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SCALING OF STORAGE RING PARAMETERS

OUTLINE

1. Beam-beam tune shift parameter $\xi_{\max}$ expected for VLHC.
2. Empirical scaling relation between Q and R.
3. Scaling relations that follow from linear and near-linear optics of the arcs.
4. "Nominal" scaling, $L_c \sim \sqrt{R}$, and the resulting B-dependencies.
5. Adiabatic scaling.
6. Radiation-dominated scaling.
7. Scaling of magnet costs. Minimization of total VLHC cost.
8. Determination of integral and fractional tune values.
9. Parameter list.
10. Conclusions.

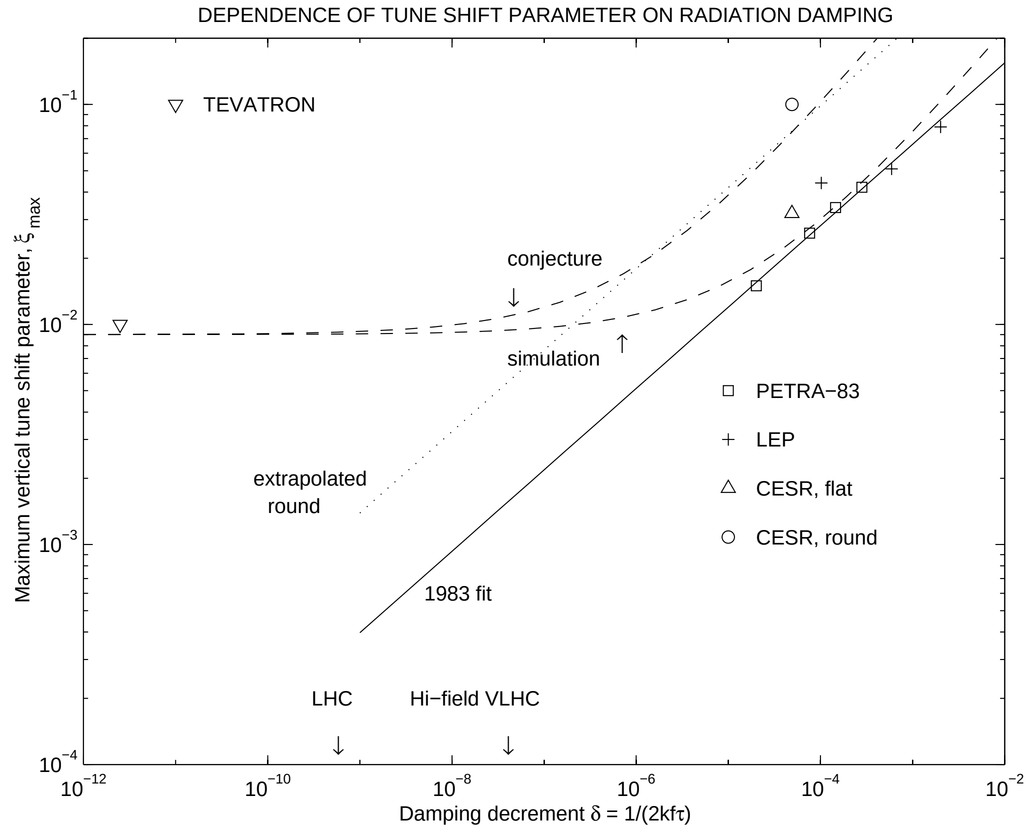

Figure 1: Dependence of maximum vertical tuneshift parameter $\xi_{\rm max}$ on damping decrement $1/(2kf\tau)$, where k is number of bunches, f is revolution frequency, and τ is damping time. The line labeled "1983 fit" was conjectured in 1983 by Keil and Talman based on data available at the time; it describes recent LEP data well. The curve labeled "simulation" linking the ultralow (proton) and ultrahigh (electron) regions is due to Peggs. The curve labeled "conjecture" is my fit (adjusting a parameter in the Peggs formula) to the Tevatron point and a (slightly downward adjusted) round beam CESR point.

- Conclusion: for likely proton colliders $\xi_{\rm max} \approx 0.01$.
- Synchrotron radiation makes flat beam operation practical in the VLHC. This makes it possible to lower the product $\beta_x^*\beta_y^*$ at the collision point, which increases the luminosity. But (based on the CESR data) this supposed gain may be largely nullified by a roughly threefold reduction in $\xi_{\rm max}$.

