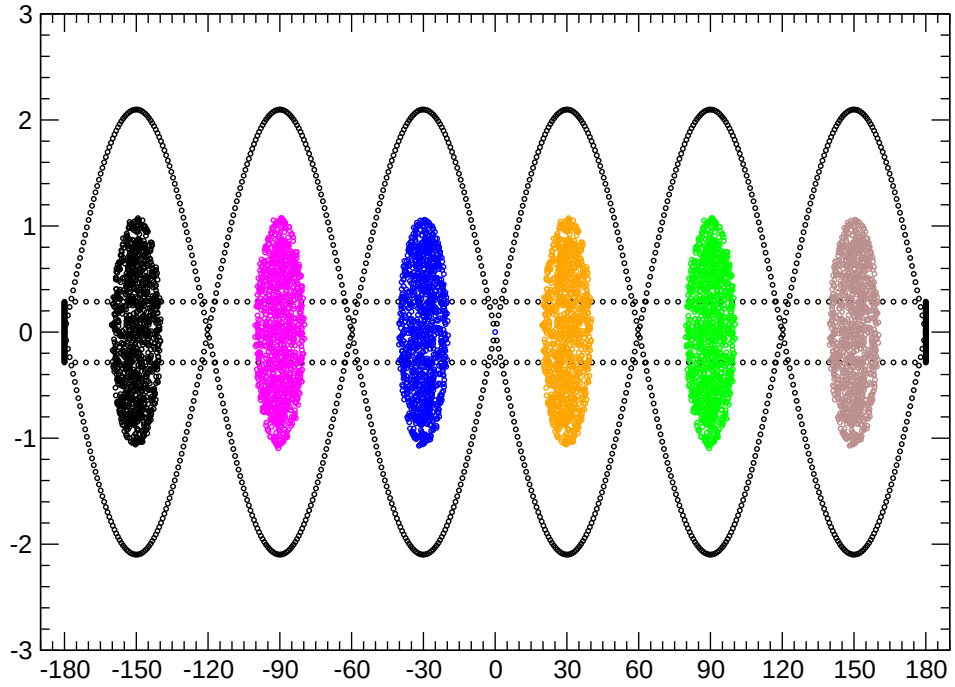

Figure 15: Six bunches of helions in harmonic 12 buckets on the AGS injection porch. The harmonic 12 frequency is $12f_s = 3.765$ MHz. The RF voltage is $V_C = 47$ kV. The black rectangle is the border of the uniform distribution used to make the bunches. The longitudinal emittance of the distribution is 1.20 eV s per nucleon. One can see by inspection that the emittance of the 6 bunches is close to that of the initial distribution. This gives an emittance of 0.2 eV s per nucleon per bunch. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

