ANALYTICAL DESIGN OF SUPERCONDUCTING MULTIPOLAR MAGNETS*

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INTRODUCTION

The analysis of magnetic fields produced by currents is an essential part of the design and construction of superconducting magnets. Beyond this, of course, a broad range of engineering judgements is involved, such as those concerning cost, materials, mechanical design, machining, production, assembly, testing, etc. These are not taken up in this summary, nor is the use of iron in the magnetic field.

A static magnetic field is always a three-dimensional configuration in space. While the basic physical principles are well understood their detailed analytical application to a whole three-dimensional magnet can rapidly become unmanageable, even with computers, unless appropriate simplifications and idealizations are introduced.

For accelerator and many other applications transverse magnetic fields are used to guide and focus beams of charged particles. When the beam cross section dimensions are small compared to the radius of curvature of the beam it is often an appropriate simplification to use a two-dimensional analysis to make the initial design of a magnet. Three-dimensional features, such as end effects, can later be taken into account as necessary along with engineering modifications once an idealized two-dimensional design has been chosen.

It should be clearly recognized, however, that certain, possibly very useful, three-dimensional patterns, such as spiral or alternating spiral fields, are expressly left out of consideration when we restrict ourselves to an idealized two-dimensional design.

The main purpose of the present paper is to describe and illustrate some of the methods now available for the analysis of two-dimensional fields.

I. TWO-DIMENSIONAL FIELDS PRODUCED BY CURRENTS

Required Field and Aperture

The components, H_X and H_Y, of the transverse field required in a beam handling magnet will lie in an X,Y plane normal to the beam and the beam cross section will lie within a specified "aperture" region in this X,Y plane. The primary problem is to find an arrangement of longitudinal currents, assumed infinitely long, straight and perpendicular to the X,Y plane (current filaments, current sheets, or solid current "blocks") lying outside the aperture which will produce the required two-dimensional field within the aperture. Usually this primary problem is solved in an inverse way — i.e., by assuming a distribution of current magnitudes and locations and calculating the field that would be produced, then modifying the assumed currents and/or their locations until the calculations give the required field with sufficient

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Obviously the efficiency of this procedure will be greatly enhanced if the assumed current distributions can be guided by idealized but mathematically precise analytic solutions such as those which can be obtained by the methods described here.

These methods also enable us to obtain analytical solutions to a wide range of secondary problems in the idealized case — for example, finding the external field, the field within conductors, the field forces acting on conductors, and the energy stored in the magnetic field.

Complex Representation

Maxwell's equations for a static two-dimensional magnetic field parallel to the X,Y plane may be written in the form

$$\begin{aligned} \frac{\partial H_Y}{\partial X} - \frac{\partial H_X}{\partial Y} &= 4\pi\sigma(X,Y) \\ \frac{\partial H_X}{\partial X} + \frac{\partial H_Y}{\partial Y} &= 0 \quad , \end{aligned} \tag{1}$$

where $\sigma(X,Y)$ is the density of current normal to the X,Y plane and $B = H$. When σ = constant (including 0), these <u>two</u> Maxwell equations may be identified[1] with the <u>two</u> Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial U}{\partial X} &= \frac{\partial V}{\partial Y} \\ \frac{\partial V}{\partial X} &= - \frac{\partial U}{\partial Y} \end{aligned} \tag{2}$$

which are necessary and sufficient for

$$F \equiv U + iV$$

to be an analytic function, $F(Z)$, in any region of the $Z = X + iY$ plane. It is easily seen that the identification may be made by setting

$$\begin{aligned} U &= H_Y - 2\pi\sigma X \\ V &= H_X + 2\pi\sigma Y \quad , \end{aligned}$$

or

$$F = H - 2\pi\sigma Z^* \quad , \tag{3}$$

where $Z^* = X - iY$ and

$$H \equiv H_Y + iH_X = i(H_X - iH_Y) \tag{4}$$

is taken, by definition, as the complex field.

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1. R.A. Beth, J. Appl. Phys. <u>38</u>, 4689 (1967).

Field as a Complex Power Series

In any region without currents (e.g., in a magnet aperture), $\sigma = 0$ and $H(Z)$ is itself an analytic function without singularities. Hence any two-dimensional field that satisfies Maxwell's equations can be written as a complex power series

$$H(Z) = H_1 + H_2 Z + H_3 Z^2 + \ldots\ldots = \sum_{n=1}^{\infty} H_n Z^{n-1} \tag{5}$$

about any point within the regular region as origin. The complex coefficients, H_n, completely specify $H(Z)$; thus H_1 specifies the dipole component, H_2 the quadrupole, H_3 the sextupole, and, in general, H_n the 2n-pole component.

For many applications the desired field will be antisymmetric above and below some "median plane" through the origin. When the X-axis represents the median plane all of the H_n coefficients are real.

Current Filaments

The magnetic field at Z due to a filament current I at z is[2]

$$H(Z) = \frac{2I}{Z - z} = H_Y + iH_X \quad . \tag{6}$$

Thus an isolated filament current I constitutes a simple pole with residue 2I for the two-dimensional magnetic field defined by Eq. (4).

Since the integral of H(Z) around any closed contour C in the Z-plane is $2\pi i$ times the sum of the residues within C, it follows that

$$\oint_C H dZ = 4\pi i I_C \quad , \tag{7}$$

where I_C is the total current within C.

Current Sheets

If dI is the filament current flowing along the elements of a cylinder perpendicular to the Z-plane in the interval dz then the field discontinuity between the right and left sides of dz can be shown from (6) to be[3]

$$H_R(z) - H_L(z) = 4\pi i \frac{dI}{dz} \quad , \tag{8}$$

where $H_R(z)$ and $H_L(z)$ are the limit values at the cylinder where $Z = z$ of the fields $H_R(Z)$ and $H_L(Z)$ in the regions to the right and left of the cylindrical current sheet.

