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Determinations of V_{us} using inclusive hadronic τ decay data

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Two methods for determining $|V_{us}|$ employing inclusive hadronic τ decay data are discussed. The first is the conventional flavor-breaking sum-rule determination whose usual implementation produces results $\sim 3\sigma$ low compared to three-family unitary expectations. The second is a novel approach combining experimental strange hadronic τ distributions with lattice light-strange current-current two-point function data. Preliminary explorations of the latter show the method promises $|V_{us}|$ determinations competitive with those from $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$. For the former, systematic issues in the conventional implementation are investigated. Unphysical dependences of $|V_{us}|$ on the choice of sum rule weight, w , and upper limit, s_0 , of the weighted experimental spectral integrals are observed, the source of these problems identified and a new implementation which overcomes these problems developed. Lattice results are shown to provide a tool for quantitatively assessing truncation uncertainties for the slowly converging $D = 2$ OPE series. The results for $|V_{us}|$ from this new implementation are shown to be free of unphysical w - and s_0 -dependences, and ~ 0.0020 higher than those produced by the conventional implementation. With preliminary new $K\pi$ branching fraction results as input, we find $|V_{us}|$ in excellent agreement with that obtained from $K_{\ell 3}$, and compatible within errors with expectations from three-family unitarity.

Keywords: CKM; sum rules; τ decay.

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1. Introduction

Three-family unitarity, together with the superallowed $0^+ \rightarrow 0^+$ nuclear β decay result $|V_{ud}| = 0.97417(21)$ ¹, leads to the expectation $|V_{us}| = 0.2258(9)$. This agrees, within errors, with both (1) the recent $K_{\ell 3}$ determinations, $0.2235(4)_{exp}(9)_{th}$ ², $0.2231(4)_{exp}(7)_{th}$ ³ and $0.2230(4)_{exp}(11)_{th}$ ⁴, obtained using updated 2014 FlaviaNet data⁵ and recent $n_f = 2+1$ ² and $n_f = 2+1+1$ ^{3,4} lattice input for $f_+(0)$, and (2) the recent $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ determinations, $0.2251(3)_{exp}(9)_{th}$ ⁶ and $0.2250(3)_{exp}(7)_{th}$ ⁷, obtained using updated 2014 FlaviaNet experimental input⁵ and recent $n_f = 2+1$ ⁶ and $n_f = 2+1+1$ ⁷ lattice determinations of f_K/f_π .

An alternate approach to determining $|V_{us}|$ is provided by flavor-breaking (FB) finite-energy sum rule (FESR) analyses employing experimental non-strange and strange hadronic τ decay distributions as input^{8,9}. A long-standing problem is the low values of $|V_{us}|$ produced by the conventional implementation of this approach⁸, the most recent such determination, $|V_{us}| = 0.2176(21)$ ¹⁰, e.g., lying 3.6σ below three-family unitarity expectations. Generalized versions of this analysis, employing alternate weight choices, w , and variable upper endpoints, $s_0 \leq m_\tau^2$, of the relevant weighted spectral integrals, moreover, produce results for $|V_{us}|$ displaying unphysical w - and s_0 -dependences⁹.

In this paper, we investigate the systematics of the FB FESR approach, and propose an alternate implementation which cures the observed systematic problems. We also introduce a new approach to determining $|V_{us}|$ which relies only on the inclusive strange hadronic τ decay distribution and in which lattice, rather than OPE, results are employed on the theory side of the relevant sum rules.

The rest of the paper is organized as follows. In the next section the conventional implementation and systematic problems of the FB FESR approach are briefly reviewed, the source of the systematic problems identified, and a new implementation of the FB FESR approach proposed and implemented. In the subsequent section, the new approach combining lattice and inclusive strange τ decay data is outlined and some preliminary results on the expected accuracy of this approach are presented. The final section provides a brief discussion of future prospects.

2. The FB FESR Approach to $|V_{us}|$

In the Standard Model (SM), the differential distribution, $dR_{V/A;ij}/ds$, with $ij = ud, us$ the flavor of the hadronic vector (V) or axial vector (A) current of interest, and $R_{V/A;ij} \equiv \Gamma[\tau^- \rightarrow \nu_\tau \text{ hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma)]$, is related to the spectral functions, $\rho_{V/A;ij}^{(J)}$, of the spin $J = 0, 1$ polarizations, $\Pi_{V/A;ij}^{(J)}$, associated with the flavor ij V or A current-current two-point function, by¹¹

$$\frac{dR_{V/A;ij}}{ds} = c_\tau^{EW} |V_{ij}|^2 \left[w_\tau(s) \rho_{V/A;ij}^{(0+1)}(y_\tau) - w_L(y_\tau) \rho_{V/A;ij}^{(0)}(s) \right] \quad (1)$$

where $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, c_τ^{EW} is a known constant, and V_{ij} is the flavor ij CKM matrix element. With f_π and f_K (which de-

termine the dominant, non-chirally-suppressed $J = 0$ contributions in this decomposition) accurately known, and highly constrained, mildly model-dependent input for the small, doubly-chirally-suppressed continuum contributions to $\rho_{V/A;us}^{(0)}(s)$ ^{12,13}, results for $dR_{V/A;ud,us}/ds$ provide experimental determinations of $\rho_{V/A;ud,us}^{(0+1)}(s)$.