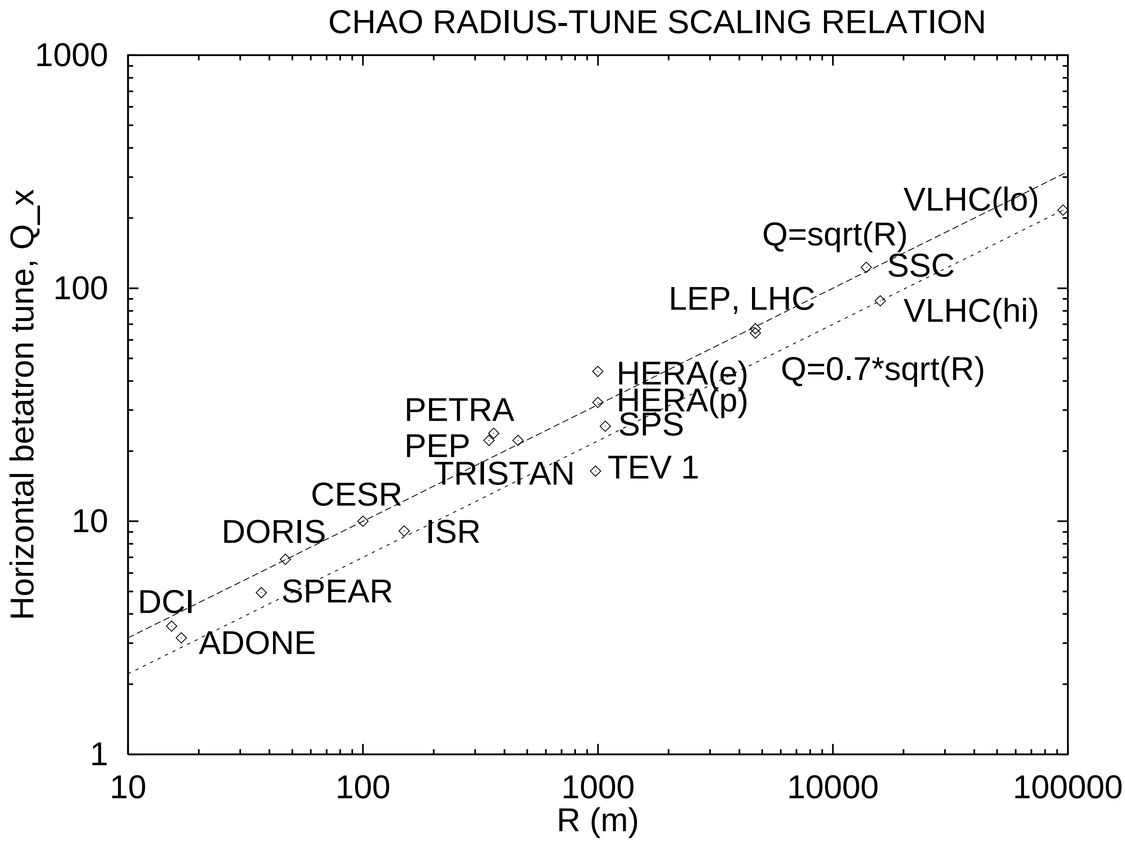

Figure 2: Empirical relation $Q \sim R^{1/2}$ between radius R and tune Q for existing high energy accelerators. Copied from Chao. The lower line has been added because the focusing seems to be systematically lower for proton than for electron accelerators. High and low field VLHC design options have been placed on the curve extrapolated to large R.

- For FODO cells of constant phase advance and length L_c, the relations $Q \sim R^{1/2}$ and $L_c \sim R^{1/2}$ are equivalent. This will be referred to as "nominal scaling".
- This $Q \sim R^{1/2}$ scaling is (empirically) the same for hadrons and electrons. But the energy scaling is different. For hadrons (at fixed B) $R \sim \gamma$. For electrons $R \sim \gamma^2$.
- The SSC point has been added in response to a question at the session.

Table 1: Linear Optics scaling with cell length L_c and mean radius R, of various beam parameters. Relativistic factor γ, "fill factor" ρ/R and emittances ϵ_x and ϵ_s are held constant. Where equality (=) or approximate equality ($\approx$) are given, 90 degree cells are assumed. The factors ($\approx 10^{-2}$) are order of magnitude factors by which a dimensionally-consistent estimate (the amplitude at which quadrupole and sextupole deflections are equal) tends to overestimate the dynamic aperture. All quantities, including multipole coefficients, are in MKS units. Based on table due to Tanaji Sen and my paper, NIM, A450,207(2000). This table is supposed to be helpful in choosing L_c with R fixed.