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2. R.A. Beth, J. Appl. Phys. 37, 2568 (1966).

3. R.A. Beth, Brookhaven National Laboratory, Accelerator Dept. Report AADD-102 (1966).

Current Blocks

When the current density is uniform (σ = constant), a straight conductor, represented by its cross section in the Z-plane, may be called a current block. For such a current block the function F(Z) in Eq. (3) can be shown to be[1]

$$F(Z) = i\sigma \oint_C \frac{z^* \, dz}{z - Z} \quad , \tag{9}$$

where $z = x + iy$ represents the points of the cross-section boundary, and $z^* = x - iy$. Then the fields inside and outside the conductor are

$$\begin{aligned} H_{in} &= F(Z_{in}) + 2\pi\sigma Z^*_{in} \\ H_{out} &= F(Z_{out}) \quad . \end{aligned} \tag{10}$$

These are both given by the single formula

$$H = i\sigma \oint_C \left(\frac{z^* - Z^*}{z - Z} \right) dz \tag{11}$$

since the residue of $Z^*/(z - Z)$ is Z^* for $Z = Z_{in}$ and is zero for $Z = Z_{out}$.

Field Forces

The resultant field force acting on a unit length of all the currents within an arbitrary contour C in the Z-plane can be shown[4,5] to have X and Y components which are given by the contour integral

$$f \equiv f_Y + if_X = -\frac{1}{8\pi} \oint_C H^2 \, dZ \quad . \tag{12}$$

Similarly the force df acting on unit length of a current sheet in the interval dz is given by[6]

$$\frac{df}{dz} = \frac{1}{8\pi} \left[H_L^2(z) - H_R^2(z) \right] \tag{13}$$

which, using (8), can be written in the form

$$\frac{df}{dz} = - i\overline{H}(z) \frac{dI}{dz} \tag{14}$$

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4. R.A. Beth, Brookhaven National Laboratory, Accelerator Dept. Report AADD-107 (1966).

5. R.A. Beth, in Proc. 2nd Intern. Conf. Magnet Technology, Oxford, 1967, p. 135.

6. R.A. Beth, in Proc. 6th Intern. Conf. High Energy Accelerators, Cambridge, Mass., 1967, p. 387.

where

$$\overline{H}(z) = \frac{1}{2}\left[H_L(z) + H_R(z) \right] \tag{15}$$

is the mean of the limits of the left and right fields at the current sheet.

Potentials

In any simply connected region without currents the integral of the analytic function H(Z) between two points is independent of the path of integration. The vector and scalar potentials, A(X,Y) and Ω(X,Y), are then given by

$$- A - i\Omega = W(Z) = \int_o^Z H\, dZ = \sum_{n=1}^{\infty} \frac{1}{n} H_n Z^n \quad , \tag{16}$$

where H is defined as in (4) and Z = 0 lies in the regular region. Conversely, the field is given by

$$H = \frac{dW}{dZ} \quad . \tag{17}$$

The curves A = constant give the lines of force of the magnetic field and are everywhere orthogonal to the scalar equipotentials, Ω = constant.

It will be seen that the vector potential A(X,Y) in this two-dimensional case is really only the component of the three-dimensional vector potential normal to the field plane; the other two components lie in the field plane and are constant.

Field Energy

The vector and scalar potentials

$$A = A(X,Y) \qquad \text{and} \qquad \Omega = \Omega(X,Y) \tag{18}$$

specify a transformation from the X,Y plane to an A,Ω plane whose Jacobian is

$$J = \frac{\partial(A,\Omega)}{\partial(X,Y)} = \begin{vmatrix} \frac{\partial A}{\partial X} & \frac{\partial A}{\partial Y} \\ \frac{\partial \Omega}{\partial X} & \frac{\partial \Omega}{\partial Y} \end{vmatrix} = \begin{vmatrix} -H_Y & H_X \\ -H_X & -H_Y \end{vmatrix} = H_Y^2 + H_X^2 = HH^* \quad , \tag{19}$$

i.e., J is proportional to the field energy density. Hence the field energy per unit thickness in any region R of the X,Y plane is proportional to the area of the transformed region R′ in the A,Ω plane,[5,7] i.e.,

$$E_R = \frac{1}{8\pi} \iint_R J dX dY = \frac{1}{8\pi} \iint_{R'} dA d\Omega \quad . \tag{20}$$

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7. R.A. Beth, Brookhaven National Laboratory, Accelerator Dept. Report AADD-106 (1966).

When the potential function W(Z), defined in (15), is known in complex form it is often convenient to calculate the area of the region R' in the W plane as a contour integral around its boundary C':

$$\iint_{R'} dA d\Omega = \frac{1}{2i} \oint_{C'} W^* dW = \frac{i}{2} \int_{C'} W dW^* \quad , \tag{21}$$

where W^* is the complex conjugate of W.[7]

II. ILLUSTRATIVE EXAMPLES

Multipole Field in a Circular Aperture

The most general nonsingular Maxwell field in two dimensions may be represented as a superposition of multiple fields as in (5).
multipole

Suppose we wish to produce the 2n-pole component

$$H_{in}(Z) = H_n Z^{n-1} \tag{22}$$

within a circular aperture of radius a by an array of current filaments.

The array of minimum lateral size and minimum field energy storage will be obtained with a cylindrical current sheet tightly surrounding the required circular aperture. Let the points z of the cylinder cross section, and the arc length s measured around the circumference be

$$\begin{aligned} z &= ae^{i\theta} \qquad 0 \le \theta < 2\pi \\ s &= a\theta \quad . \end{aligned} \tag{23}$$

Then $z^* = ae^{-i\theta} = a^2/z$ and $dz = izd\theta = i(z/a)ds$.

