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POWER SERIES SOLUTION TO THE CLASSICAL THREE BODY PROBLEM

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Abstract

We present a general solution to the classical problem of three bodies, interacting under their mutual gravitational attractions, as a set of nine power series for the rectangular coordinates as functions of time. The coupled recurrence relations for the expansion coefficients are derived directly from the differential equations of motion by integration in the complex t-plane. This solution is applied in two illustrative cases: the first, a restricted problem with one of the three masses equal to zero, the second, a more general example with the three bodies having nonzero masses.

In our accompanying note,[1] we have shown how the initial value problems that arise in nonlinear mechanics can be solved by power series. The recurrence relations for the expansion coefficients were obtained directly from the differential equations by integration in the complex time-plane. Here we would like to present this type of solution for the system of three masses moving under the mutual action of their Newtonian gravitational attractions - the three body problem.[2-4] Since the notation for three bodies, each with three rectangular coordinates, can get rather confusing, we shall use Greek subscripts for the designation of the bodies and Latin superscripts for the rectangular coordinates. Thus the coordinates of the body named μ are written as

$$f_\mu^k = (f_\mu^1,\ f_\mu^2,\ f_\mu^3) = (f_\mu,\ g_\mu,\ h_\mu)\ , \tag{1}$$

where μ can take on the values 1, 2, or 3. The acceleration of a body under the gravitational action of the other two bodies is accordingly

$$f_\mu^{k(2)} = -k^2 \sum_{\nu \neq \mu} \frac{m_\nu (f_\mu^k - f_\nu^k)}{\left[(f_\mu^\ell - f_\nu^\ell)(f_\mu^\ell - f_\nu^\ell)\right]^{3/2}}\ . \tag{2}$$

The repeated Latin index indicates a summation over ℓ = 1, 2, and 3, while for the Greek indices a sum is indicated explicitly. The units employed in Eq. (2) are those adopted in celestial mechanics:[2,3,5] all masses are in units of the solar mass, distances are in astronomical units, and times are in mean solar days. The multiplying factor (k^2) is the square of the Gaussian gravitational constant $\left(k = 0.01720209895\ \mathrm{day}^{-1}\right)$. For the solutions of these nine second-order differential equations, we assume nine power series for the rectangular coordinates of the three bodies (μ = 1, 2, 3), that is,

$$f_\mu = \sum_{n=0} a_{\mu,n}(t-z_0)^n , \tag{3a}$$

$$g_\mu = \sum_{n=0} b_{\mu,n}(t-z_0)^n , \tag{3b}$$

and

$$h_\mu = \sum_{n=0} c_{\mu,n}(t-z_0)^n . \tag{3c}$$

It is the coupled recurrence relations for the expansion coefficients ($a_{\mu,n}$; $b_{\mu,n}$; $c_{\mu,n}$) that we must derive. To illustrate the procedure, we write out the first equation ($\mu = 1$, $k = 1$) of Eq. (2) as

$$f_1^{(2)} = -\frac{k^2 m_2(f_1-f_2)}{\left((f_1-f_2)^2+(g_1-g_2)^2+(h_1-h_2)^2\right)^{3/2}} - \frac{k^2 m_3(f_1-f_3)}{\left((f_1-f_3)^2+(g_1-g_3)^2+(h_1-h_3)^2\right)^{3/2}} . \tag{4}$$

Integrating this differential equation according to the method of Ref. (1), we obtain

$$n(n+1)a_{1,n+1} = -\frac{k^2 m_2}{2\pi i}\oint \frac{dw}{(w-z_0)^n}\frac{f_1-f_2}{\left((f_1-f_2)^2+(g_1-g_2)^2+(h_1-h_2)^2\right)^{3/2}} - \frac{k^2 m_3}{2\pi i}\oint \frac{dw}{(w-z_0)^n}\frac{f_1-f_3}{\left((f_1-f_3)^2+(g_1-g_3)^2+(h_1-h_3)^2\right)^{3/2}} . \tag{5}$$

To proceed further, it is necessary to find the power series expansions for some of the factors within the contour integrals of this equation. Thus, a typical one is

$$f_1 - f_2 = \sum_{n=0} (a_{1,n} - a_{2,n})(w-z_0)^n . \tag{6}$$

For the denominator it is best to go step-by-step. Firstly,

$$(f_1-f_2)^2 + (g_1-g_2)^2 + (h_1-h_2)^2 = \sum_{p=0} A_p(12)(w-z_0)^p \ , \tag{7}$$

wherein

$$A_p(12) = \sum_{\ell=0}^{p} \Big\{ (a_{1,\ell} - a_{2,\ell})(a_{1,p-\ell} - a_{2,p-\ell}) + (b_{1,\ell} - b_{2,\ell})(b_{1,p-\ell} - b_{2,p-\ell}) + (c_{1,\ell} - c_{2,\ell})(c_{1,p-\ell} - c_{2,p-\ell}) \Big\} . \tag{8}$$