Parameter	Symbol		coefficient	dependence
Bend angle/half-cell	$\Delta\theta$	=	$1/2$	$L_c R^{-1}$
Number of half cells	n	=	4π	$L_c^{-1} R$
Arc tune	Q_a	=	$\pi/2$	$L_c^{-1} R$
Strength of half quad	q	$\approx$	$\sqrt{2}$	L_c^{-1}
Beta function (mean/maximum)	β_x	$\approx$	$2/\pi, 1.71$	L_c
Horizontal betatron size (ϵ_x fixed)	σ_β	$\sim$		$L_c^{1/2}$
Dispersion (mean/maximum)	η	$\approx$	$4/\pi^2, 0.66$	$L_c^2 R^{-1}$
Slip factor $\approx$ momentum compaction	$\eta_{syn} \approx 1/\gamma_t^2$	$\sim$		$L_c^2 R^{-2}$
Matched (fractional) momentum spread	$\sigma_\delta^{(\mathrm{match})} = \sigma_\beta/\eta$	$\sim$		$L_c^{-3/2} R$
Natural chromaticity (arcs only)	Q'_a	$\approx$	$\pi/2$	$L_c^{-1} R$
Chromaticity due to b_2 in dipoles	Q'_{b2}	$\approx$	$(8/\pi^3)\, b_2$	$L_c^3 R^{-1}$
Rough average half-sext.(chr-corr)	S	$\approx$	$\pi^2/2^{3/2}$	$L_c^{-3} R$
Tune shift with amplitude due to b_n		$\sim$		$b_n L_c^{(n+1)/2}$
Dynamic aperture due to	x_{da}		big is good	
(i) chromaticity sextupoles		$\approx$	$(\approx 10^{-2})(8/\pi^2)$	$L_c^2 R^{-1}$
(ii) systematic b_2 error		$\approx$	$(\approx 10^{-2})2\sqrt{2}(1/b_2)$	$L_c^{-2} R$
(iii) random b_2 error	?	$\approx$	$(\approx 10^{-2})\sqrt{2/\pi}(1/\sigma_{b2})$	$L_c^{-3/2} R^{1/2}$

- The numerical value of the scaling ratio that corresponds to the lower line in Fig. 2 is $L_c^2/R = 5.0\,\mathrm{m}$.
- Note that, for "nominal" scaling, mean dispersion $\eta \approx 5.0 \times 4/\pi^2 \approx 2\,\mathrm{m}$ is constant, independent of R.
- The dynamic aperture due to chromaticity sextupoles is also approximately constant $x_{\mathrm{da}} \approx (\approx 10^{-2})5.0 \times 8/\pi^2 \approx 4\,\mathrm{cm}$. (Of course this neglects strong tune-dependent, resonant, sensitivity to tune values.)
- It follows that any design in which magnet costs are reduced by drastic reduction in bore diameter, will entail drastic deviation from the nominal $L_c \sim \sqrt{R}$ design.
- Since $x_{\mathrm{da}}^{chr.}$ and $x_{\mathrm{da}}^{b_2}$ depend inversely on L_c^2/R the dynamic aperture is maximized when they are equal. Taking the factors ($\approx 10^{-2}$) equal for the two cases (which is certainly unjustified) yields

$$b_2 \approx \frac{2\sqrt{2}\,\pi^2}{8} \left(\frac{L_c^2}{R}\right)^{-2} \approx 0.1\,\mathrm{m}^{-2}$$

which is the same as 0.1 "units", i.e. 0.1 parts per 10^4 at one centimeter. This estimate is "in the ballpark" of modern magnet technology. This means that the general line of reasoning is at least not ruled out.

- (Whether or not it reflects historical design decisions accurately) the "nominal" $L_c \sim \sqrt{R}$ proportionality causes the dynamic apertures due to chromaticity sextupoles and due to systematic dipole nonlinearity to be "matched".
- "Better" (i.e. smaller systematic b_2) magnets permit L_c to be *increased,* but this is probably not the direction one wants to go to reduce emittance growth due to stochastic effects.

Table 2: Nominal ($L_C \sim \sqrt{R}$) scaling, expressed as exponent of the power law dependence on magnetic field B, for various beam parameters. This table is identical to Table 1 except $L_c \to \sqrt{R}$ followed by $R \to B^{-1}$, and all coefficients are suppressed. This table is supposed to be helpful in choosing B assuming $L_C \sim \sqrt{R}$, which is the "nominal" dependence.

Parameter	Symbol	scaling	B-exponent after $L_c \to \sqrt{R}, R \to B^{-1}$
Radius	R	R	-1
Cell length	L_c	L_c	$-1/2$
Bend angle/half-cell	$\Delta\theta$	$L_c R^{-1}$	$1/2$
Number of half cells	n	$L_c^{-1} R$	$-1/2$
Arc tune	Q_a	$L_c^{-1} R$	$-1/2$
Strength of half quad	q	L_c^{-1}	$1/2$
Beta function	β_x	L_c	$-1/2$
Horizontal betatron size (ϵ_x fixed)	σ_β	$L_c^{1/2}$	$-1/4$
Dispersion	η	$L_c^2 R^{-1}$	0
Slip factor $\approx$ momentum compaction	η_{syn}	$L_c^2 R^{-2}$	1
Matched (fractional) momentum spread	$\sigma_\delta^{(\mathrm{match})}$	$L_c^{-3/2} R$	$-1/4$
Natural chromaticity (arcs only)	Q'_a	$L_c^{-1} R$	$-1/2$
Chromaticity due to b_2 in dipoles	Q'_{b2}	$L_c^3 R^{-1}$	$(b_2), -1/2$
Rough average half-sext.(chr-corr)	S	$L_c^{-3} R$	$1/2$
Tune shift with amplitude due to b_n		$b_n L_c^{(n+1)/2}$	$(b_2), -(n+1)/4$
$\sqrt{\text{Dynamic acceptance}}$ due to	$x_{\mathrm{da}}/\sqrt{\beta_x}$	big is good	
(i) chromaticity sextupoles		$L_c R^{-1}$	$1/2$
(ii) systematic b_2 error		$L_c^{-3} R$	$(1/b_2), 1/2$
(iii) random b_2 error	?	$L_c^{-5/2} R^{1/2}$	$(1/\sigma_{b2}), 3/4$

- "Second order" effects are neglected. e.g. increasing B reduces L_c and increases q which would necessitate reducing the filling factor.
- Increasing B helps dynamic acceptance. This will translate to substantially greater luminosity!