By means of the current sheet theorem (8) we can see that the required current distribution is

$$\frac{dI}{ds} = -\frac{1}{4\pi a}\left(H_n z^n + H_n^* z^{*n} \right) = -\frac{|H_n|}{2\pi} a^{n-1} \cos\,(n\theta + \theta_n) \quad , \tag{24}$$

where $H_n = |H_n|\, e^{i\theta_n}$, and the external field is

$$H_{out}(Z) = -H_n^* \frac{a^{2n}}{Z^{n+1}} \quad . \tag{25}$$

Thus the required linear current density in the cylinder is proportional to a^{n-1} and varies sinusoidally as an n^{th} harmonic of the central angle θ around the circumference of the cylinder. Note that the phase angle θ_n merely specifies the orientation of the multipole field with respect to the direction chosen for θ = 0.

The field force acting on unit area of the current sheet, df/ds, can be evaluated by using the force theorem (14). It turns out that the radial component is everywhere zero and the tangential component is

$$\frac{df_\theta}{ds} = \frac{dI}{ds} \cdot |H_n| \, a^{n-1} \sin (n\theta + \theta_n) \quad . \tag{26}$$

Thus the tangential force density (26) is zero where the current density (24) is zero or has maximum magnitude. The tangential force is directed toward the nearest absolute maximum of (24) at intermediate points.

It can be shown from the energy theorem (20) that the internal and external field energies are equal for a circular multipole current sheet and that

$$E_{in} = E_{out} = \frac{|H_n|^2 \, a^{2n}}{8n} \quad . \tag{27}$$

For an elliptic aperture and current sheet the formulas are somewhat more complicated; they have been worked out together with the case of two confocal elliptic (or concentric circular) cylinders required to produce a prescribed field within the inner cylinder while cancelling the field external to the outer cylinder.[8]

Step-Function Approximation for cos φ

We may wish to approximate the smoothly varying current density (24) by a step function made up of intervals of constant current density. We set

$$\varphi = n\theta + \theta_n \tag{28}$$

and seek to approximate $\cos \varphi$ by a function of N steps per quadrant. With cosine symmetry as shown in Fig. 1, the Fourier composition of the step function is

$$S(\varphi) = \sum_{m=1}^{\infty} C_m \cos (2m-1) \varphi \quad ,$$

where the coefficients C_m will depend on N values of g_ν and N values of φ_ν. These 2N values can be chosen to make $C_1 \neq 0$ and $C_m = 0$ for $m = 2, 3, \ldots 2N$. The solution[9] is

$$\begin{aligned} g_\nu &= \frac{\cos (\nu - \frac{1}{2}) \alpha}{\cos \frac{1}{2}\alpha} \quad , \qquad \nu = 1, 2, \ldots N \\ \varphi_\nu &= \nu\alpha \quad , \qquad \nu = 1, 2, \ldots N \quad , \end{aligned} \tag{29}$$

where

$$\begin{aligned} \alpha &= \frac{\pi}{2N+1} = \frac{2\pi}{M} \\ M &= 4N + 2 \quad . \end{aligned} \tag{30}$$

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8. R.A. Beth, IEEE Trans. Nucl. Sci. NS-14, No. 3, 386 (1967).
9. R.A. Beth, Brookhaven National Laboratory, Accelerator Dept. Report AADD-135 (1967).

With these values of g_ν and φ_ν the Fourier expansion of the step function is

$$S_N(\varphi) = \frac{M}{\pi} \tan \frac{\pi}{M} \left[\sum_{k=0}^{\infty} \frac{\cos (kM+1) \varphi}{kM + 1} - \sum_{k=1}^{\infty} \frac{\cos (kM-1) \varphi}{kM - 1} \right] \tag{31}$$

so that, after the fundamental, all harmonics are eliminated up to the cos $(4N+1)\varphi$ term — which, in view of (28), means sinusoidal functions of $n(4N+1)\theta$ and a deviation of the order of

$$\frac{1}{4N+1} \left(\frac{Z}{a} \right)^{4Nn} \quad \text{times the fundamental} \tag{32}$$

from the ideal field (22).[9] Figures 2 through 6 show the general form of $S_N(\varphi)$ and the first few cases N = 1, 2, 3, and 4.

It will be seen that the approximation is so good that practical construction inaccuracies will soon outweigh the deviation of S_N from a pure cos φ field even for N = 3 or 4. The construction of step-function dipoles and quadrupoles was described by Britton during this Summer Study.[10]

Constant Gradient Field in an Elliptic Aperture

Any desired field (5) can be produced within an elliptic aperture by providing the proper current distribution on the elliptic cylinder determined by the specified aperture; the resulting external field and field energy can be calculated.[8]

The relations for a constant gradient field in an elliptical aperture may be summarized as follows[5]:

To produce the field

$$H_{in} = B_o (1 + KZ) \quad \text{(dipole + quadrupole)} \tag{33}$$

within the elliptical cylinder whose normal section is

$$z = a \cos \theta + ib \sin \theta = re^{i\theta} + \delta e^{-i\theta} \qquad (0 \le \theta < 2\pi) \tag{34}$$

where

$$r = \tfrac{1}{2}(a + b) \qquad \delta = \tfrac{1}{2}(a - b) \tag{35}$$

requires the current distribution in the cylinder elements

$$\frac{dI}{d\theta} = - \frac{B_o}{2\pi} (r \cos \theta + r^2 K \cos 2\theta) \tag{36}$$

with

$$c^2 = a^2 - b^2 = 4r\delta \qquad W_1 = 2r (r + \delta)$$

$$\xi = Z + \sqrt{Z^2 - c^2} \qquad W_2 = 2r^2 (r^2 + \delta^2) K \ .$$

10. R.B. Britton, these Proceedings, p. 893.

The external field is

$$H_{out} = - B_o \left[W_1 \xi^{-1} + 2W_2 \xi^{-2} \right] \Big/ \sqrt{Z^2 - c^2} \quad . \tag{37}$$

The complex potentials are $W = -(A + i\Omega)$ such that $H = dW/dZ$

$$W_{in} = B_o \left[Z + \frac{1}{2} KZ^2 \right] \qquad W_{out} = B_o \left[W_o + W_1 \xi^{-1} + W_2 \xi^{-2} \right] \quad . \tag{38}$$

Vector potential A is continuous across the ellipse when $W_o = r\delta = c^2/4$.