The conventional inclusive FB τ decay determination of $|V_{us}|$ is based on FESRs involving the FB polarization difference, $\Delta\Pi_\tau \equiv [\Pi_{V+A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)}]$ and its associated spectral function $\Delta\rho_\tau$. Explicitly,

$$\int_0^{s_0} w(s) \Delta\rho_\tau(s) ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s) \Delta\Pi_\tau(s) ds, \quad (2)$$

a relation valid for any s_0 and any analytic $w(s)$. The LHS is to be evaluated using experimental data, the RHS (for sufficiently large s_0) using the OPE.

Once the $J = 0$ contributions have been subtracted from $dR_{V/A;ij}/ds$, as indicated above, producing a $J = 0 + 1$ analogue $dR_{V/A;ij}^{(0+1)}/ds$, re-weighted versions

$$R_{V+A;ij}^w(s_0) \equiv \int_0^{s_0} ds \frac{w(s)}{w_\tau(s)} \frac{dR_{V+A;ij}^{(0+1)}(s)}{ds} \quad (3)$$

of $R_{V+A;ij}$ are constructable for any w and any $s_0 \leq m_\tau^2$. Forming the FB difference

$$\delta R_{V+A}^w(s_0) \equiv \frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \frac{R_{V+A;us}^w(s_0)}{|V_{us}|^2}, \quad (4)$$

and using the FESR relation Eq. (2) to replace the LHS with its OPE representation, one may solve for $|V_{us}|$, obtaining

$$|V_{us}| = \sqrt{R_{V+A;us}^w(s_0) / \left[\frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \delta R_{V+A}^{w,OPE}(s_0) \right]}, \quad (5)$$

where $|V_{ud}|$ is to be taken as external input. Provided any assumptions used in evaluating $\delta R_{V+A}^{w,OPE}(s_0)$ are reliable and there are no problems with the experimental data employed, the various terms on the RHS of Eq. (5) should combine to produce a $|V_{us}|$ independent of s_0 and the choice of weight, w . Performing the analysis for a range of different s_0 , as well as for different w , is thus an important way of testing for possible systematic problems. The s_0 -dependence test is particularly useful for investigating assumptions regarding higher D OPE contributions, since integrated $D = 2k + 2$ OPE contributions scale as $1/s_0^k$, and problems with such assumptions will thus lead to an unphysical s_0 -dependence of $|V_{us}|$.

The conventional implementation of Eq. (5)⁸ employs $w = w_\tau$ and $s_0 = m_\tau^2$, and hence has available only one piece of experimental information. Since w_τ has degree 3, OPE contributions up to dimension $D = 8$, unsuppressed by additional powers of α_s , appear in $\delta R_{V+A}^{w,OPE}(s_0)$. The leading $D = 2$, and sub-leading $D = 4$, contributions are fixed by α_s , the light and strange quark masses and the light and strange quark condensates¹⁴, and hence can be taken as external input^{15,16,17}. Typically^{8,9}, $D = 6$ contributions have been estimated using the vacuum saturation

approximation (VSA) and $D = 8$ contributions neglected. There is, however, a very strong double cancellation in the $D = 6$ VSA estimate, first by a factor of ~ 3 in the individual ud and us V+A sums, and then by a further factor of ~ 6 in the subsequent FB ud - us V+A difference. The reliability of this treatment of $D = 6$ and 8 contributions is thus far from clear, especially since studies in the ud sector show the VSA to be extremely crude, and violated in a channel-dependent manner¹⁸.

The restricted choice, $s_0 = m_\tau^2$ and $w = w_\tau$, in the conventional implementation has the experimental advantage of allowing the spectral integrals $R_{V+A;ud,us}^{w_\tau}(m_\tau^2)$ to be determined using inclusive non-strange and strange hadronic τ branching fractions alone, without need for the differential distributions. This advantage, however, comes at a cost: fixing s_0 and w precludes one from testing the rather strong assumptions concerning $D = 6$ and 8 OPE contributions for self-consistency.

