Harmonic Description of 2-Dimensional Fields

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Fields in Free Space: Scalar Potential

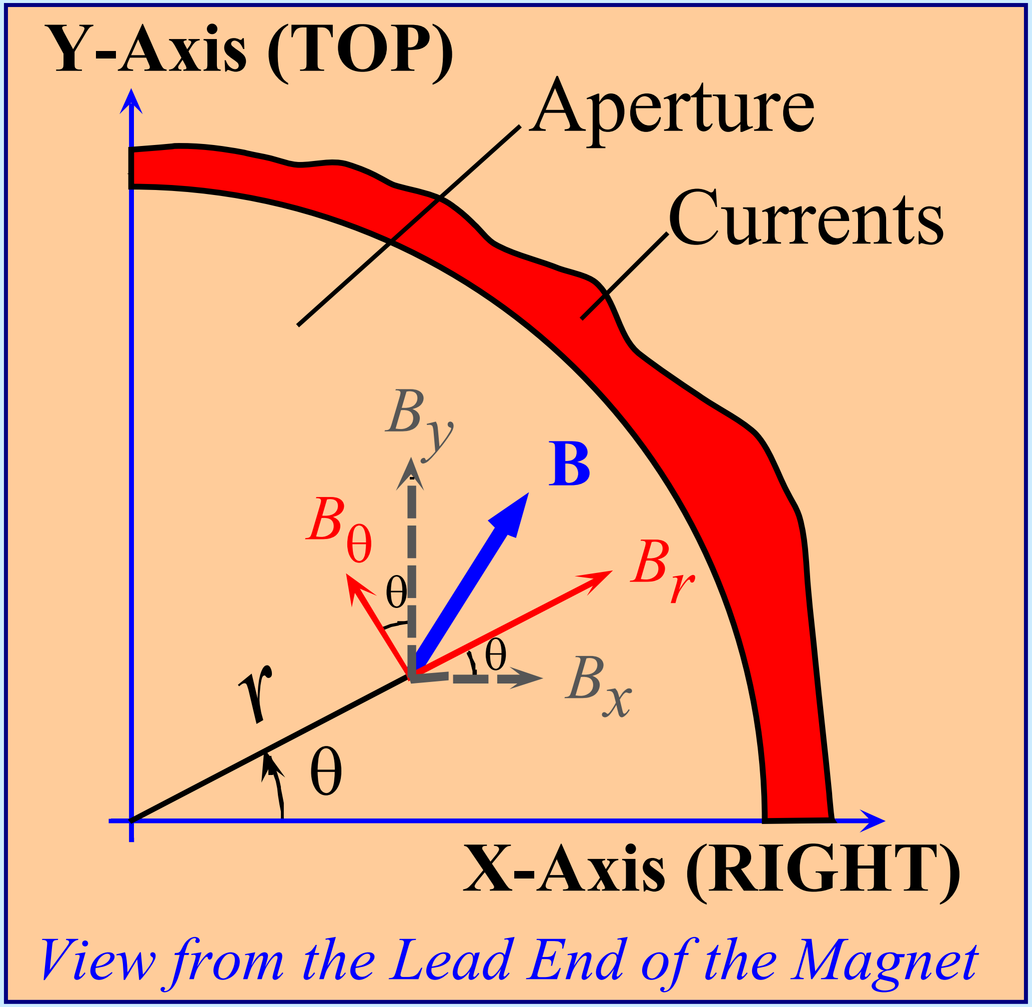
- $\boxed{\nabla.\mathbf{B} = 0}$ (Always true)
- In a region free of any currents or magnetic material, $\boxed{\nabla \times \mathbf{B} = 0}$, and **B** may be written as the gradient of a scalar potential, $\boxed{\mathbf{B} = \nabla\Phi_m}$
- The two equations above may be combined to obtain the Laplace's equation for the scalar potential, Φ_m,

$$\boxed{\nabla^2\Phi_m = 0}$$

2-D Fields in Free Space

- and $\boxed{\nabla^2\Phi_m = 0}$
- Most accelerator magnet apertures have a cylindrical symmetry, with a length much larger than the aperture. In such situations, the field away from the ends can be considered 2-dimensional, and the general solution can be expressed in a relatively simple ***harmonic series***.