Circular cylinder case: $r = a = b$, $\delta = 0$, $c = 0$.

The field energies per unit length are:

$$\begin{aligned} E_{in} &= \frac{1}{8} B_o^2 \left[ab + \frac{1}{4} K^2 ab(a^2 + b^2) \right] \\ E_{out} &= \frac{1}{8} B_o^2 \left[a^2 + \frac{1}{8} K^2 (a^2 + b^2)^2 \right] \\ E_{total} &= \frac{1}{8} B_o^2 (a + b) \left[a + \frac{1}{8} K^2 (a + b)(a^2 + b^2) \right] \quad . \end{aligned} \tag{39}$$

The ring magnets of the Brookhaven Alternating-Gradient Synchrotron provide a constant gradient field with $K = 0.0425\ \text{cm}^{-1}$ within a roughly elliptical aperture for which $a = 8.8$ cm, $b = 4.0$ cm.

Such a field can be produced by an elliptic cylinder current sheet chosen to fit the aperture. The equipotential curves $U = - A(X,Y) = \text{const}$ and $V = - \Omega(X,Y) = \text{const}$ are plotted in the left side of Fig. 7 and the corresponding A,Ω plot with areas proportional to field energy is shown in the right side of the same figure.[5]

The $U = \text{const}$ curves show lines of force of the magnetic field in the space plot. Since the total field energy is always finite, the potential plot will always cover only finite regions of the U,V (or A,Ω) plane. Areas can be calculated by the area theorem (21). Selected corresponding regions have been crosshatched similarly to elucidate the interrelations of the two plots.

Superposition of Elliptical Current Blocks

By means of the integral formula (9) the fields (10) inside as well as outside an elliptical conductor bounded by

$$z = a \cos \theta + ib \sin \theta$$

and carrying a uniform current density, $\sigma = \text{const}$, can be evaluated[1]:

$$\begin{aligned} H_{in} &= \frac{4\pi\sigma}{a + b} (bX - iaY) \\ H_{out} &= \frac{4\pi\sigma\, ab}{Z + \sqrt{Z^2 - c^2}} \quad , \end{aligned} \tag{40}$$

where

$$c^2 = a^2 - b^2 \quad .$$

If we superpose two equal area elliptical conductors with $\sigma' = -\sigma$, $ab = a'b'$, and centers at $Z_o = -X_o$, $Z_o' = X_o$, as shown in Fig. 8(a), then the overlap region forms an empty aperture (since $\sigma + \sigma' = 0$) with the resultant interior constant gradient field:

$$H_{in} = \frac{4\pi\sigma}{(a+b)(a'+b')}\left[(ab' + a'b + 2bb')\, X_o + (a'b - ab')\, Z\right] \tag{41}$$

where

$$ab = a'b' \quad .$$

We obtain a pure dipole field

$$H_{in} = \frac{8\pi\sigma\, b\, X_o}{a + b} = \text{const} \tag{42}$$

for equal ellipses, $a' = a$ and $b' = b$, as in Fig. 8(b).

We obtain a pure quadrupole field

$$H_{in} = \frac{4\pi\sigma\,(a'b - ab')}{(a+b)(a'+b')}\,\bar{Z} \tag{43}$$

when both ellipses are centered at the origin, $X_o = 0$, as in Fig. 8(c).

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Remarks on Complex Methods

The complex variable methods for two-dimensional fields described and illustrated in this paper go beyond the older methods which focus on setting up a potential that satisfies Laplace's equation in a region without currents. Here the natural emphasis is on the pair of field components which have direct physical significance everywhere — even within current bearing regions where both potentials cannot be defined. Currents are systematically taken into the theory as singularities and all three aspects of analytic functions — Cauchy-Riemann equations, Cauchy integrals, and power series representations — turn out to have useful physical applications. Field forces and field energy storage can be calculated. In these and other ways the methods described form useful extensions of the usual complex treatment of two-dimensional fields.

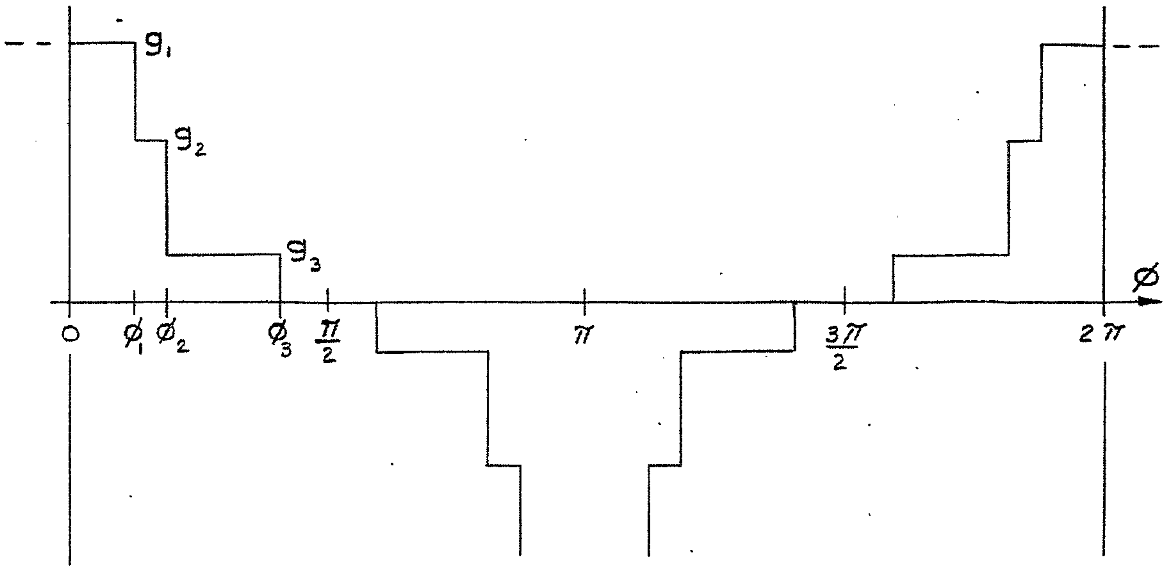

Fig. 1. General form of step function.