Secondly, we write

$$\big((f_1-f_2)^2 + (g_1-g_2)^2 + (h_1-h_2)\big)^{3/2} = \sum_{n=0} B_n(12)(w-z_0)^n \ , \tag{9}$$

with the sequence of coefficients, $B_n(12)$, given by

$$B_0(12) = \Big(A_0(12)\Big)^{3/2} \ , \tag{10a}$$

$$B_1(12) = \frac{3}{2B_0(12)} A_1(12) \Big(A_0(12)\Big)^2 \ , \tag{10b}$$

and for $n \geq 2$

$$B_n(12) = \frac{1}{nB_0(12)} \Big\{ - \sum_{\ell=1}^{n-1} \ell B_\ell(12) B_{n-\ell}(12) + \frac{3}{2} \sum_{p=0}^{n-1} (n-p) A_{n-p}(12) \sum_{\ell=0}^{p} A_\ell(12) A_{p-\ell}(12) \Big\} \ . \tag{10c}$$

Thirdly, we find for the reciprocal of Eq. (9) the series

$$\left[(f_1-f_2)^2 + (g_1-g_2)^2 + (h_1-h_2)^2\right]^{-3/2} = \sum_{n=0} C_n(12)(w-z_0)^n \ , \tag{11}$$

with coefficient

$$C_0(12) = \frac{1}{B_0(12)} \ , \tag{12a}$$

and for $n \geq 1$

$$C_n(12) = -\frac{1}{B_0(12)} \sum_{k=1}^{n} C_{n-k}(12) B_k(12) \ . \tag{12b}$$

Substitution of Eq. (11) and the similar relation involving $C_n(13)$ into Eq. (5) now yields

$$n(n+1)a_{1,n+1} = -\frac{k^2 m_2}{2\pi i} \sum_{\ell=0} C_\ell(12) \oint \frac{dw(f_1-f_2)}{(w-z_0)^{n-\ell}}$$
$$- \frac{k^2 m_3}{2\pi i} \sum_{\ell=0} C_\ell(13) \oint \frac{dw(f_1-f_3)}{(w-z_0)^{n-\ell}} \ . \tag{13}$$

Whence the first recurrence relation ($n \geq 1$) follows, i.e.

$$n(n+1)a_{1,n+1} = -k^2 m_2 \sum_{\ell=0}^{n-1} C_\ell(12)(a_{1,n-\ell-1} - a_{2,n-\ell-1})$$

$$-k^2 m_3 \sum_{\ell=0}^{n-1} C_\ell(13)(a_{1,n-\ell-1} - a_{3,n-\ell-1}) \ . \tag{14a}$$

The remaining eight recurrence relations are similar to Eq. (14a) with the indices appropriately permuted. For the sake of some completeness we list those for the other two rectangular coordinates of the body numbered one:

$$n(n+1)b_{1,n+1} = -k^2 m_2 \sum_{\ell=0}^{n-1} C_\ell(12)(b_{1,n-\ell-1} - b_{2,n-\ell-1})$$

$$-k^2 m_3 \sum_{\ell=0}^{n-1} C_\ell(13)(b_{1,n-\ell-1} - b_{3,n-\ell-1}) , \tag{14b}$$

and

$$n(n+1)c_{1,n+1} = -k^2 m_2 \sum_{\ell=0}^{n-1} C_\ell(12)(c_{1,n-\ell-1} - c_{2,n-\ell-1})$$

$$-k^2 m_3 \sum_{\ell=0}^{n-1} C_\ell(13)(c_{1,n-\ell-1} - c_{3,n-\ell-1}) . \tag{14c}$$

In short, the power series expansions, Eq. (3), combined with the nine coupled recurrence relations [see Eq. (14)], are the general solutions to the ordinary differential equations, Eq. (2). These series are all convergent within the circle of convergence of smallest radius [see Ref. (1)]. The initial conditions of the three masses are specified by the eighteen coefficient ($a_{1,0}$; $a_{1,1}$: $b_{1,0}$: $b_{1,1}$; $c_{1,0}$; $c_{1,1}$; $a_{2,0}$; $a_{2,1}$; ... ; $c_{3,0}$; $c_{3,1}$). All the higher order expansion coefficients are evaluated sequentially in groups of nine by employing the full set of recurrence relations like Eq. (14), in conjunction with the subsidiary relationships like Eqs. (8, 10, 12). Following the trajectories of the three bodies for times beyond the circle of convergence entails the process of analytically continuing[4,6] these power series, as discussed in Ref. (1).