In order to test the conventional implementation, we consider FB FESRs for a range of $s_0 \leq m_\tau^2$, and involving, in addition to $w_\tau(y)$, the weights $w_N(y) = 1 - \left(\frac{N}{N-1}\right)y + \left(\frac{y^N}{N-1}\right)$, $N = 2, 3, 4$ and $\hat{w}(y) = (1-y)^3$ (where $y = s/s_0$). OPE input is as outlined above, spectral integral input as follows: π and K pole contributions from $\pi_{\mu 2}$, $K_{\mu 2}$ and SM expectations, recent ALEPH data for the continuum ud V+A distribution¹⁹, BaBar^{20,21} and Belle²² results for the $K^-\pi^0$ and $\bar{K}^0\pi^-$ distributions, BaBar results²³ for the $K^-\pi^+\pi^-$ distribution, Belle results²⁴ for the $\bar{K}^0\pi^-\pi^0$ distribution, a combination of BaBar²⁵ and Belle²⁶ results for the very small $\bar{K}\bar{K}K$ distribution and normalization, and 1999 ALEPH results²⁸ for the combined distribution of those strange modes not remeasured by the B-factory experiments^a.

The left panel, and solid lines in the right panel, of Fig. 1 show the results of these expanded tests of the conventional implementation. The significant s_0 - and w -dependence of the results for $|V_{us}|$ signals the existence of systematic problems, to be investigated further below. The dashed lines in the right panel show the results of an alternate implementation, also discussed in more detail below.

The weight dependence seen in the left panel of Fig. 1 is particularly illuminating since, with $w_\tau(y) = 1 - 3y^2 + 2y^3$ and $\hat{w}(y) = 1 - 3y + 3y^2 - y^3$, the integrated $D = 6$ OPE contributions to the w_τ and $\hat{w}$ FESRs are equal in magnitude but opposite in sign, while the $\hat{w}$ $D = 8$ contribution is $-1/2$ that of w_τ . If, as conventionally assumed, the $D = 8$ contribution is negligible and the very small VSA estimate

^aNote that, for the $K^-\pi^0$ mode, there are two possibilities for the choice of the branching fraction $B[\tau^- \rightarrow K^-\pi^0\nu_\tau]$ which normalizes the distribution, 0.00433(15) from the 2014 HFAG summer fit²⁷, and the preliminary BaBar thesis result 0.00500(14)²¹. While the difference might appear small, the preliminary BaBar result raises the total strange branching fraction by $\sim 2.4\%$ and hence increases the $|V_{us}|$ obtained from the conventional implementation of Eq. (5) by $\sim 1.2\%$ or ~ 0.0026 . In what follows we employ the preliminary BaBar result to normalize the $K^-\pi^0$ distribution since it leads to a sum of $K^-\pi^0$ and $\bar{K}^0\pi^-$ branching fractions in agreement with the expectations of the dispersive analysis of Ref. 29.

Fig. 1. Left panel: $|V_{us}|$ from the w_τ and $\hat{w}$ FESRs with conventional OPE input (including CIPT for the $D = 2$ series). Right panel: Comparison of conventional implementation results (solid lines) with those obtained using central fitted $C_{6,8,10}$ values and the FOPT $D = 2$ prescription favored by lattice results, for the weights $w_{2,3,4}$ defined in the text (dashed lines).

for the $D = 6$ contribution is reliable for the w_τ FESR, these assumptions should also be equally valid for the $\hat{w}$ FESR. In that case, the two FESRs should produce compatible values of $|V_{us}|$, both showing good s_0 -stability. In contrast, if one or both of these assumptions is/are not reliable, one should see s_0 -instabilities of opposite sign in the $|V_{us}|$ results obtained from the w_τ and $\hat{w}$ FESRs. It is obviously the latter scenario which is realized, strongly suggesting the treatment of $D = 6$ and 8 contributions in the conventional implementation is unreliable.

Another potential problem for the FB FESR approach is the slow $D = 2$ OPE series convergence. To four loops, neglecting $O(m_{u,d}^2/m_s^2)$ corrections, one has ¹⁴

$$[\Delta\Pi_\tau(Q^2)]_{D=2}^{OPE} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[1 + \frac{7}{3}\bar{a} + 19.93\bar{a}^2 + 208.75\bar{a}^3 + \dots \right], \quad (6)$$

where $\bar{a} = \alpha_s(Q^2)/\pi$, and $m_s(Q^2)$ and $\alpha_s(Q^2)$ are the running strange quark mass and coupling in the $\overline{MS}$ scheme. With $\bar{a}(m_\tau^2) \simeq 0.1$, convergence at the spacelike point on $|s| = s_0$ is marginal at best, even for the largest s_0 accessible in τ decays.