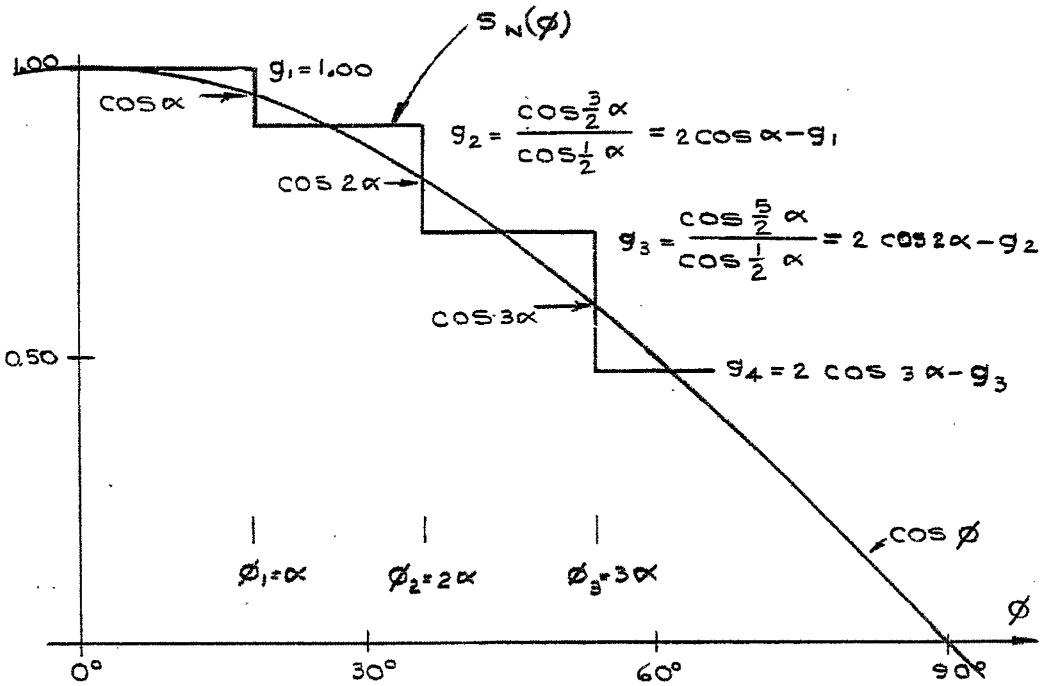

Fig. 2. Form of $S_N(\varphi)$ for arbitrary N.

N steps $\quad \alpha = \dfrac{\pi}{2N+1} \quad M = 4N + 2$

$$\frac{2}{\alpha}\tan\frac{\alpha}{2} = \frac{M}{\pi}\tan\frac{\pi}{M}$$

$$S_N(\varphi) = \frac{M}{\pi}\tan\frac{\pi}{M}\Big[\cos\varphi - \frac{\cos(M-1)\varphi}{M-1} + \frac{\cos(M+1)\varphi}{M+1} - \frac{\cos(2M-1)\varphi}{2M-1} + \frac{\cos(2M+1)\varphi}{2M+1} - \frac{\cos(3M-1)\varphi}{3M-1} + \frac{\cos(3M+1)\varphi}{3M+1} - \cdots + \cdots\Big]$$

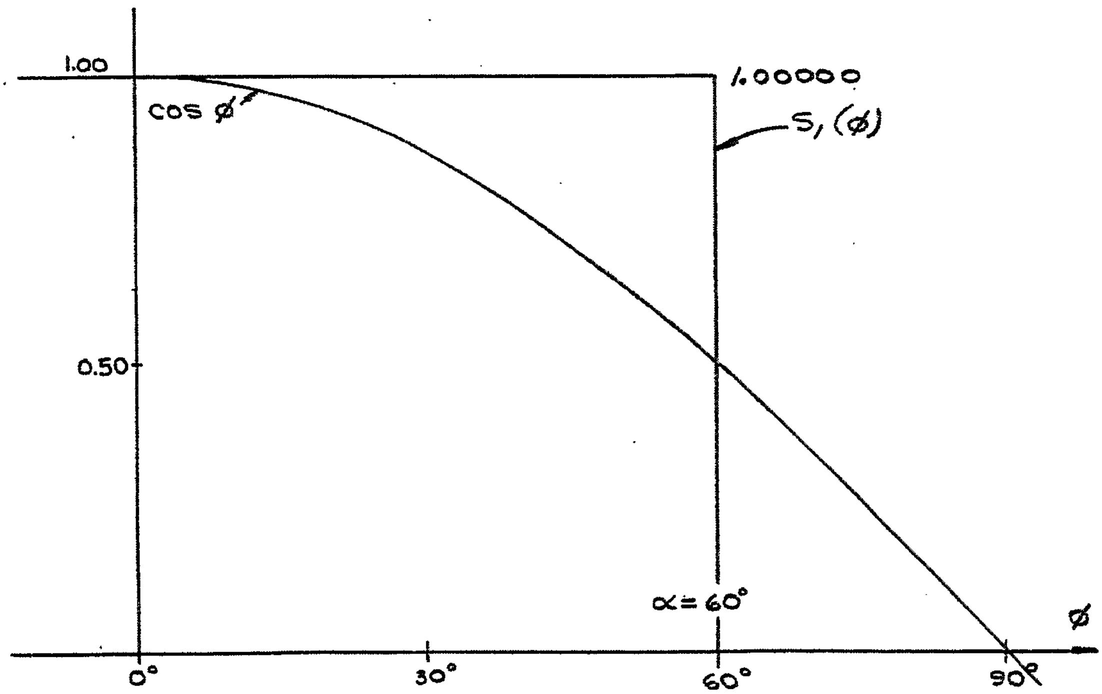

Fig. 3. $S_1(\varphi)$.