Table 3: Energy dependent scaling ("adiabatic damping") with relativistic factor γ. RF frequency ω_{rf} and synchronous phase ϕ_0, are held constant.

Adiabatically varying quantity	Symbol	dependence
Horizontal emittance	$\epsilon_x^{(ad)}$	γ^{-1}
Betatron beam size	$\sigma_\beta^{(ad)}$	$L_c^{1/2}\gamma^{-1/2}$
Betatron matched momentum spread	$\sigma_\beta^{(ad)}/\eta$	$L_c^{-3/2}R\gamma^{-1/2}$
RF voltage V_{rf} constant		
Bunch length	$\sigma_z^{(ad)}$	$L_c^{1/2}R^{-1/2}\gamma^{-1/4}$
Fractional momentum spread	$\sigma_\delta^{(ad)}$	$L_c^{-1/2}R^{1/2}\gamma^{-3/4}$
$V_{rf} \sim \gamma$		
Bunch length	$\sigma_z^{(ad)}$	$L_c^{1/2}R^{-1/2}\gamma^{-1/2}$
Fractional momentum spread	$\sigma_\delta^{(ad)}$	$L_c^{-1/2}R^{1/2}\gamma^{-1/2}$

- These dependencies are mainly applicable to hadron accelerators.
- Comparing the momentum spread entries, $\sigma_\delta^{(ad)}$ does not naturally stay "matched" to $\sigma_\beta^{(ad)}$ during energy "ramping", unless the RF voltage satisfies $V_{rf} \sim \gamma$.
- It would not be valid, in general, to assume quantities held fixed in Table 3 are equal in different machines. But if they were held fixed, matching betatron and energy contributions to transverse displacement ((third entry)/(last entry)$=\sqrt{R}/L_c$) would yield the scaling relation $L_c \sim \sqrt{R}$. This is therefore another possible explanation for the "nominal" L_c, R scaling.

Review the Bases for "Nominal" $L_c \sim R^{1/2}$ scaling.

Since the arc tune Q_a dominates the total tune Q, the "Chao relation" $Q \sim R^{1/2}$ implies $L_c \sim R^{1/2}$ (for cells with fixed phase advance.) But what is the basis for this scaling?

- "Nominal" scaling tends to balance dynamic aperture restrictions due to chromaticity correction and due to magnet nonuniformity.
- In hadron accelerators "nominal" scaling is consistent with keeping betatron oscillation demand for aperture matched to synchrotron oscillation demand for aperture.
- No one ever got fired for using the same algorithm for designing the next accelerator as was used for designing the present one.
- Further suggestions are welcome.

Table 4: Radiation-dominated scaling of equilibrium values of various beam parameters.

Parameter	Symbol	dependence	B, γ-dep.
Critical energy	u_c	$R^{-1}\gamma^3$	$B\gamma^2$
Energy loss per turn	U_o	$R^{-1}\gamma^4$	$B\gamma^3$
Damping decrement	$\alpha^{(\mathrm{rad})} = U_0/(2\gamma mc^2)$	$R^{-1}\gamma^3$	$B\gamma^2$
Emittance growth per turn	$\Delta\epsilon_x^{(\mathrm{rad})}$	$L_c^3 R^{-4}\gamma^5$	$B^{5/2}\gamma^{5/2}$
Horizontal emittance	$\epsilon_x^{(\mathrm{rad})} \approx \Delta\epsilon_x^{(\mathrm{rad})}/(4\alpha^{(\mathrm{rad})})$	$L_c^3 R^{-3}\gamma^2$	$B^{3/2}\gamma^{1/2}$
Betatron part of beam width	$\sigma_\beta^{(\mathrm{rad})}$	$L_c^2 R^{-3/2}\gamma$	$B^{1/2}\gamma^{1/2}$
(Energy growth)-squared per turn	$\Delta\epsilon_\delta^{(\mathrm{rad})}$	$R^{-2}\gamma^5$	$B^2\gamma^3$
Energy spread	$\sigma_\delta^{(\mathrm{rad})} \approx \sqrt{\Delta\epsilon_\delta^{(\mathrm{rad})}/(8\alpha^{(\mathrm{rad})})}$	$R^{-1/2}\gamma$	$B^{1/2}\gamma^{1/2}$
Energy part of beam width	$\eta\sigma_\delta^{(\mathrm{rad})}$	$L_c^2 R^{-3/2}\gamma$	$B^{1/2}\gamma^{1/2}$