To investigate further the possibility of non-negligible $D > 4$ OPE contributions and the impact of the slow convergence of the $D = 2$ OPE series, we compare lattice data for $\Delta\Pi_\tau(Q^2)$ over a range of Euclidean $Q^2 = -q^2 = -s$ with variously truncated versions of the OPE. For this study, we employ $\Delta\Pi_\tau(Q^2)$ results obtained using the RBC/UKQCD $n_f = 2+1$, $32^3 \times 64$, $1/a = 2.38$ GeV, domain wall fermion ensemble with $m_\pi \sim 300$ MeV ³⁰. A tight cylinder cut is employed to avoid lattice artifacts at higher Q^2 . The radius of this cut was determined in a recent study of the determination of α_s from lattice current-current two-point function data ³¹.

We first investigate the reliability of the $D = 2$ OPE series by comparing lattice and OPE results for Q^2 sufficiently large that $[\Delta\Pi_\tau]_{OPE}$ will be dominated by its $D = 2$ and $D = 4$ contributions. The $D = 2$ OPE contribution is determined using ensemble values of m_u and m_s ³⁰, the most recent central PDG value for α_s ¹⁶,

and considering two-, three- and four-loop truncation of the $D = 2$ series. We also consider either a fixed scale choice $\mu^2 = 4 \text{ GeV}^2$ (the analogue of the fixed order FOPT prescription for the FESR contour integrals) or the local scale choice $\mu^2 = Q^2$ (the analogue of the FESR contour improved CIPT prescription) in evaluating the truncated $D = 2$ series. For the $D = 4$ contribution, which is determined by the FB difference $\langle m_\ell \bar{\ell} \ell \rangle - \langle m_s \bar{s} s \rangle$, we employ GMOR for $\langle m_\ell \bar{\ell} \ell \rangle$ and fix $\langle m_s \bar{s} s \rangle$ using the HPQCD result for $\langle \bar{s} s \rangle / \langle \bar{\ell} \ell \rangle$ at physical quark masses¹⁷, NLO ChPT³² to translate this ratio to the ensemble masses, and the ensemble mass ratio m_s/m_ℓ .

Fig. 2. Comparison of lattice data and $D = 2 + 4$ OPE expectations for $Q^2 \Delta \Pi_\tau(Q^2)$, for various $D = 2$ truncation orders and either fixed-scale (left panel) or local-scale (right panel) treatments of the $D = 2$ series.

As shown in the left panel of Fig. 2, the three-loop fixed-scale $D = 2$ truncation provides an excellent OPE representation of the lattice data over a wide range of Q^2 , extending from near $\sim 10 \text{ GeV}^2$ down to just above $\sim 4 \text{ GeV}^2$. This is true, somewhat surprisingly, in spite of the slow convergence of the $D = 2$ series. The right panel shows the analogous comparison for the alternate local-scale choice. The Q^2 dependence of the lattice data clearly favors the fixed-scale treatment.

The comparison between the lattice data and the three-loop-truncated fixed-scale $D = 2$ version of the $D = 2 + 4$ OPE sum in the region below $Q^2 \sim 4 \text{ GeV}^2$ is shown in the left panel of Fig. 3. The $D = 2 + 4$ OPE sum, which provides an excellent representation of the lattice data at higher Q^2 , begins to pull away from the lattice data in this region. Adding the VSA version of the $D = 6$ contributions does nothing to improve the agreement. These observations are in accord with what one would expect if it is $D > 4$ OPE contributions significantly larger than those implied by the VSA $D = 6$ estimate which are responsible for the s_0 -instabilities

Fig. 3. Left panel: Comparison of lower- Q^2 lattice data with $D = 2 + 4$ and $D = 2 + 4 + 6$ OPE expectations, both obtained using the three-loop-truncated fixed-scale treatment of the $D = 2$ series, and, for the $D = 2 + 4 + 6$ sum, the VSA estimate for the $D = 6$ contributions. Right panel: Lattice data and the $D = 2 + 4$ OPE sum at larger Q^2 , here with conventional OPE error estimates, for the 3-loop-truncated, fixed-scale $D = 2$ treatment.

and differences in $|V_{us}|$ values obtained using the $\hat{w}$ and w_τ FESRs, and seen in the left panel of Fig. 1.

Finally, in the right panel of Fig. 3, we display the comparison of the lattice data and $D = 2 + 4$ OPE sum corresponding to the three-loop-truncated, fixed-scale treatment of the $D = 2$ series, now with the errors for the latter also displayed. These errors are obtained using standard estimates for the $D = 2$ and $D = 4$ truncation uncertainties (in the $D = 2$ case, employed despite the very slow convergence of the $D = 2$ series). The truncation error estimates are combined in quadrature, in the usual manner, with uncertainties produced by those on the input OPE parameters (the quark masses, α_s , and light and strange condensates). From the figure we conclude that, in spite of the very marginal nominal convergence of the $D = 2$ series, using the three-loop-truncated, fixed-scale version in the $D = 2 + 4$ sum yields a very reliable assessment of $D = 2 + 4$ OPE contributions, one for which conventional error estimates, moreover, provide a very conservative assessment of the associated error.