$N = 1$ step $\quad \alpha = \frac{\pi}{3} \rightarrow 60^{\circ} \quad M = 6$

$$\frac{2}{\alpha} \tan \frac{\alpha}{2} = \frac{6}{\pi} \tan 30^{\circ} = 1.10266$$

$$S_1(\varphi) = 1.10266 \Big[\cos \varphi - \frac{1}{5} \cos 5\varphi + \frac{1}{7} \cos 7\varphi - \frac{1}{11} \cos 11\varphi + \frac{1}{13} \cos 13\varphi - \frac{1}{17} \cos 17\varphi + \frac{1}{19} \cos 19\varphi - \dots + \dots \Big]$$

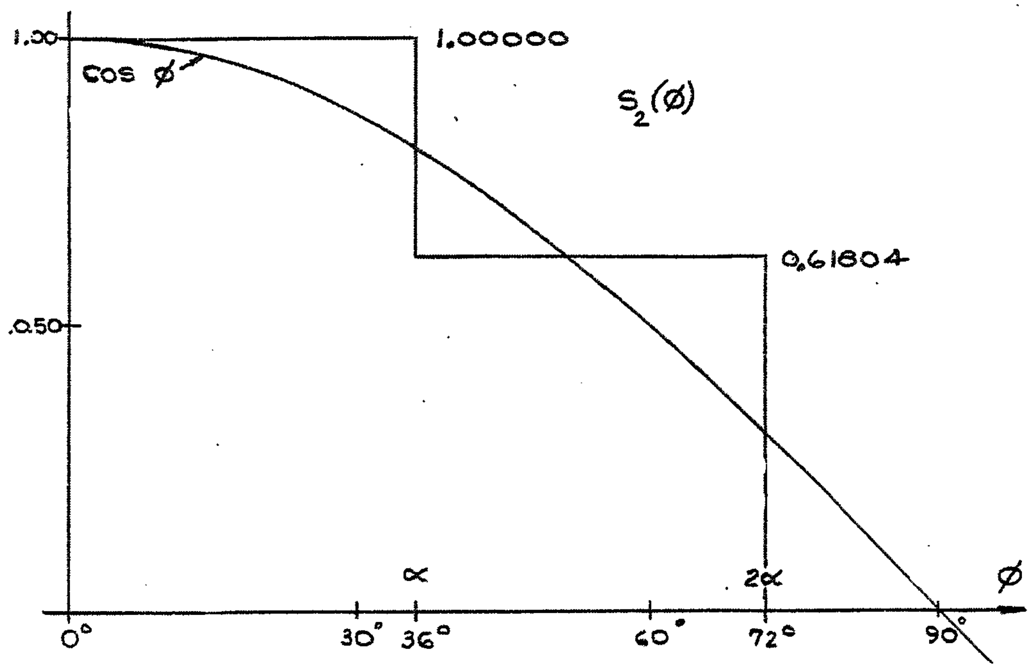

Fig. 4. $S_2(\varphi)$

$N = 2$ steps $\qquad \alpha = \frac{\pi}{5} \rightarrow 36^{\circ} \qquad M = 10$

$$\frac{2}{\alpha} \tan \frac{\alpha}{2} = \frac{10}{\pi} \tan 18^{\circ} = 1.03425$$

$$S_2(\varphi) = 1.03425 \Big[\cos \varphi - \frac{1}{9} \cos 9\varphi + \frac{1}{11} \cos 11\varphi - \frac{1}{19} \cos 19\varphi + \frac{1}{21} \cos 21\varphi - \frac{1}{29} \cos 29\varphi + \frac{1}{31} \cos 31\varphi - \; \cdots \; + \; \cdots \Big]$$

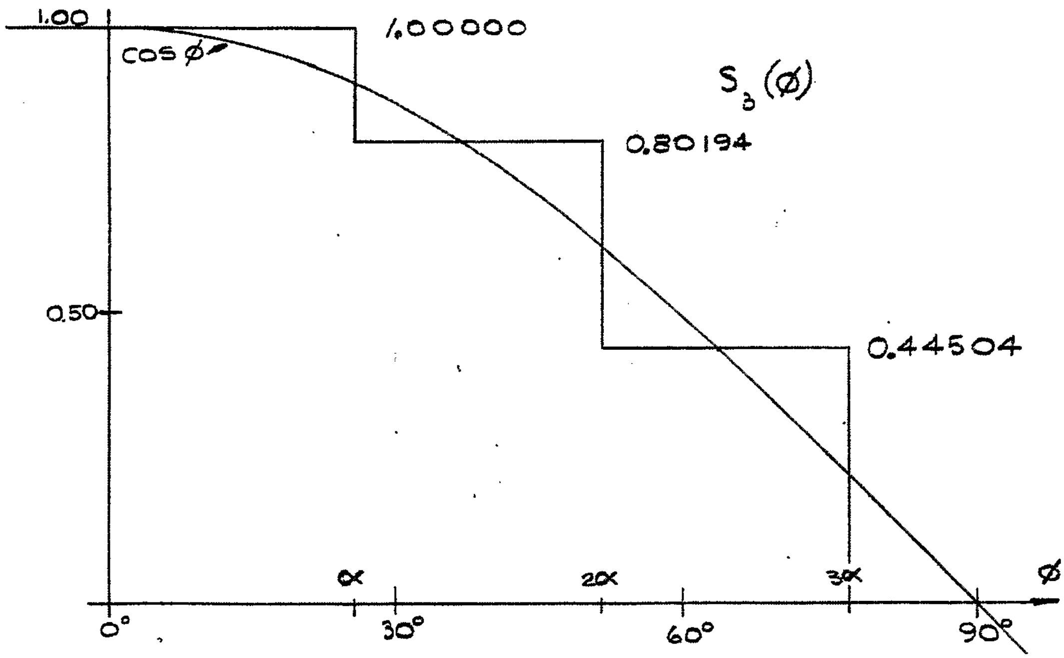

Fig. 5. $S_3(\varphi)$

N = 3 steps $\quad \alpha = \frac{\pi}{7} \rightarrow 25\,\frac{5}{7}\,^{o}$ $\quad$ M = 14