- These dependencies apply to electrons and, some hours after filling (in the absence of other stochastic effects), to protons.
- The beam sizes $\sigma_\beta^{(\mathrm{rad})}$, due to betatron oscillation, and $\eta\sigma_\delta^{(\mathrm{rad})}$, due to energy oscillation, scale identically. This includes the result that their ratio is (approximately) invariant around the ring (arc sections) since $\eta(s) \sim \beta^{1/2}(s)$.
- Review of familiar argument: for electron rings the scaling relation $R \sim \gamma^2$ is usually said to be the result of minimizing (at fixed γ) the sum of lattice cost C_L and RF cost C_rf;

$$\frac{d\left(C_\mathrm{L}+C_\mathrm{rf}\right)}{dR} = \frac{d}{dR}\left(2\pi c_L R + c_\mathrm{rf}\gamma^4 R^{-1}\right) = 0\ .$$

Because the terms have inverse R dependencies, the minimum occurs with circumferential cost and RF cost equal,

$$C_\mathrm{L} = C_\mathrm{rf} \quad \text{and} \quad R = \sqrt{\frac{c_\mathrm{rf}\gamma^4}{2\pi c_L}} \sim \gamma^2\ .$$

Note: *ratio of terms at optimum = (inverse) ratio of exponents.*

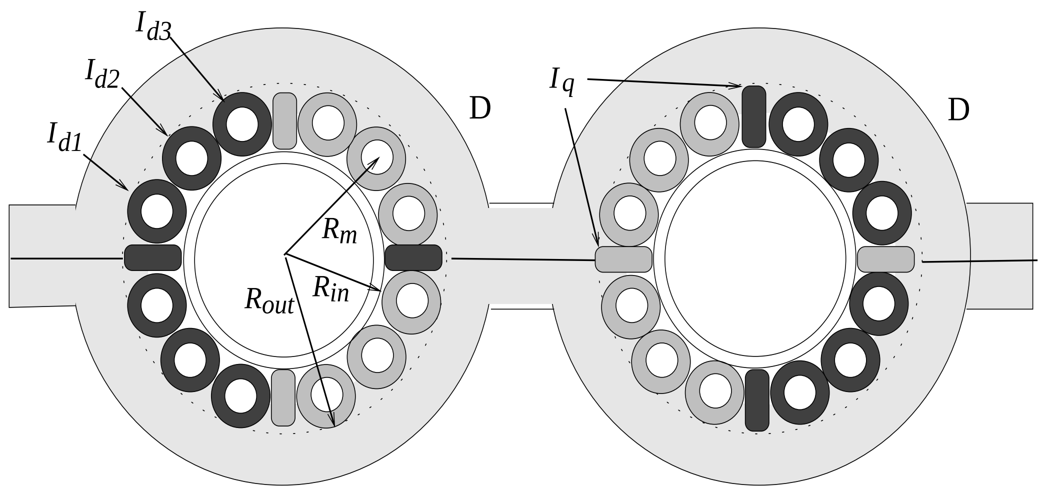

Figure 3: Suggested current distribution for ironless, "end-free", combined function VLHC magnets. There are three independent bend-like currents, I_{d1}, I_{d2}, I_{d3}, and one quad-like current I_q.

- This design will next be used to estimate the dependence of stored magnetic energy (and hence cost) on magnetic field B.
- The magnet is squashed or stretched sideways to produce chromaticity compensating sextupole field. Less than one percent deformation is actually required.
- Quadrupole windings have to be crossed at cell ends to convert from D to F focusing.