In light of the above observations, it is obvious how to go about modifying the conventional implementation of the FB FESR approach³³. First, relying on the observations from the comparison to high- Q^2 lattice data, we employ, in obtaining our central analysis results, the three-loop-truncated FOPT treatment of the $D = 2$ OPE integrals favored by the lattice data. We do, however, also consider the three-loop-truncated CIPT version for comparison. Second, since both lattice and continuum investigations suggest the assumptions about $D > 4$ OPE contributions employed in the conventional implementation are unreliable, we drop all such assumptions and attempt to fit the relevant effective $D > 4$ condensates, C_D , using data, where the C_D are defined such that the $D > 4$ OPE contribution to $\Delta\Pi_\tau(Q^2)$

has the form

$$[\Delta\Pi_\tau(Q^2)]_{D>4} = \sum_{D=6,8,\dots} \frac{C_D}{Q^D}. \quad (7)$$

In attempting to fit the effective $D > 4$ condensates C_D , it is convenient to consider FESRs for the weights $w_N(y)$ noted above. These weights, like w_τ , have a double zero at $s = s_0$, strongly suppressing possible duality violating contributions to the associated FESRs³⁴. In addition, by construction, the OPE side of the w_N FESR has only a single $D > 4$ contribution, equal to

$$\frac{(-1)^N}{N-1} \frac{C_{2N+2}}{s_0^N}. \quad (8)$$

The s_0 dependence of the w_N -weighted spectral integral in the region above $s_0 \sim 2 \text{ GeV}^2$, where residual duality violations remain very small, is then sufficient to allow us to solve for both unknowns, $|V_{us}|$ and C_{2N+2} , which enter the w_N FESR. Checking that the $|V_{us}|$ values obtained from the different w_N FESRs are compatible provides a further useful self-consistency test.

The results of these analyses, employing the preliminary BaBar update of the $K^-\pi^0$ branching fraction²¹ for the normalization of the $K^-\pi^0$ distribution, are presented in Table 1. The results in Column 2 were obtained using the CIPT prescription for the integrated $D = 2$ series, those in column 3 using the FOPT prescription, the latter being the prescription favored by the comparison to lattice data. The consistency between the results obtained from the different w_N is obviously excellent.

Table 1. Results for $|V_{us}|$ from analyses of the w_N , $N = 2, 3, 4$, FESRs involving fits of both C_{2N+2} and $|V_{us}|$ and employing either the CIPT (column 2) or FOPT (column 3) prescriptions for the 3-loop-truncated $D = 2$ OPE series

Weight	$ V_{us} $ (CIPT $D = 2$ treatment)	$ V_{us} $ (FOPT $D = 2$ treatment)
w_2	0.22271(228)	0.22252(228)
w_3	0.22271(228)	0.22282(228)
w_4	0.22271(229)	0.22296(229)

Although not shown here, the results for C_6 , C_8 and C_{10} obtained from the w_2 , w_3 and w_4 analyses also pass the qualitative self-consistency tests of showing FB reductions relative to the corresponding effective condensates for the flavor ud $V + A$ channel obtained in Ref.¹⁸.

The excellent consistency between the results for $|V_{us}|$ obtained from the different w_N FESRs shows that alternate implementation of the FB FESR approach, in which data is used to fit the effective $D > 4$ OPE condensates, C_D , has succeeded

in curing the unphysical weight-dependence of $|V_{us}|$ seen in the conventional implementation of the same approach. It turns out that it also cures the problem of the unphysical s_0 -dependence, seen earlier in the left panel, and the solid lines of the right panel, of Figure 1. This can be seen by displaying the results for $|V_{us}|$ as a function of s_0 obtained from the w_2 , w_3 and w_4 FESRs using now, in place of the conventional implementation treatment of $D = 6$ and 8 OPE contributions (which lead to the solid lines of the left panel of Fig. 1), the central values for C_6 , C_8 and C_{10} obtained from the alternate implementation fits. The results are those given by the dashed lines of the right panel of Fig. 1. The very good s_0 -stability of the results is immediately evident, confirming that the previously seen s_0 -instabilities can indeed be attributed to the use of incorrect values for $D > 4$ effective condensates C_D produced by the assumptions employed in the conventional implementation.