Commonly Used Coordinate System



Users of magnetic measurements data may use a system oriented differently, often requiring suitable transformations of the measured harmonics.

$$B_x(r,\theta) = B_r \cos\theta - B_\theta \sin\theta$$

$$B_y(r,\theta) = B_r \sin\theta + B_\theta \cos\theta$$

Solution in Cylindrical Coordinates

For no *z*-dependence (2-D fields),

$$\boxed{\nabla^2 \Phi_m = \left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_m}{\partial r}\right) + \left(\frac{1}{r^2}\right)\left(\frac{\partial^2 \Phi_m}{\partial \theta^2}\right) = 0}$$

writing $\Phi_m(r,\theta) = R(r)\Theta(\theta)$, and imposing the conditions

$$\boxed{\Theta(\theta + 2\pi) = \Theta(\theta); \quad R(r) = \text{finite at } r = 0}$$

we can get the solution of the Laplace's equation in terms of a harmonic series.

2-D Fields: Harmonic Series

- Components of 2-D fields in cylindrical coordinates:

$$B_r(r,\theta) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \sin[n(\theta-\alpha_n)]$$

$$B_\theta(r,\theta) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \cos[n(\theta-\alpha_n)]$$

- $C(n)$ = ***Amplitude***, α_n = ***phase angle*** of the ***2n-pole term*** in the expansion.
- R_{ref} = ***Reference radius***, arbitrary, typically chosen ~ the region of interest.

C(n) scales as R_{ref}^{n-1}

2-D Fields: Cartesian Components

- Cartesian components of **B** may be written as:

$$B_x(r,\theta) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \sin[(n-1)\theta - n\alpha_n]$$

$$B_y(r,\theta) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \cos[(n-1)\theta - n\alpha_n]$$

- A ***Complex field***, $\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x$, where $\boldsymbol{z} = x + iy$, combines the 2 equations above:

$$\boxed{\boldsymbol{B}(\boldsymbol{z}) = \sum_{n=1}^{\infty} [C(n)\exp(-in\alpha_n)]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}}$$

2-D Fields: Normal & Skew Terms

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=1}^{\infty}\left[C(n)\exp(-in\alpha_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

may be written as:

$$\boldsymbol{B}(\boldsymbol{z}) = \sum_{n=1}^{\infty}\left[B_n + iA_n\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

Simple power series, valid within source-free zone.

where:

$$B_n \equiv C(n)\cos(n\alpha_n) = 2n\text{ - pole NORMAL Term}$$

$$A_n \equiv -C(n)\sin(n\alpha_n) = 2n\text{ - pole SKEW Term}$$

In the US, the 2n-pole terms are denoted by B_{n-1} and A_{n-1}.

Sometimes, the skew terms are defined without the negative sign, but the above form is the most common now.

Analytic Functions of a Complex Variable

Any function of the complex variable, z, given by

$$\boxed{F(z) = U(x,y) + i\,V(x,y)}$$

is an ***Analytic*** function of z, if

$$\boxed{\left(\frac{\partial U}{\partial x}\right) = \left(\frac{\partial V}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial U}{\partial y}\right) = -\left(\frac{\partial V}{\partial x}\right)}$$

Cauchy-Riemann Conditions.

An analytic function can be expressed as a power series in z. This series is valid within the ***circle of convergence***, which extends to the nearest singularity. Analytic function does not depend on z^*.

Analyticity of Complex Field

$$\left(\frac{\partial U}{\partial x}\right) = \left(\frac{\partial V}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial U}{\partial y}\right) = -\left(\frac{\partial V}{\partial x}\right)$$

Cauchy-Riemann Conditions.

Maxwell's equations in source free region:

$$\nabla . \boldsymbol{B} = 0 \Rightarrow \left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right)$$

$$(\nabla \times \boldsymbol{B})_z = 0 \Rightarrow \left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right)$$