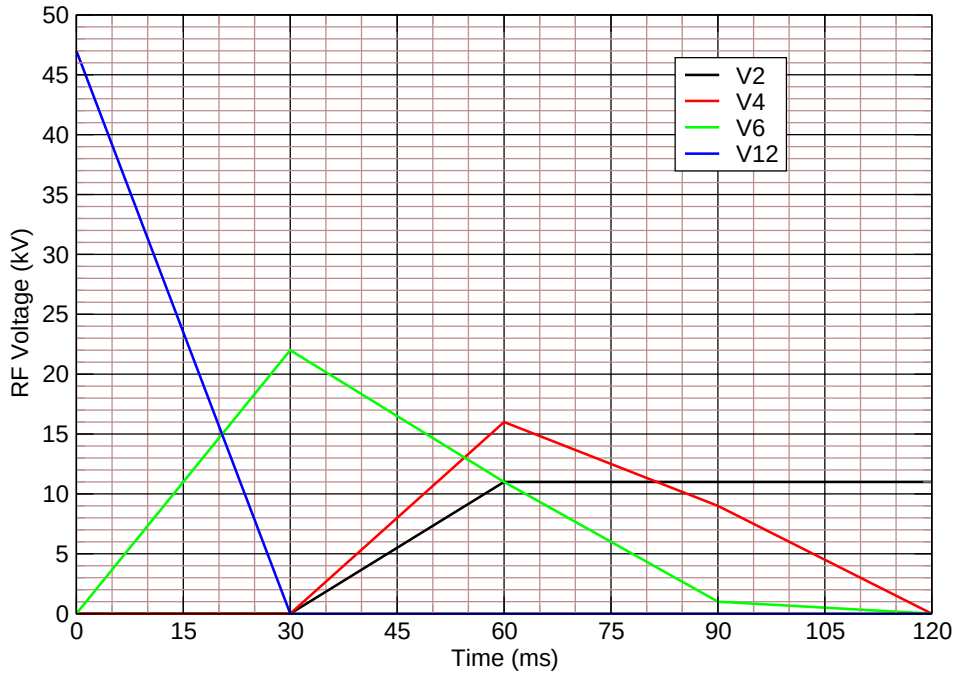

Figure 16: RF voltages for the 6 to 1 merge. Here the black, red, green, and blue curves are the harmonic $2 = 1h$, $4 = 2h$, $6 = 3h$, and $12 = 6h$ voltages respectively. These are given by the formulae of Section 12 with h set equal to 2. Here the harmonic 12 voltage $V_C = 47$ kV. The harmonic 6 voltages $V_M = 22$ kV, $V_{M1} = 11$ kV, and $V_{M2} = 1.0$ kV. The harmonic 4 voltages $V_{K1} = 16$ kV and $V_{K2} = 9$ kV. The harmonic 2 voltage $V_{L1} = 11$ kV. The times T_K, T_{L1}, T_{L2}, and T_F are 30, 60, 90, and 120 ms respectively. The 6 to 3 merge takes place over the time interval $0 \leq T \leq 30$ ms. The 3 to 1 merge takes place over the interval $30 \leq T \leq 120$ ms.

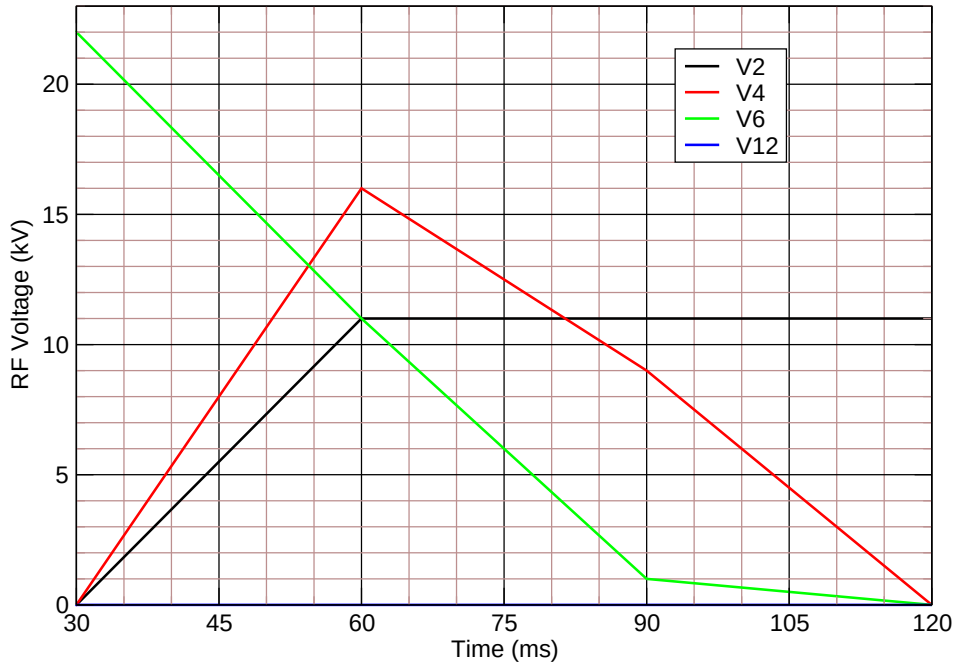

Figure 17: Magnified view of **Figure 16** showing the RF voltages for the 3 to 1 portion of the 6 to 1 merge.