$$\frac{2}{\alpha}\tan\frac{\alpha}{2} = \frac{14}{\pi}\tan 12\,\frac{6}{7}\,^{o} = 1.01712$$

$$S_3(\varphi) = 1.01712\left[\cos\varphi - \frac{1}{13}\cos 13\varphi + \frac{1}{15}\cos 15\varphi - \frac{1}{27}\cos 27\varphi + \frac{1}{29}\cos 29\varphi - \frac{1}{41}\cos 41\varphi + \frac{1}{43}\cos 43\varphi - \ldots + \ldots\right]$$

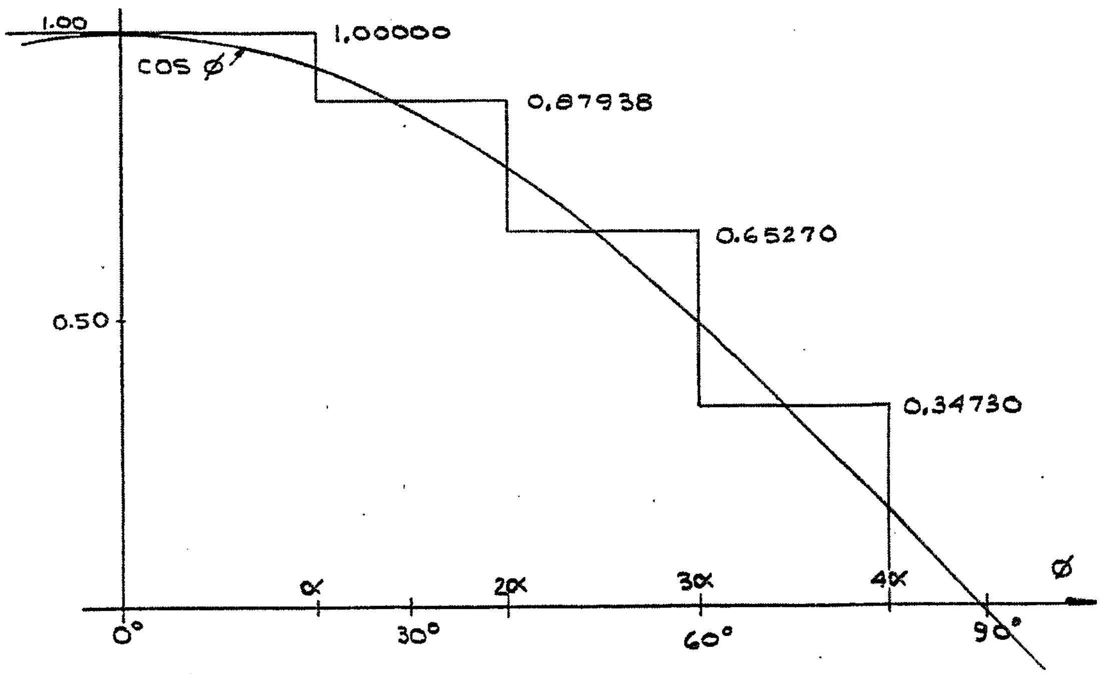

Fig. 6. $S_4(\varphi)$

$N = 4$ steps $\qquad \alpha = \frac{\pi}{9} \rightarrow 20^\circ \qquad M = 18$

$$\frac{2}{\alpha} \tan \frac{\alpha}{2} = \frac{18}{\pi} \tan 10^\circ = 1.01030$$

$$S_4(\varphi) = 1.01030 \left[\cos \varphi - \frac{1}{17} \cos 17\varphi + \frac{1}{19} \cos 19\varphi - \frac{1}{35} \cos 35\varphi + \frac{1}{37} \cos 37\varphi - \frac{1}{53} \cos 53\varphi + \frac{1}{55} \cos 55\varphi - \ldots + \ldots \right]$$