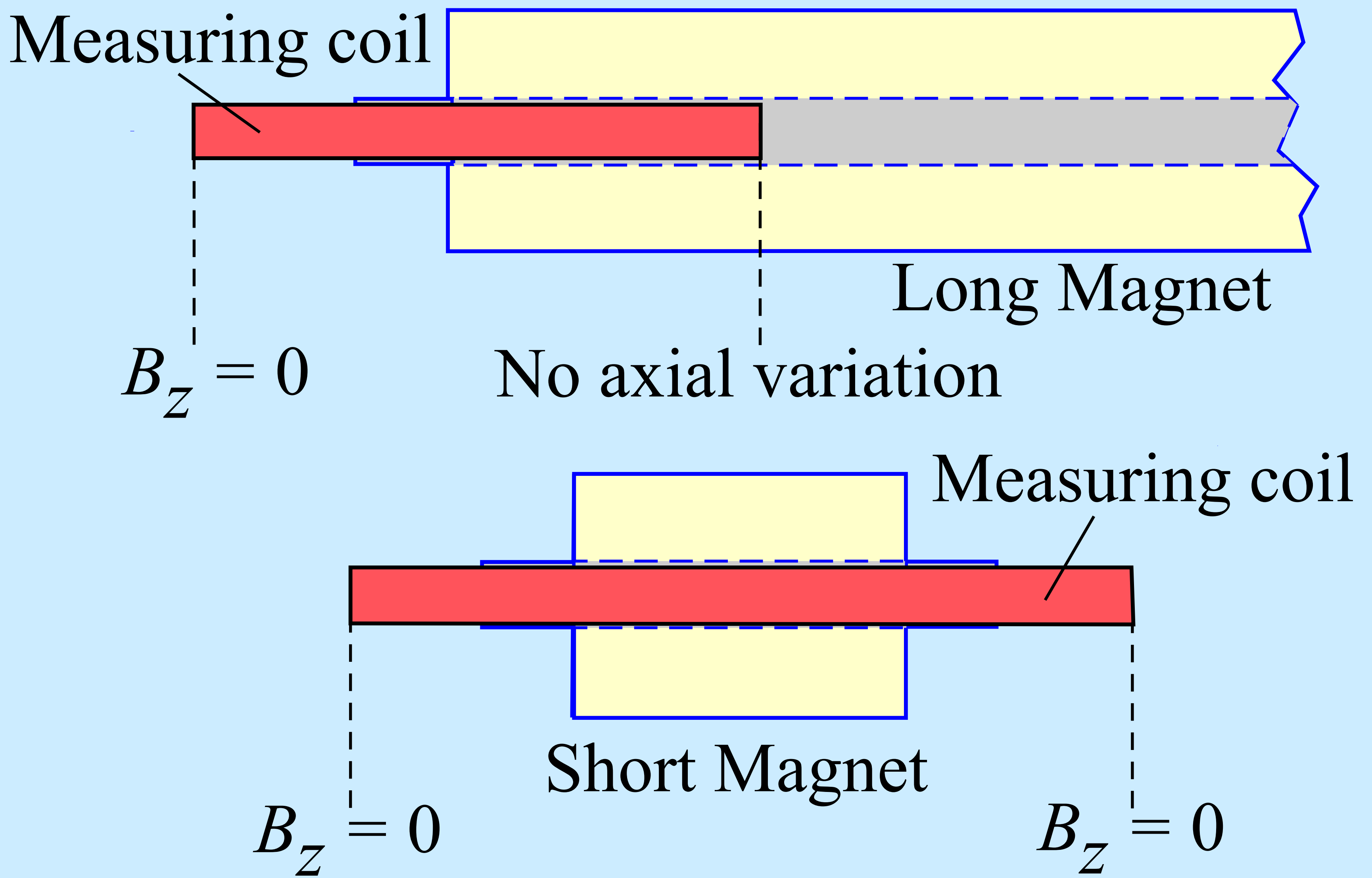
Maxwell's equations = Cauchy-Riemann conditions if we choose: $U(x,y) = B_y(x,y)$ and $V(x,y) = B_x(x,y)$

Thus, $\boldsymbol{B}(z) = B_y(x,y) + i\, B_x(x,y)$ is an analytic function of z. The analyticity is useful in dealing with 2-D problems in magnetostatics.

End Fields & Short Magnets

- The field near the ends of a long magnet, or everywhere in a short magnet, has all three components. The simple 2-D expansion is not valid in these cases. However, if one considers only integrated values of field components, a similar 2-D expansion can be shown to be valid.
- For components of field at a point, a more complex expansion must be used.