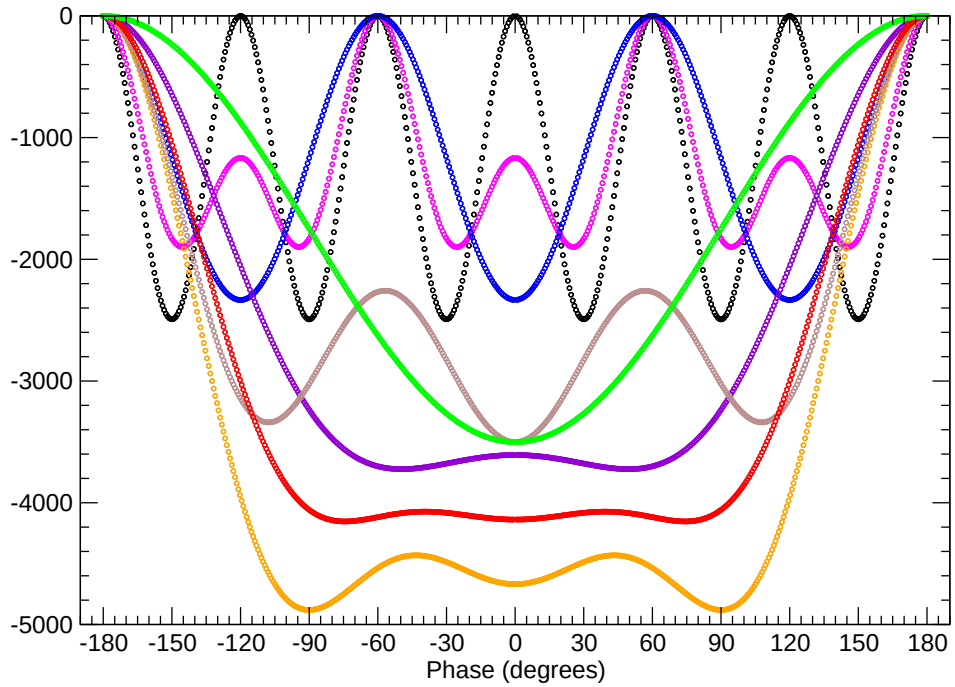

Figure 18: Plots of the "potential" $U(\phi_u) - U(\phi)$ at various times during the 6 to 1 merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 12. The black, magenta, blue, brown, orange, red, violet, and green curves are $U(\phi_u) - U(\phi)$ at times 0, 15, 30, 45, 60, 75, 90, and 120 ms respectively. The local minima of the curves correspond to stable fixed points. The local maxima correspond to unstable fixed points.

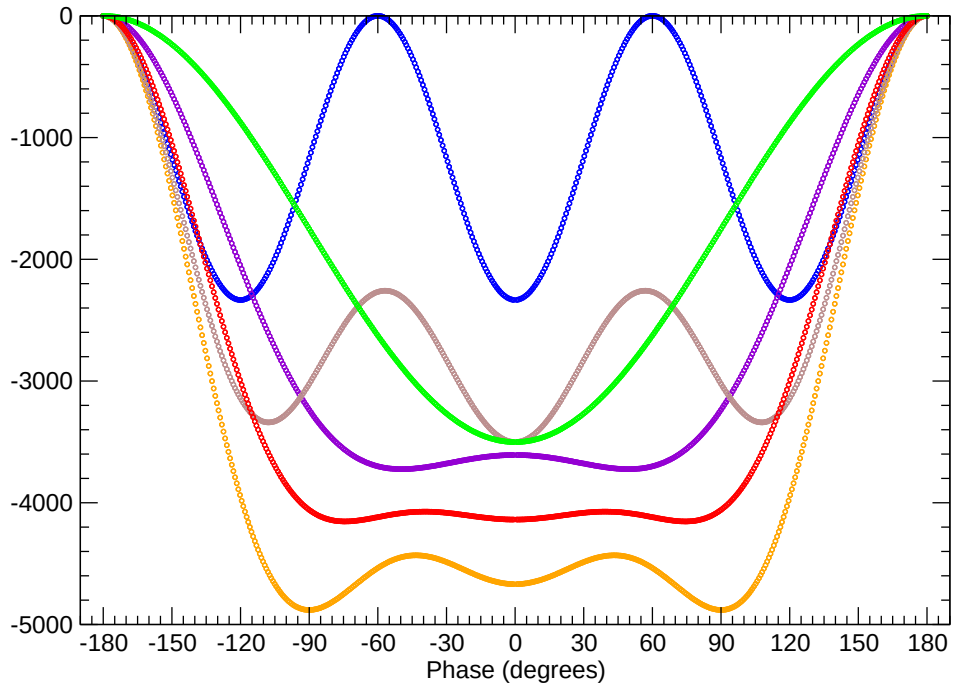

Figure 19: Plots of the "potential" $U(\phi_u) - U(\phi)$ at various times during the 3 to 1 portion of 6 to 1 merge. Here $U(\phi_u)$ and $U(\phi)$ are defined in Sections 8 and 12. The blue, brown, orange, red, violet, and green curves are $U(\phi_u) - U(\phi)$ at times 30, 45, 60, 75, 90, and 120 ms respectively. The local minima of the curves correspond to stable fixed points. The local maxima correspond to unstable fixed points.