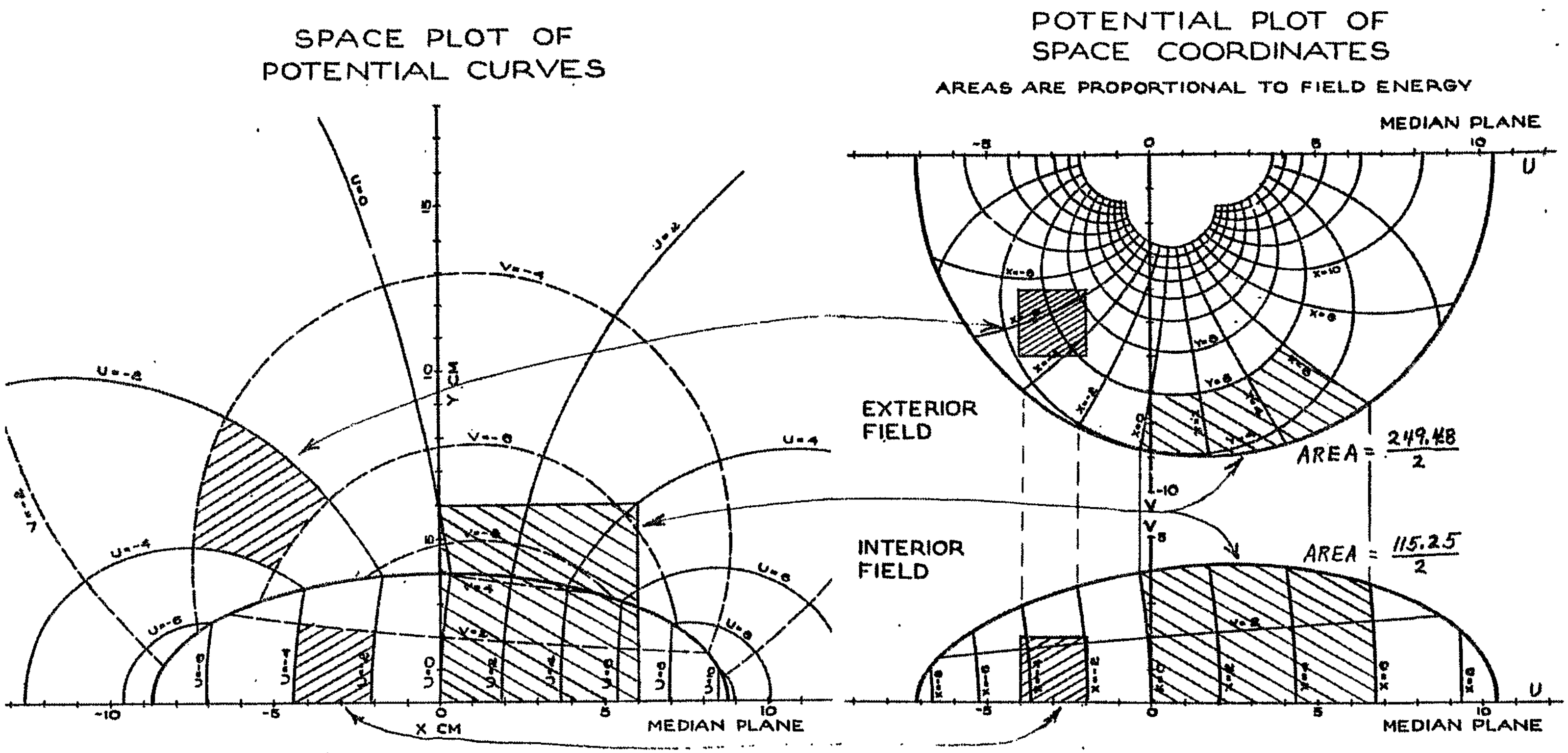

Fig. 7. Constant gradient field within an elliptical current sheet.

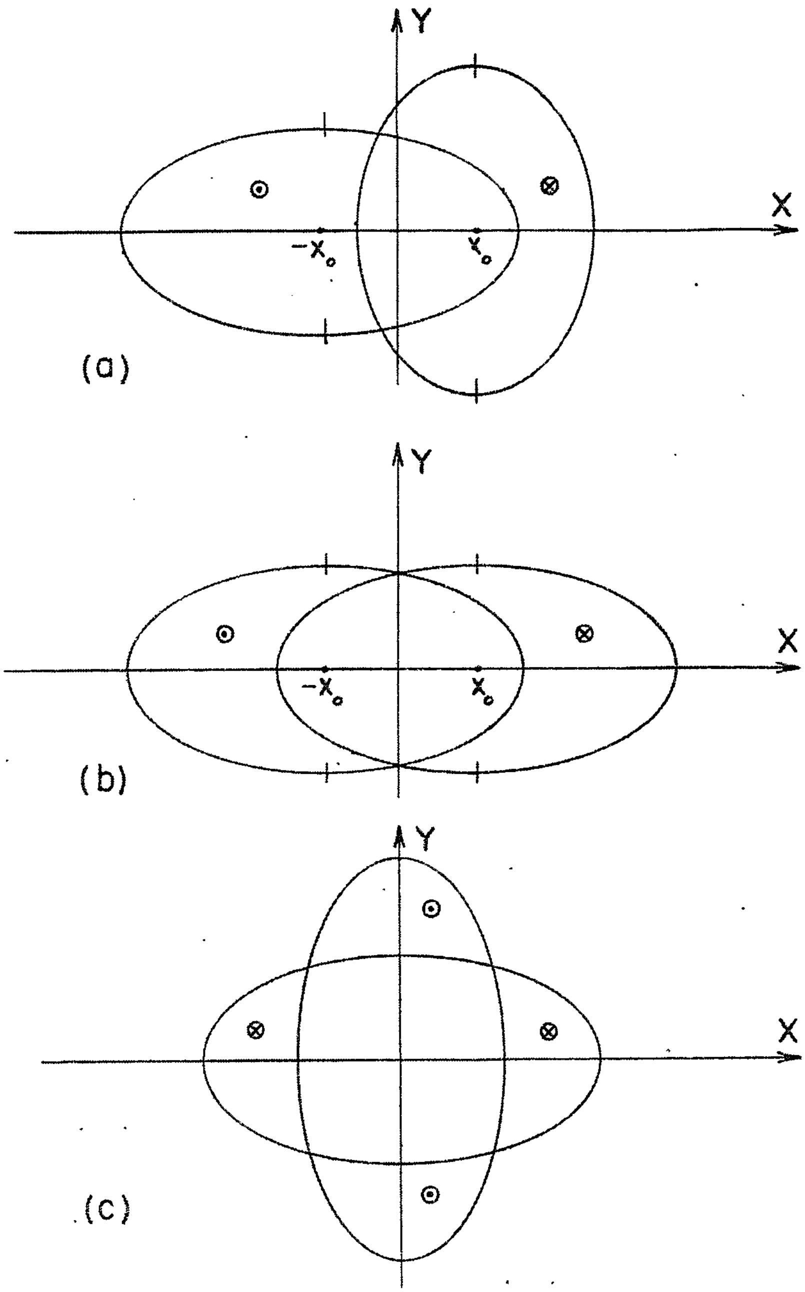

Fig. 8. Apertures formed by overlapping elliptical conductors.
(a) Constant gradient field.
(b) Dipole field.
(c) Quadrupole field.