Table 2 gives the error budget for the FOPT $D = 2$ based w_N fits. The dominant error is that due to the uncertainties in the strange experimental distribution. The $D = 2$ truncation uncertainty (“ $D = 2$ trunc”) represents the difference in the results obtained using four- and three-loop truncation. From the lattice-OPE comparison discussed above, this, in fact, represents an exceptionally conservative assessment of this uncertainty. Combining the tabulated theory errors in quadrature, one obtains total theory errors on $|V_{us}|$ of 0.0006, 0.0007 and 0.0008 for the w_2 , w_3 and w_4 determinations. Such (very conservative) theory errors are comparable to, or slightly smaller than, the theory errors of the alternative $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ determinations.

Table 2. Contributions to the errors on $|V_{us}|$ obtained from the w_2 , w_3 and w_4 FESRs using the $D = 2$ FOPT prescription. Errors induced by theoretical uncertainties in the input OPE parameters α_s , $m_s(2 \text{ GeV})$ and $\langle m_s \bar{s}s \rangle$, as well as the small longitudinal continuum $J = 0$ subtraction (“long corr”), are listed above the horizontal line, those induced by the errors/covariances of the non-strange (“ ud ”) and strange (“ us ”) experimental distributions $dR_{V+A;ud}/ds$ and $dR_{V+A;us}/ds$ below the horizontal line.

Source	$\delta V_{us} $ (w_2 FESR)	$\delta V_{us} $ (w_3 FESR)	$\delta V_{us} $ (w_4 FESR)
$\delta\alpha_s$	0.00001	0.00004	0.00004
$\delta m_s(2 \text{ GeV})$	0.00017	0.00019	0.00019
$D = 2$ truncation	0.00047	0.00062	0.00067
$\delta\langle m_s \bar{s}s \rangle$	0.00035	0.00035	0.00035
$\delta(\text{long corr})$	0.00009	0.00009	0.00009
ud exp	0.00027	0.00028	0.00028
us exp	0.00226	0.00227	0.00227

Given the excellent consistency of the results of the single-weight w_N analyses, it makes sense to perform a combined fit to the w_2 , w_3 and w_4 FESRs. Our final

result for $|V_{us}|$,

$$|V_{us}| = 0.2228(23)_{exp}(6)_{th}, \quad (9)$$

is obtained from the version of this fit employing the FOPT treatment of the $D = 2$ OPE series favored by lattice dat^b

3. A new inclusive hadronic τ decay approach to V_{us}

As shown in the last section, the alternate implementation of the FB FESR determination of $|V_{us}|$ has a rather small theoretical error. At present, however, the uncertainties in the weighted us spectral integrals completely dominate the total $|V_{us}|$ error, and are sufficiently large that the method is a factor of more than 2 from being fully competitive with the alternate determinations based on $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma\pi_{\mu 2}$.

To understand the path to improving this situation it is useful to look at the relative contributions of the different exclusive mode distributions to the w_N -weighted us inclusive spectral integrals in the s_0 window employed in the alternate implementation fits described in the last section. Table 3 lists these contributions at the low and high end of the s_0 interval employed (the results grow monotonically with increasing s_0 between these points).

Table 3. Relative contributions to the w_N -weighted us spectral integrals in the s_0 fit window employed in the alternate implementation of the FB FESR approach. $K\pi$ column entries are the sum of the $K^-\pi^0$ and $\bar{K}^0\pi^-$ contributions, $K\pi\pi$ (B factory) column entries the sum of the $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ contributions. To simplify presentation, the very small $K\bar{K}\bar{K}$ contributions generated using the recent BaBar distribution results have been included with the residual mode contributions in the “Other” column. This contribution is 0.0% at $s_0 = 2.15 \text{ GeV}^2$ and 0.1% at $s_0 = 3.15 \text{ GeV}^2$ for all three weights.

Weight	s_0 [GeV ²]	K	$K\pi$	$K\pi\pi$ (B-factory)	Other
w_2	2.15	0.496	0.426	0.062	0.017
	3.15	0.360	0.414	0.162	0.065
w_3	2.15	0.461	0.446	0.073	0.019
	3.15	0.331	0.415	0.182	0.074
w_4	2.15	0.441	0.456	0.082	0.021
	3.15	0.314	0.411	0.194	0.081

^bThis result would be reduced to $|V_{us}| = 0.2200(23)_{exp}(6)_{th}$ if we had used the HFAG 2014 ²⁷ result for the $K^-\pi^0$ mode branching fraction. The qualitative conclusions of the analysis are, however, unchanged by such an alteration: the conventional implementation still produces serious unphysical weight- and s_0 -dependences in $|V_{us}|$, both of which are cured once the $D > 4$ C_D are fit to data.