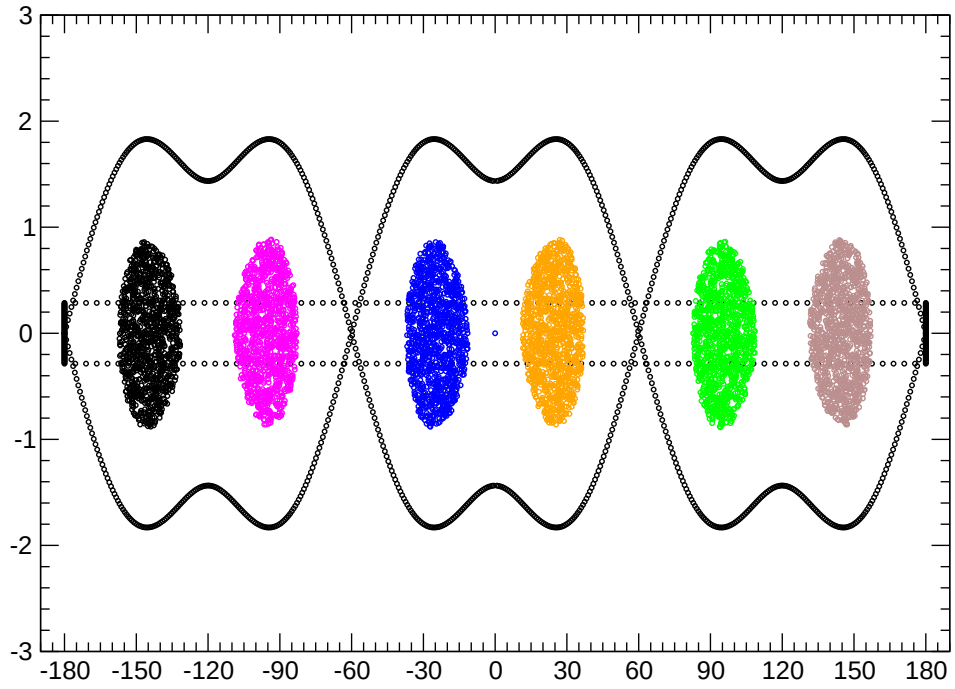

Figure 20: Halfway through the 6 to 3 merge of the bunches in **Figure 15**. This is at time 15 ms in **Figure 16** and corresponds to the magenta curve in **Figure 18**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

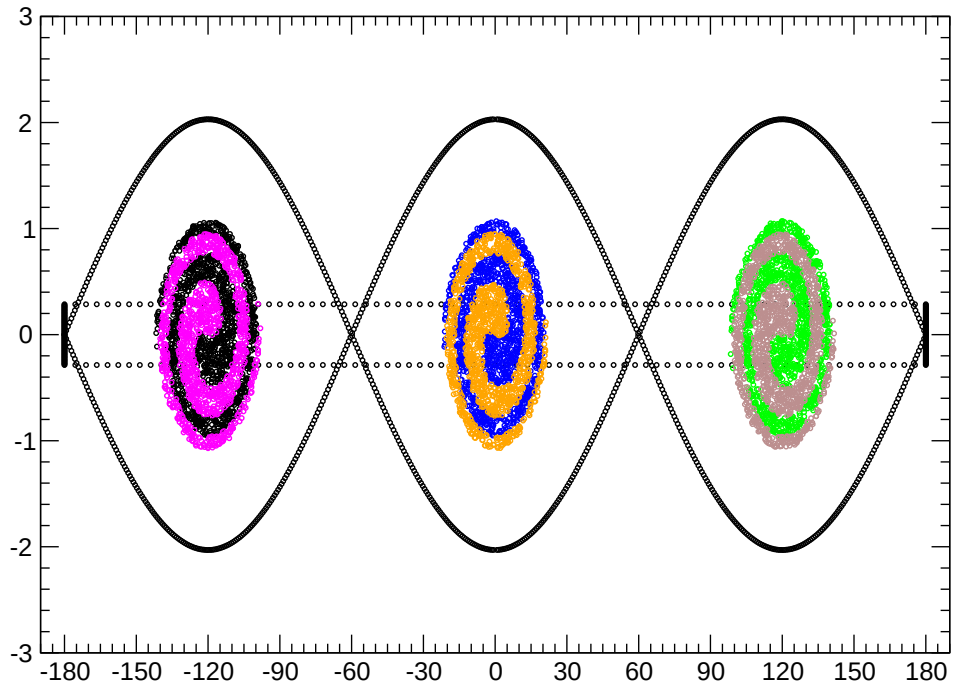

Figure 21: Completion of the 6 to 3 merge. This is at time 30 ms in **Figure 16** and corresponds to the blue curve in **Figure 18**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