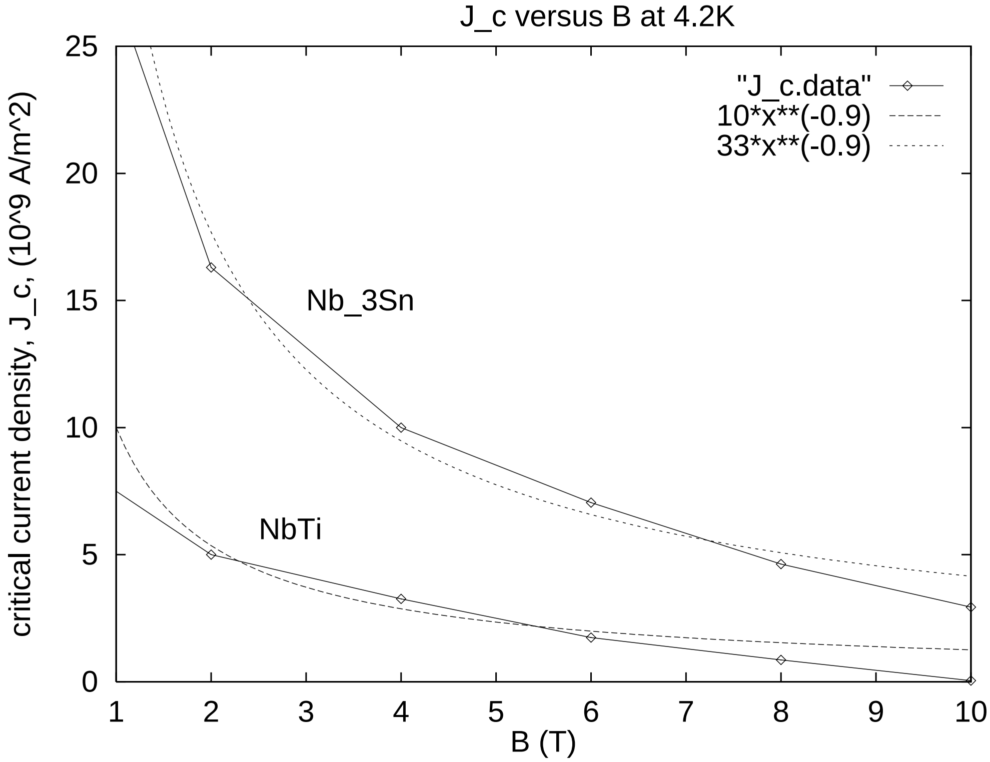

Figure 4: Critical current for NbTi and Nb_3Sn at 4.2K, over likely range for B.

- The LHC magnet uses NbTi at a magnetic field $B \approx 8\,\mathrm{T}$. This would be impractical without lowering the temperature to $1.8\,\mathrm{K}$ to increase J_c.
- Switching from NbTi to Nb_3Sn in the subsequent optimization would "buy" a factor of $3.3^{1/1.9} = 1.87$ increase in the optimal B field.

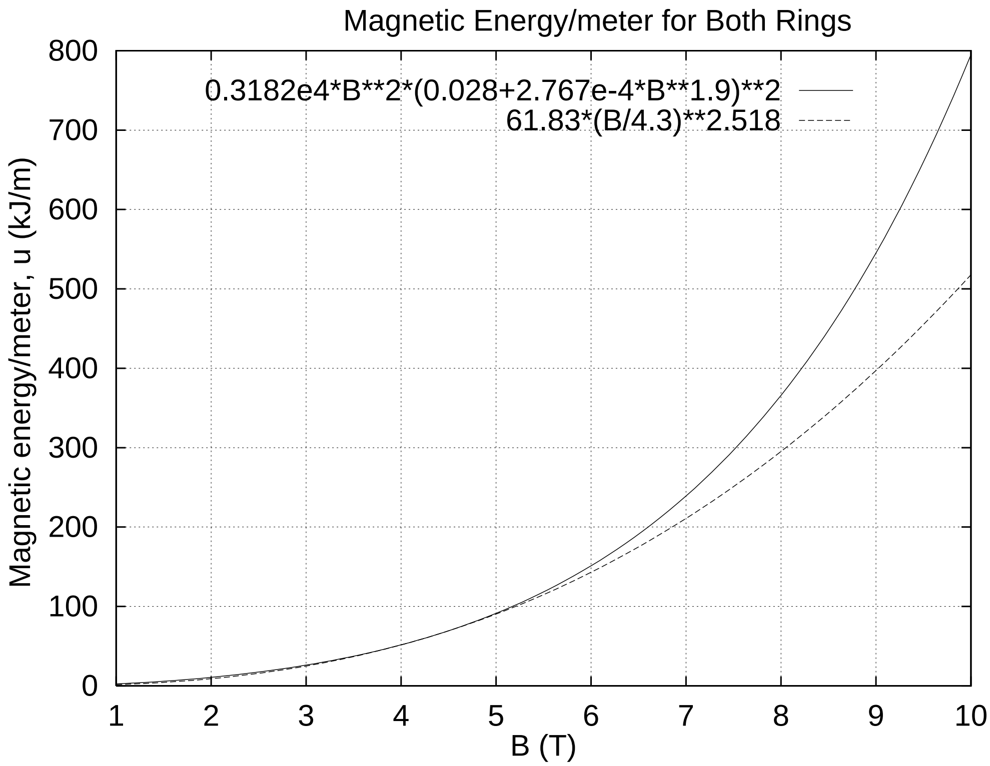

Figure 5: Stored energy per meter u (both rings). Also shown is a fit to the dependence $u \sim B^m$, yielding $m = 2.518$.