The errors on the lower-multiplicity exclusive mode $K^-\pi^0$, $\bar{K}^0\pi^-$, $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ contributions based on BaBar and Belle distribution data are, at present, dominated by the uncertainties on the corresponding branching fractions (which normalize the unit-normalized experimental distributions) and hence improvable through improvements in the exclusive-mode branching fractions. More difficult to improve in the near term are the residual mode contributions, currently based on the old low-statistics 1999 ALEPH data, which, in addition to having low statistics, involves Tauola-based Monte Carlo input for a number of the higher-multiplicity mode distributions. The contributions of these residual modes to the inclusive w_N -weighted us spectral integrals at $s_0 = m_\tau^2$ are 6.4%, 7.3% and 8.0% for w_2 , w_3 and w_4 , respectively. The relative errors on these numbers are all $\sim 25\%$. To make the FB FESR approach fully competitive requires achieving sub-0.5% precision on the experimental component of the $|V_{us}|$ error, and hence sub-1% errors on the weighted us spectral integrals. To achieve this goal with analyses based on the existing weight choices would thus unavoidably require improvements by a factor > 2 in the residual mode contribution errors, and hence significant improvements to the experimental analyses of the higher multiplicity strange τ decay mode *differential distributions*. This is not likely to be forthcoming in the near future. One might consider adding additional multiplicative factors of $1 - (s/s_0)$ to existing weights in order to reduce the relative size of residual mode contributions, but this necessarily leads to higher degree polynomials and hence to the necessity of fitting a larger number of $D > 4$ effective condensates in the limited s_0 window available. This is unlikely to be feasible without significant reductions in the errors on the lower-multiplicity exclusive mode distributions.

Further progress is, however, possible by shifting to an analysis in which lattice data is used in place of the OPE. The basic idea is to consider generalized dispersion relations for the product of the experimentally determined us V+A inclusive distribution with weights consisting of products of factors having poles at Euclidean values of Q^2 . The dispersive integral over the product of the weight with the relevant us V+A spectral function combination is then given by a sum of residues involving the corresponding $J = 0$ and 1 us V+A polarizations evaluated at Euclidean Q^2 values. Provided the pole locations are at moderate Q^2 , these polarizations can be evaluated with good accuracy on the lattice.

Explicitly, this works as follows. From Eq. (1), the combination $|V_{us}|^2 \tilde{\rho}(s)$, with

$$\tilde{\rho}_{us;V+A}(s) \equiv \left(1 + 2\frac{s}{m_\tau^2}\right) \rho_{us;V+A}^{(J=1)}(s) + \rho_{us;V+A}^{(J=0)}(s), \quad (10)$$

is directly determinable from the experimental $dR_{us;V+A}/ds$ distribution, *without the need of any (albeit mildly model-dependent) continuum $J = 0$ subtraction*. $\tilde{\rho}(s)$ is the spectral function of the kinematic-singularity-free combination,

$$\tilde{\Pi}_{us;V+A}(Q^2) \equiv \left(1 - 2\frac{Q^2}{m_\tau^2}\right) \Pi_{us;V+A}^{(J=1)}(Q^2) + \Pi_{us;V+A}^{(J=0)}(Q^2), \quad (11)$$

of $J = 0$ and 1 us V+A polarizations, where $Q^2 = -s$. For weights $W_N(s)$ defined by $W_N(s) \equiv 1/\prod_{k=1}^N (s + Q_k^2)$ with $N \geq 3$ and all $Q_k^2 > 0$, one has the convergent, unsubtracted dispersion relation

$$\int_{th}^{\infty} ds W_N(s) \tilde{\rho}_{us;V+A}(s) = \sum_{k=1}^N \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j \neq k} (Q_j^2 - Q_k^2)}. \quad (12)$$

The idea is then to use lattice data for $\tilde{\Pi}_{us;V+A}(Q_k^2)$ on RHS of Eq. (12) and evaluate the LHS, up to the unknown factor $|V_{us}|^2$, from the experimental $dR_{us;V+A}/ds$ distribution. The spectral integral on the LHS, of course, extends up to infinity, while the $dR_{us;V+A}/ds$ data from τ decays is available only up to $s = m_\tau^2$, with, moreover, significant errors for s in the upper part of the kinematically allowed region. The goal is then to choose the number, N , of poles, and the locations, $Q_1^2, \dots, Q_N^2$, of these poles, in such a way that the contributions on the LHS from both the region where the us data errors are large and the region $s > m_\tau^2$ are very small. This can certainly be accomplished by choosing N large enough and keeping all Q_k^2 below $\sim 1 \text{ GeV}^2$. Such an increase in N , however, comes at the cost of increasing errors on the RHS of the Eq. (12) since the level of cancellation involved in the sum of residues on the RHS can be shown to grow with increasing N . This increased cancellation leads to an increase in the errors on the lattice determination of the RHS. The error on the extracted $|V_{us}|$ is then to be minimized by optimizing the choice of N and the pole locations, subject to these competing constraints.