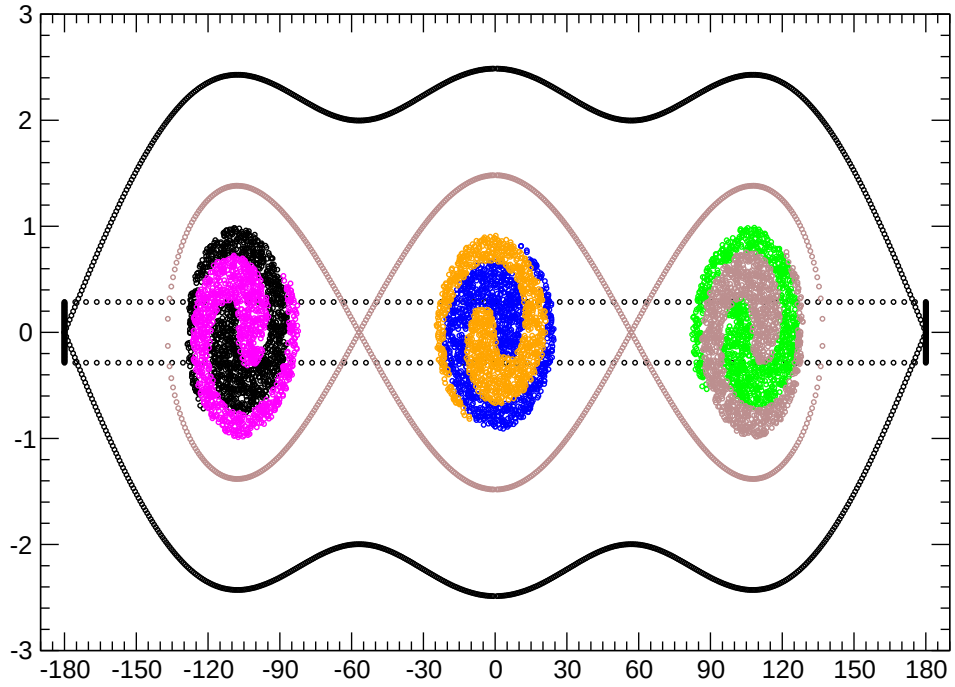

Figure 22: One sixth of the way through the 3 to 1 merge of the bunches in **Figure 21**. This is at time 45 ms in **Figure 16** and corresponds to the brown curve in **Figure 18**. The bunches are kept separated by the brown separatrix. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

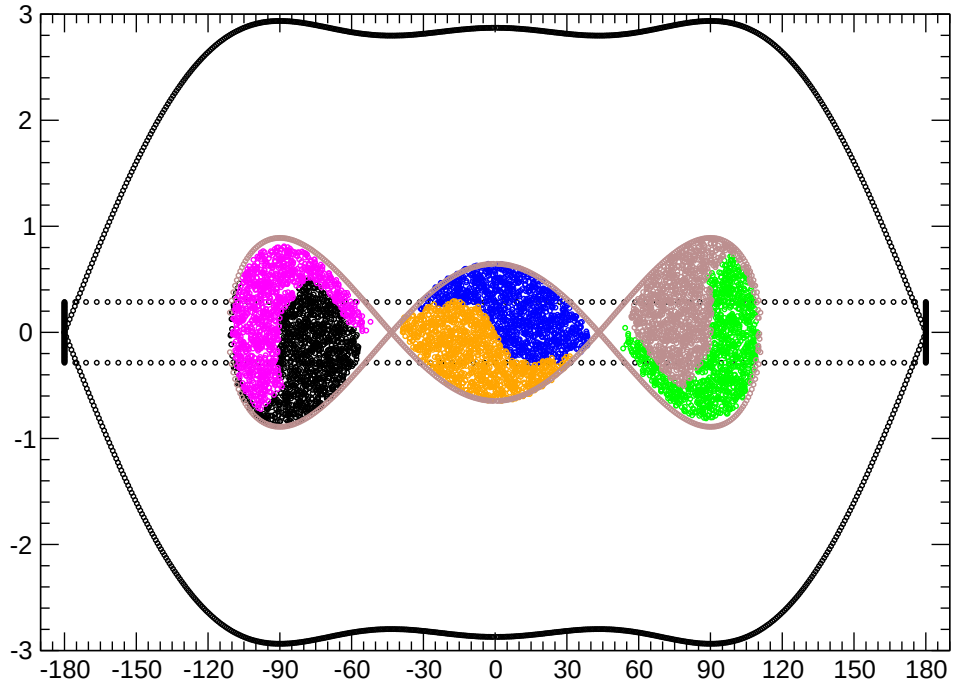

Figure 23: One third of the way through the 3 to 1 merge of the bunches in **Figure 21**. This is at time 60 ms in **Figure 16** and corresponds to the orange curve in **Figure 18**. The bunches are nearly touching but are still in separate buckets. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