Validity of 2-D Field Expansion



3-D Field Expansion

If the field harmonics vary along the axial direction, Z:

$$B_r(r,\theta,z) = \sum_{n=1}^{\infty}\left[B_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l (n-1)!(2l+n)}{2^{2l}\, l!(l+n)!} B_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1} \sin(n\theta)$$

$$+ \sum_{n=1}^{\infty}\left[A_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l (n-1)!(2l+n)}{2^{2l}\, l!(l+n)!} A_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1} \cos(n\theta)$$

$$B_\theta(r,\theta,z) = \sum_{n=1}^{\infty}\left[B_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l n!}{2^{2l}\, l!(l+n)!} B_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1} \cos(n\theta)$$

$$- \sum_{n=1}^{\infty}\left[A_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l n!}{2^{2l}\, l!(l+n)!} A_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1} \sin(n\theta)$$

where the index [2ℓ] denotes (2ℓ)th derivative with respect to z. If integral values are considered between Z_1 and Z_2 such that all derivatives are zero at the ends, then the above expression reduces to the 2-D expansion.

Interpretation of Harmonics

$$B_{n+1}(\text{European}) = B_n(\text{US}) = \frac{R_{ref}^n}{n!}\left(\frac{\partial^n B_y}{\partial x^n}\right)\Bigg|_{x=0;y=0}$$

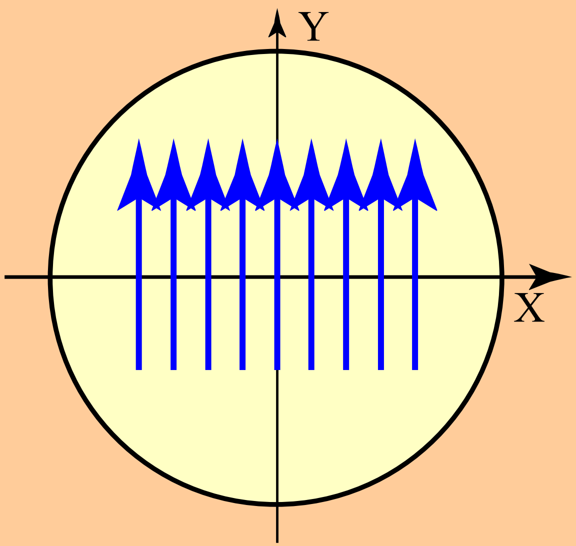
$$A_{n+1}(\text{European}) = A_n(\text{US}) = \frac{R_{ref}^n}{n!}\left(\frac{\partial^n B_x}{\partial x^n}\right)\Bigg|_{x=0;y=0} \qquad n \geq 0$$

$B_y = \text{Constant} \Rightarrow$ Dipole Only

$(dB_y/dx) = \text{Constant} \Rightarrow$ Dipole plus Quadrupole

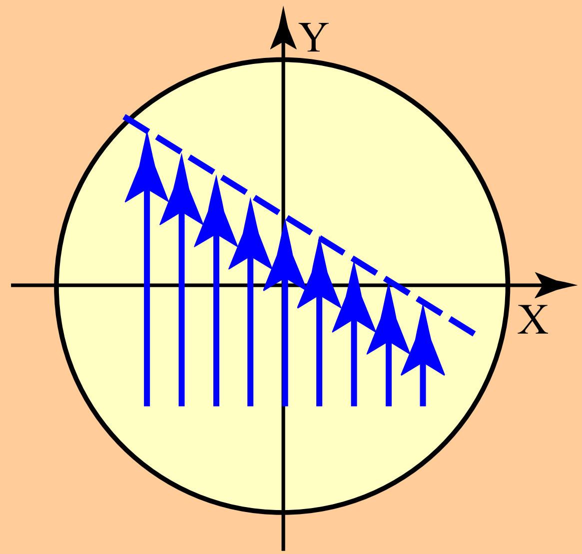
and so on ...