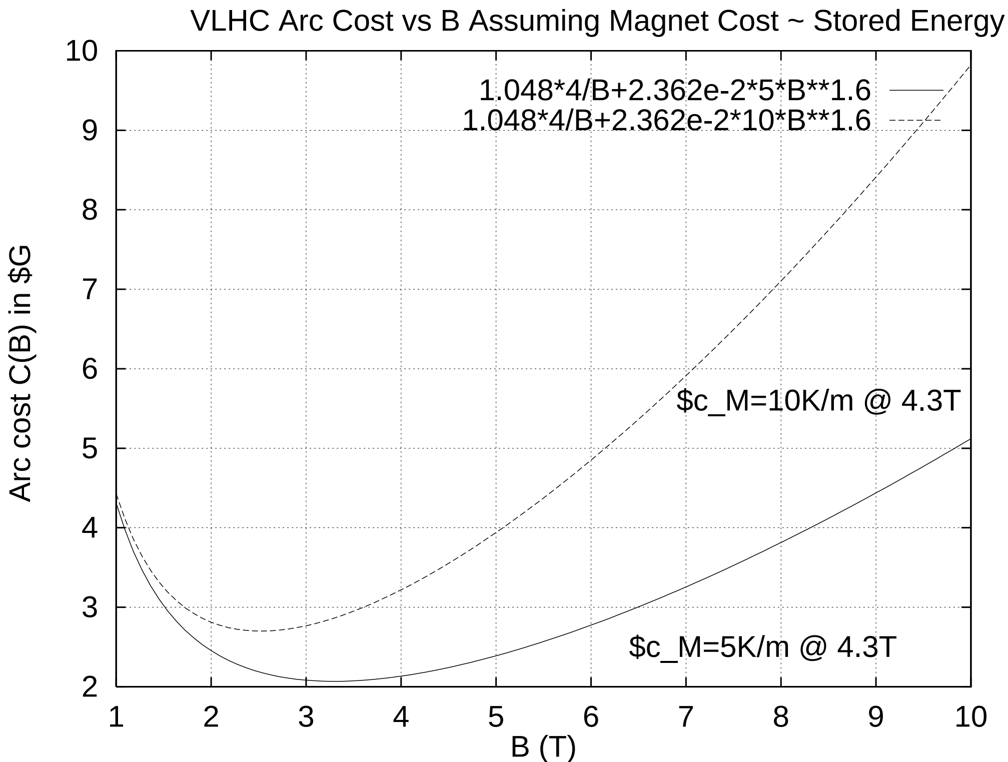

Figure 6: Arc cost $C_T + C_M$ (tunnel and magnet) in billions of dollars assuming magnet cost is proportional to stored magnetic energy.

Assume that magnet cost per meter scales as $c_M = c'_M B^m$, where $m = 2.6$. Total tunnel and magnet costs are

$$C_T = c_T\, 2\pi R \quad \left(= 1.048 \times 10^6\, c_T\, B^{-1} \right),$$
$$C_M = c'_M B^m\, 2\pi R \quad \left(= 2.362 \times 10^4\, c_{M,ref}\, B^{1.6} \right).$$

ratio of terms at optimum = (inverse) ratio of exponents.

$$B_{\rm opt} = \left(\frac{c_T}{(m-1)\, c'_M} \right)^{1/m}, \quad \frac{C_M}{C_T} = \frac{1}{m-1}. \qquad (1)$$

CHOOSING FRACTIONAL TUNES TO MINIMIZE FUNCTION OF (DE)MERIT (FOM) FOR GIVEN RANDOM ERRORS

100*FOM = percentage acceptance reduction at 10 sigma due to randoms

qx= \ qy =	0.270	0.278	0.286	0.294	0.302	0.310	0.318	0.326	0.334	0.342	0.350
0.260	11.5	9.5	8.8	8.5	8.6	8.9	9.7	11.1	34.4	12.4	13.6
0.264	14.3	10.1	9.0	8.7	8.7	9.0	10.0	11.3	34.7	12.8	14.5
0.268	28.9	11.3	9.5	8.9	8.9	9.2	10.6	11.5	35.0	13.3	15.6
0.272	28.9	14.3	10.2	9.3	9.1	9.4	xxxx	11.8	35.4	14.0	17.0
0.276	14.3	29.0	11.5	9.8	9.4	9.6	11.1	12.1	35.9	14.8	18.9
0.280	11.4	29.0	14.5	10.6	9.9	10.0	11.0	12.5	36.4	15.7	21.6
0.284	10.2	14.4	29.3	12.0	10.5	10.4	11.2	13.0	37.1	17.0	25.5
0.288	9.6	11.6	29.4	15.1	11.3	10.9	11.6	13.6	37.9	18.5	32.1
0.292	9.2	10.5	14.9	29.9	12.8	11.6	12.1	14.3	38.8	20.6	45.0
0.296	9.1	10.0	12.2	30.1	16.0	12.6	12.8	15.3	40.0	23.5	83.3
0.300	9.0	9.7	11.1	15.7	30.9	14.2	13.8	16.7	41.5	27.7	xxxx

- The nominal LHC fractional tunes $Q_x = 0.28$, $Q_y = 0.31$, are good because a 10 sigma scraper is "fuzzed out" by only 10% due to nonlinear motion. Some nearby tunes are (just a little bit) better.
- The same considerations and tunes are applicable to VLHC.