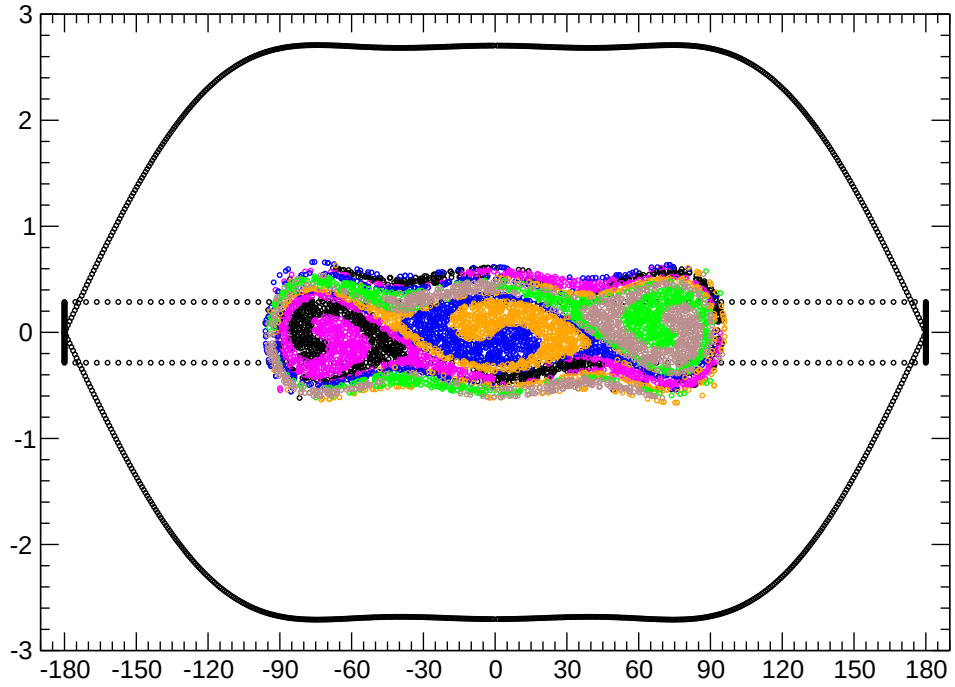

Figure 24: Halfway through the 3 to 1 merge of the bunches in **Figure 21**. This is at time 75 ms in **Figure 16** and corresponds to the red curve in **Figure 18**. Here the bunches have started to mix with one another. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

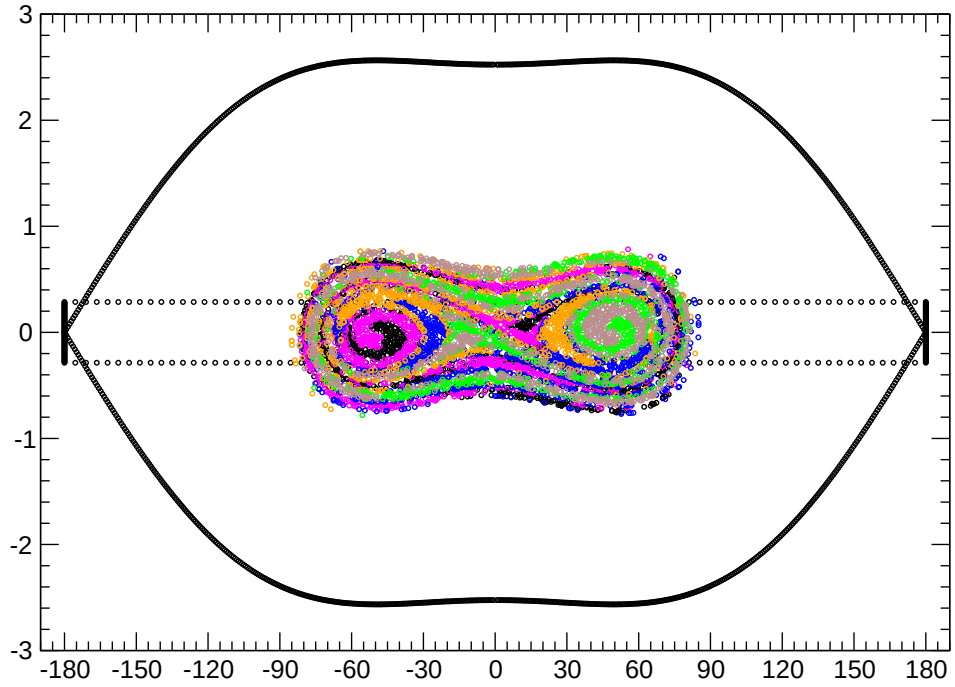

Figure 25: Two thirds of the way through the 3 to 1 merge of the bunches in **Figure 21**. This is at time 90 ms in **Figure 16** and corresponds to the violet curve in **Figure 18**. The bunches continue to mix and merge. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