Examples of Harmonics



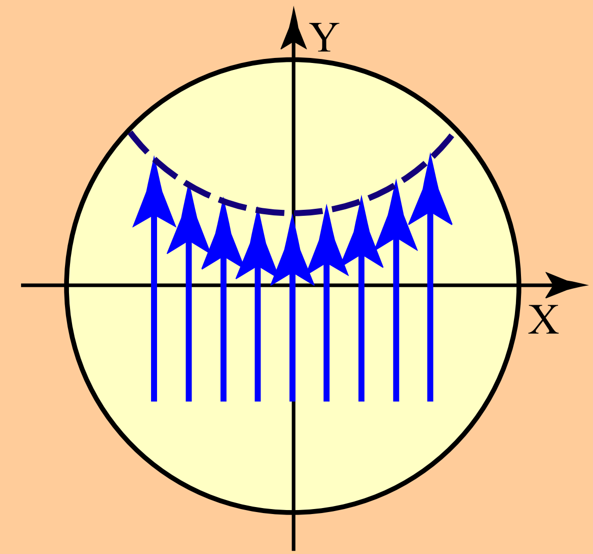
$$B_y = B_0 \text{ (Constant)}$$

Normal Dipole



$$B_y = B_0 + G.x$$

Normal Dipole
+
Normal Quadrupole



$$B_y = B_0 + A.x^2$$

Normal Dipole
+
Normal Sextupole

Normalized Coefficients: Multipoles

- The coefficients B_n and A_n denote the absolute strength of the *n*-th harmonic, and are thus a function of the magnet excitation.
- The variation in the *shape* of the field as a function of excitation is best described using coefficients normalized by a suitable ***reference field***, often chosen to be the amplitude of the most dominant term in the harmonic expansion. The normalized coefficients are also referred to as ***multipoles***.

Normalized Coefficients: Multipoles

$$B_y + iB_x = \sum_{n=n_0}^{\infty}\left[B_n + iA_n\right]\left(\frac{x+iy}{R_{ref}}\right)^{n-n_0} \quad ;$$

$$\boxed{\begin{array}{c} n_0 = 0 : \text{US} \\ n_0 = 1 : \text{European} \end{array}}$$

$$= B_{ref}\sum_{n=n_0}^{\infty}\left[b_n + ia_n\right]\left(\frac{x+iy}{R_{ref}}\right)^{n-n_0} \quad ; \text{where}$$

$$b_n = B_n / B_{ref}; \qquad a_n = A_n / B_{ref}$$

For a 2m-pole magnet, $B_{ref} = |B_m + iA_m|$

(b_n,a_n) independent of current: LINEAR SYSTEM

$(b_n,a_n)\times 10^4$ = Normal & Skew Multipoles in "UNITS"

Properties of Harmonics

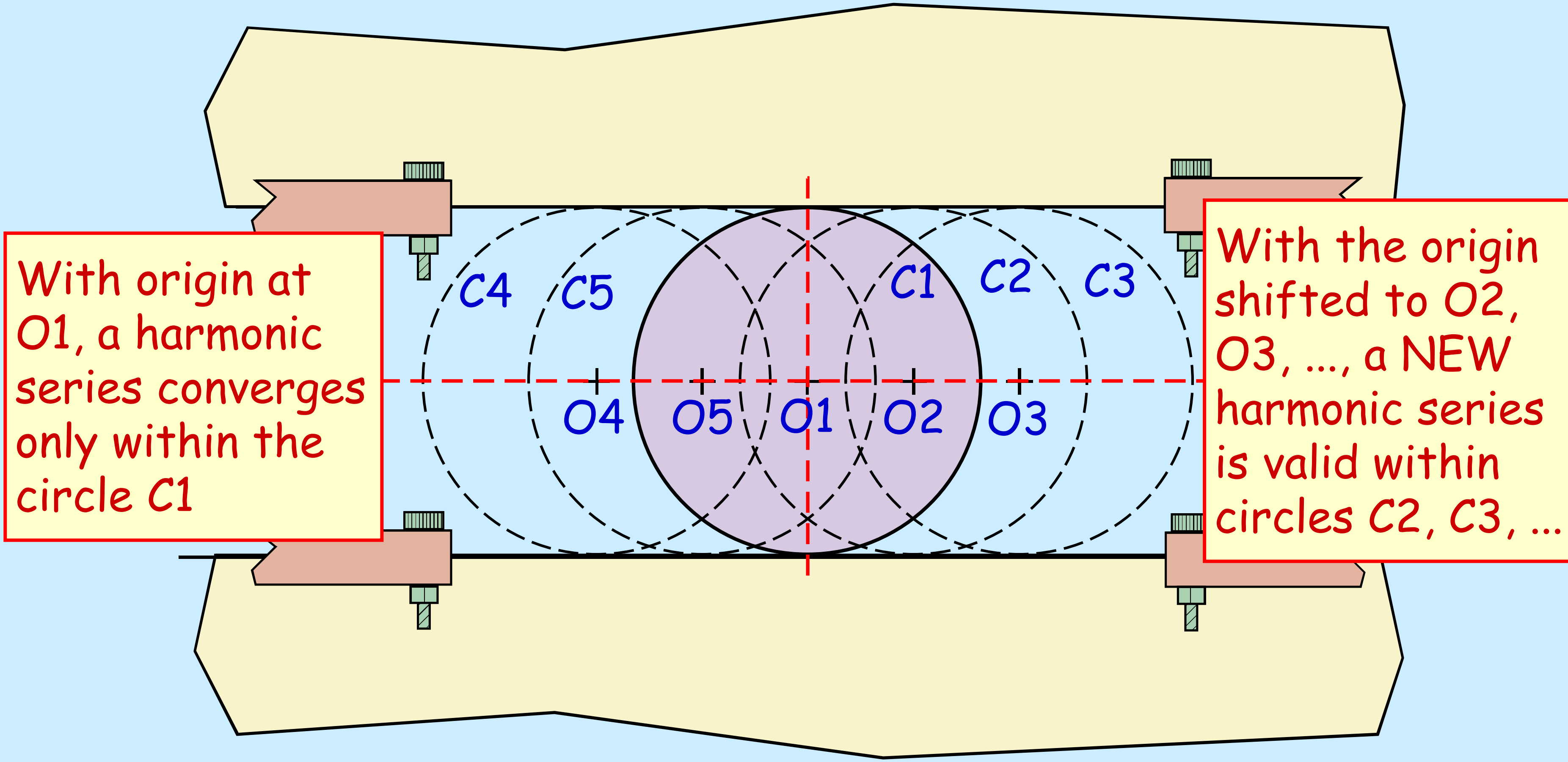
- The Normal and Skew harmonics represent coefficients of expansion in a power series for the field components.
- The harmonics allow computation of field everywhere in the aperture (within a circle of convergence) using only a few numbers.
- These coefficients obviously depend on the choice of origin and orientation of the coordinate system. Measured harmonics, therefore, often need to be ***centered*** and ***rotated***.