There are a number of advantages of the approach just described over the FB FESR approach. The first, that one avoids the small continuum us $J = 0$ subtraction necessary in the FB FESR approach, is relatively minor, from a practical point of view, since this subtraction is strongly constrained by the known value of m_s through scalar and pseudoscalar sum rule determinations of m_s , and, as seen in Table 2, the associated uncertainty on $|V_{us}|$ is very small. The second is that, by using lattice data, we avoid the necessity of using the OPE and fitting the $D > 4$ C_D . The third, which turns out to be especially significant, is that, through the possibility of varying N and the choice of pole locations, Q_k^2 , the weights $W_N(s)$ are tunable to suppress contributions from the region of the spectrum where errors are large in a way that is not possible using the $w_N(y)$ of the FB FESR approach. It turns out that, for choices of N and the Q_k^2 which keep lattice statistical errors for the RHS small, it is possible to have the LHS strongly dominated by the K and $K\pi$ contributions, with the latter at the level of $\sim 10 - 15\%$ of the former, and all higher multiplicity contributions safely below 1%. In what follows we restrict our attention to $W_N(s)$ with uniformly spaced poles, characterizing such weights by the number of poles, N , the nearest-neighbor spacing, Δ , and the center, $Q^2 = C$, of the interval in which the Q_k^2 lie.

Figure 4 illustrates the suppression of high- s , higher-multiplicity contributions to the weighted us V+A spectral integral, for 4-pole weights with $\Delta = 0.05 \text{ GeV}^2$. The figure shows the relative contributions of the different exclusive (or semi-exclusive)

modes as a function of C . The results are very insensitive to modest increases in Δ . “ $K\pi$ ” labels the sum of $K^-\pi^0$ and $\bar{K}^0\pi^-$ contributions, “residual” the sum of contributions from $K^-\pi^0\pi^0$ and higher multiplicity strange modes, and “pQCD” the contribution from $s > m_\tau^2$ where no experimental data exist. The latter has been evaluated using the 5-loop-truncated $D = 0$ OPE form³⁵ for the us V+A spectral function, an approximation which should be very safe given the smallness of this contribution.

Fig. 4. Relative contributions to the weighted us V+A spectral integral as a function of C for 4-pole weights with $\Delta = 0.05 \text{ GeV}^2$. The contributions are labelled by the content of the strange hadronic state in questions, as explained in the text. For simplicity of presentation the very tiny contributions from the two $K\bar{K}K$ modes have been omitted.

As an illustration of expected precision of this new approach, we note that, for $N = 4$, $\Delta = 0.05 \text{ GeV}^2$ and $C = 0.325 \text{ GeV}^2$, i.e., $\{Q_k^2\} = \{0.25, 0.30, 0.35, 0.40 \text{ GeV}^2\}$, the combined spectral integral error is $\sim 0.4\%$, and the statistical error on the lattice evaluation of the sum of residues for the two near-physical-point RBC/UKQCD ensembles are $\sim 0.4\%$ for the $48^3 \times 96$, $1/a = 1.73 \text{ GeV}$ ensemble, and $\sim 0.5\%$ for the $64^3 \times 128$, $1/a = 2.36 \text{ GeV}$ ensemble. The quadrature sums of these errors yield contributions to the uncertainty on $|V_{us}|$ of ~ 0.0007 , ~ 0.0008 for the 48^3 and 64^3 ensembles, respectively.

The total spectral integral error and error on the sum of lattice residues are shown, somewhat more generally, as functions of C (though still for $N = 4$ and $\Delta = 0.05 \text{ GeV}^2$) in the left and right panels of Figure 5, respectively. Since the

increase in the spectral integral error is more rapid than the decrease in the error on the sum of the lattice residues with increasing C , lower values of C will be preferred for the final analysis. One will, however, also want to avoid having poles too close to $Q^2 = 0$ in order to keep residual finite volume effects under control.

Fig. 5. Left panel: Total fractional errors on the weighted us V+A spectral integrals, as a function of C , for $N = 4$ and $\Delta = 0.05 \text{ GeV}^2$. Right panel: Statistical errors on the lattice determinations of the sum of residues for the two near-physical-point RBC/UKQCD ensembles.

The errors quoted in the last paragraph are, of course, not yet complete. Systematic errors on the lattice determination of the RHS of the basic dispersion relation, due to residual finite volume effects, continuum extrapolation and corrections for the slight mis-tunings of the u , d and s masses, all of which are under current study, remain to be fully quantified. Preliminary explorations, however, suggest that these can be brought under good control, and hence that the approach outlined in this section promises to provide a new competitive approach to determining $|V_{us}|$. We will report on these further explorations, and provide further details on the optimization of the $|V_{us}|$ determination, in the near future.

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