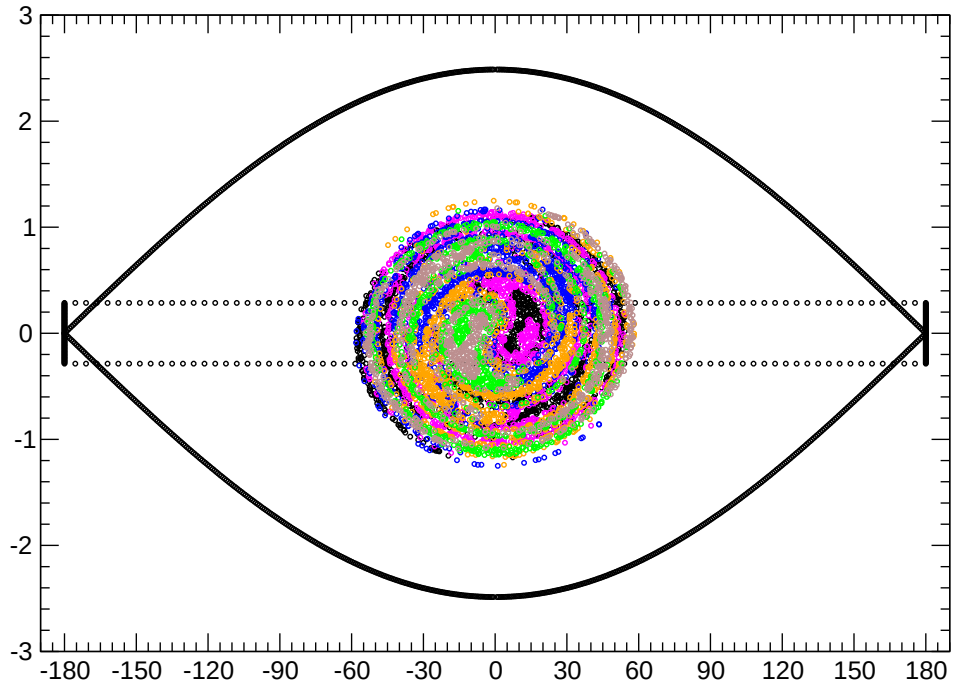

Figure 26: Completion of the 3 to 1 merge of the bunches in **Figure 21**. This is at time 120 ms in **Figure 16** and corresponds to the green curve in **Figure 18**. The horizontal axis is the phase ϕ in degrees. The vertical axis is W in units of eV s.

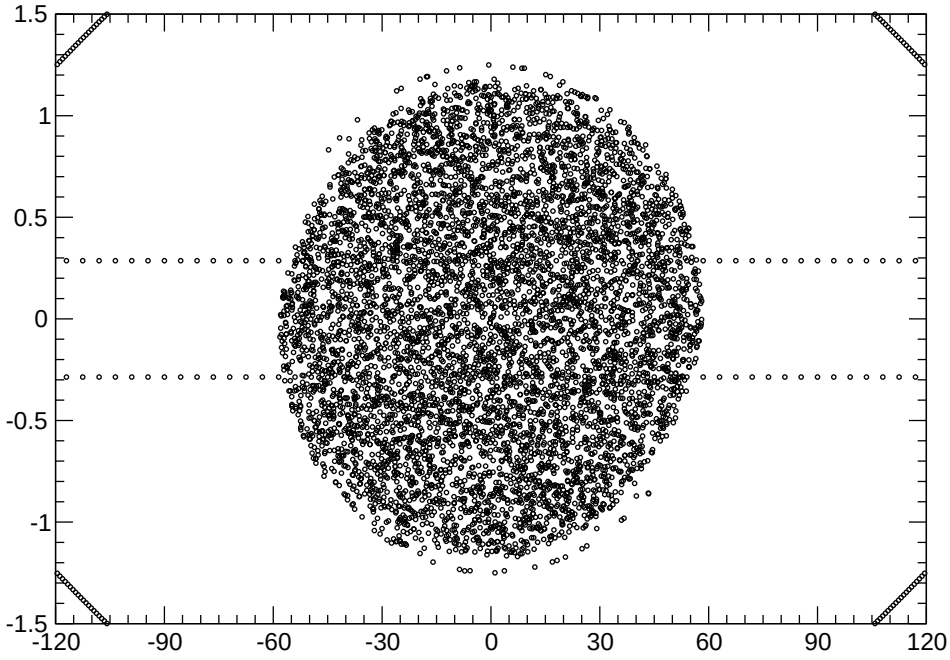

Figure 27: Magnified view of **Figure 26** with color suppressed, showing a tightly merged bunch. This is the final result of the 6 to 1 merge.