Field in a Non-circular Aperture

The 2-D field expansion in a harmonic series is valid only within the circle of convergence, which extends from the origin to the nearest current element or a magnetic material ("singularity").

For non-circular apertures, a single series expansion does not cover the entire "source-free region", even though the complex field $B_y + iB_x$ is an analytic function of $(x + iy)$ throughout the aperture. One can circumvent the problem by defining several series expansions, each centered at a different origin.

Field in a Non-circular Aperture



By having a significant overlap between the various circles of convergence, one can verify the integrity and accuracy of data by comparing results in the overlap regions.

The Vector Potential

Scalar potential approach does not provide a relationship between the currents and the field.

From Maxwell's equations:

$$\nabla.\mathbf{B} = 0; \quad \therefore \mathbf{B} = \nabla \times \mathbf{A}$$

A is called the Vector Potential

$$\nabla \times \mathbf{B} = \mu_0(\nabla \times \mathbf{H}) = \mu_0 \mathbf{J}$$

In "free space", $\mathbf{B} = \mu_0 \mathbf{H}$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \mu_0(\nabla \times \mathbf{H}) = \mu_0 \mathbf{J}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Poisson's Equation

$$\mathbf{A}(\mathbf{r}) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

Summary

- The 2-D field , away from the ends, in the aperture of a typical accelerator magnet, can be described by a simple power series, valid within a circle extending to the nearest current source or magnetic material.
- A similar 2-D expansion is also valid for 3-D fields if one considers only integrated values of the field components such that there is no axial variation at the boundaries of the integration interval.

Summary (Contd.)

- The expansion coefficients may be interpreted as spatial derivatives of the field components.
- The expansion coefficients, or harmonics, depend on the choice of coordinate frame. This demands a careful description of the frame when quoting results of measurements. Similarly, users of the data also need to pay close attention to the coordinate definition.

Summary (Contd.)

- The complex field, $\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x$, is an analytic function of the complex variable, $\boldsymbol{z}$.
- For non-circular apertures, one can describe the field in the entire aperture by defining several series expansions centered at different points in the aperture (***analytic continuation***).
- Scalar potential approach is unsuitable for establishing a relationship between the current and the field. A vector potential approach is more general.