CHOOSING INTEGER TUNES TO MINIMIZE FUNCTION OF (DE)MERIT (FOM) FOR GIVEN SYSTEMATIC ERRORS

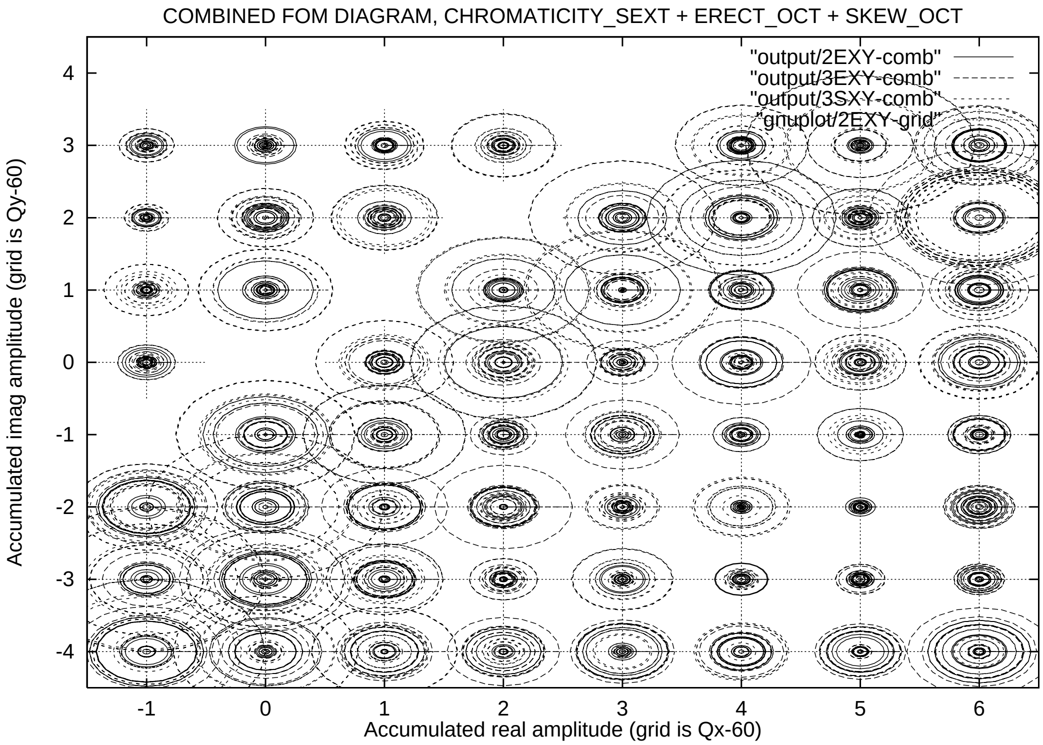

Figure 7: FOM values (plotted as ellipses centered on integer tune values (Q_x,Q_x), with chromaticity sextupoles plus erect octupole plus skew octupole systematic errors, all superimposed. Smallest blobs of ellipses are best. No points are plotted on the diagonal, which is hopelessly bad.

- A good choice of integer tunes with $Q_x > Q_y$ is $Q_x = 65$, $Q_y = 58$, but there are closer-together tune choices that are almost as good.
- These calculations were performed for the LHC.
- For the integer parts of the tunes in the following table of suggested VLHC parameters I have taken $Q_x/Q_y = 1.1$.

Table 5: VLHC PARAMETERS ($c_M = \$10\,\mathrm{K/m}$)

Parameter		units	Low field	Minimum cost	High field
B	field	Tesla	1.8	2.5	12
R	radius	km	92.7	66.7	13.9
$< Q >_{int}$	mean (int) tune		213	181	82
Q_x			224.26	190.26	86.26
Q_y			202.30	172.30	78.30
ℓ	half-cell length	m	341.6	289.9	132.3
q	half-quad str.	1/m	0.002071	0.002439	0.005345
β_{max}		m	1166	989.9	451.7
β_{typ}		m	483.1	409.9	187.1
η_{max}		m	3.342	3.342	3.342
$\sigma_{x,max}^{inj}$		mm	1.170	1.078	0.7279
σ_{δ}^{inj}		per/mil	0.3500	0.3224	0.2178
Id	dipole current	kA	74.93	106.7	1023
Iq	quad current	kA	2.102	3.070	58.74
R_{in}	min coil radius	mm	28	28	28
R_{out}	max coil radius	mm	29.69	31.16	90.15
R_m	mean coil radius	mm	28.85	39.58	59.07
A	coil 1/4 area	cm^2	0.447	0.856	33.62
u	stored energy/m	kJ/m	8.582	17.40	1599
"optical fiber"	max runout	m	40.0	26.6	3.74
	runout/cell	mm	0.88	0.812	0.55
C_T	tunnel cost	\$G	2.329	1.676	0.349
C_M	magnet cost	\$G	0.650	1.070	11.6
C_T	total cost	\$G	2.98	2.746	11.9

A Good Plan

- Phase I, low magnetic field, large radius
  - merits: cost optimized (tunnel plus ring)
  - demerits: modest energy, modest luminosity

- Phase II, high magnetic field, same tunnel
  - merits:

    high energy
    high luminosity
    high energy injection
    "inexpensive" (tunnel exists and plenty of time to develop magnet)

  - demerits: long time to completion