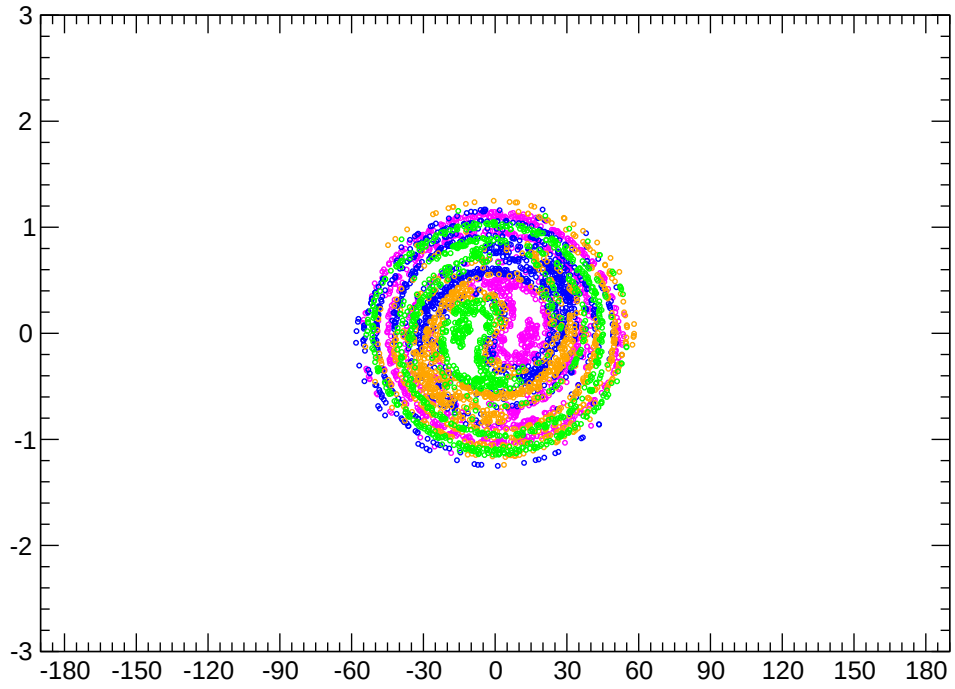

Figure 28: If the 2 outer buckets in **Figure 15** are left empty, then the 6 to 1 merge becomes another alternative scheme for merging 4 bunches into 1. In this case, however, the phase space of the 2 outer buckets mixes with the 4 inner bunches and dilutes the gross emittance. One ends up with the bunch shown above. (This was obtained by simply turning off the black and brown points in **Figure 26**.) The gross emittance of the bunch is 6/4 times larger than that obtained with the alternative 4 to 1 scheme of Section 14.

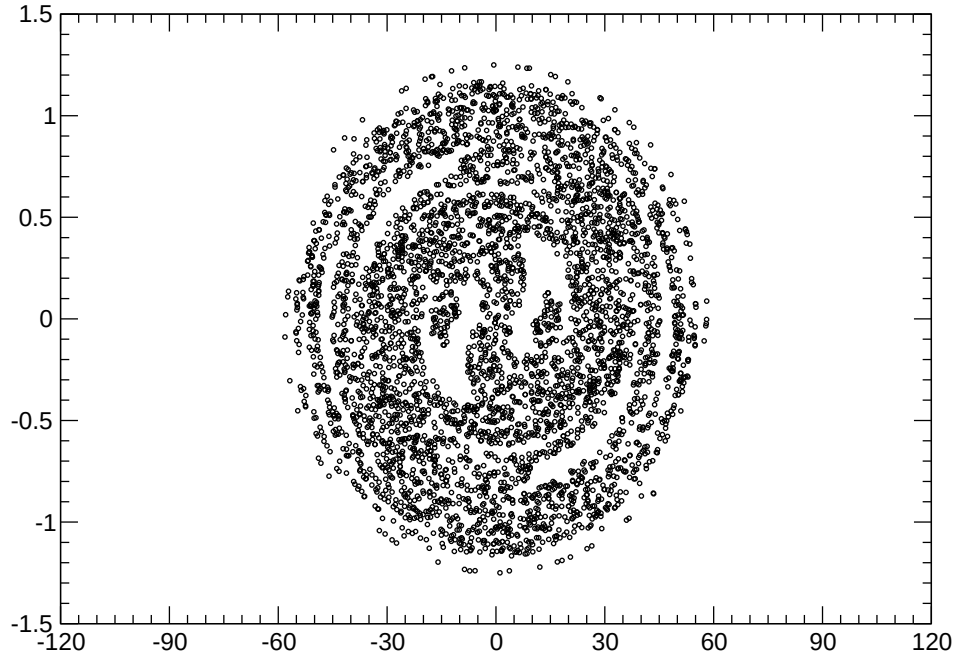

Figure 29: Magnified view of **Figure 28** with color suppressed, showing the dilution of the gross emittance.