To test the more practical aspects of the above approach, we have written a computer program which, given the initial conditions for the three masses, uses the recurrence relations to sequentially calculate a set of 44 expansion coefficients for each of the nine rectangular coordinates. By choosing successive initial times that are a small fraction ($\sim$ 1/3 to 1/10) of the radius of convergence, we have found the coordinates on the trajectories of the three bodies to an accuracy of about sixteen decimal digits (double precision on the IBM PC). At each of these times a check was made on the constancy of the ten integrals[2-5] of the motion: the location of the center of mass, the total energy, the velocity of the center of mass, and the components of the total angular momentum.

The first example is a restricted problem[2,4] with the initial parameters and values given in Table I. The zero mass body (m_2 = 0) starts at the perihelion position (corresponding to a two-body Keplerian problem) with a velocity relative to the primary body (m_1 = 1) which would, for a two-body problem, result in a sidereal period equal to (2π) Gaussian time intervals [1 Gaussian time interval = k^{-1} = 58.13244087 days]. The mass of the third body is chosen in such a way that its sidereal period equals 16 Gaussian time intervals. Two additional initial conditions are that the velocity of the center of mass be zero and that the motions start out in one plane. We should emphasize that all these restrictions on initial conditions are arbitrary and only serve to simplify our illustrative example. Figure (1) depicts the three trajectories calculated. Bodies (1) and (3) trace out the expected closed (periodic) orbits, while the massless body (2) follows a complicated spiral. The small numbers near each curve are the elapsed time (Gaussian time intervals) from the start of the motions. The number of incremental time steps (Δt = 0.1) for each trajectory was 160, and the associated radius of convergence varied from about 0.381 to about 2.47 Gaussian time intervals.

Our second illustration is more general, although again, to make the presentation easier, we have let the initial motions be in one plane. In Table II we list the appropriate initial conditions. The second body now has a mass which would, for the two-body motion relative to the primary, result in a sidereal period of 6 Gaussian time intervals. In Fig. (2) we exhibit the motion of the three interacting bodies for approximately one revolution of bodies (1) and (3). It is to be noted that none of the trajectories closes on itself. Compared to our previous examples [Fig. (1)], the motion has speeded up with body (2) continuing to spiral while the path of the primary has developed turns and even a cusp (actually a small loop). In this example it was necessary (in order to maintain the desired accuracy with 44 expansion coefficients in the power series) to change the incremental displacement in time from 0.1 to 0.05 when the radius of convergence dropped below 0.3 to as low as 0.162 Gaussian time intervals.

It is important to mention that in our two examples the small radii of convergence occur for those configurations where two of the bodies come relatively close to each other. This behavior would be expected since Newton's law is singular for a collision of two of the bodies. Under this circumstance, the power series expansions have a radius of convergence equal to zero. The best way of handling collisions and the related problem of regularization[4,7-9] have been the subjects of discussion for a long time, but we have not investigated these aspects of the three body problem.

In conclusion, we would like to point out that the basic method employed in this paper for the case of three bodies can just as well be applied to the case of n bodies moving under their mutual gravitational attractions. Though this statement may be true in principle, the actual number of bodies that can be dealt with in a practical problem remains a topic for future study.

References

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TABLE I. Parameters and Initial Values for a Restricted Three Body Problem, see Fig. (1).

Masses (Solar Masses)	
m_1; m_2; m_3	1; 0; $(\pi^2/8) - 1$
Two-Body Semi-Major Axes (AU) and Eccentricities	
a_2; a_3; e_2; e_3	1; 2; 0.2; 0.2
Two-Body Sidereal Periods (Gaussian Time Intervals)	
P_2; P_3	2π; 16
Initial Values, Coordinates (AU) and Velocities (AU/Gaussian Interval)	
$a_{1,0}$; $b_{1,0}$; $c_{1,0}$; $a_{1,1}$; $b_{1,1}$; $c_{1,1}$	0; 0; 0; 0; −0.18221557; 0
$a_{2,0}$; $b_{2,0}$; $c_{2,0}$; $a_{2,1}$; $b_{2,1}$; $c_{2,1}$	0.8; 0; 0; 0; 1.04252930; 0
$a_{3,0}$; $b_{3,0}$; $c_{3,0}$; $a_{3,1}$; $b_{3,1}$; $c_{3,1}$	1.6; 0; 0; 0; 0.77969680; 0
Total Energy, (Solar Masses) (AU/Gaussian Interval)^2	
$E = T - U$	$-5.84251375 \times 10^{-2}$
Center of Mass Coordinates (AU)	
$f_{c.m.}$; $g_{c.m.}$; $h_{c.m.}$	0.30308885; 0; 0
Center of Mass Velocity (AU/Gaussian Interval)	
v_f; v_g; v_h	0; 0; 0
Total Angular Momentum, (Solar Masses) (AU)^2 (Gaussian Interval)$^{-1}$	
J_f; J_g; J_h	0; 0; 0.29154491

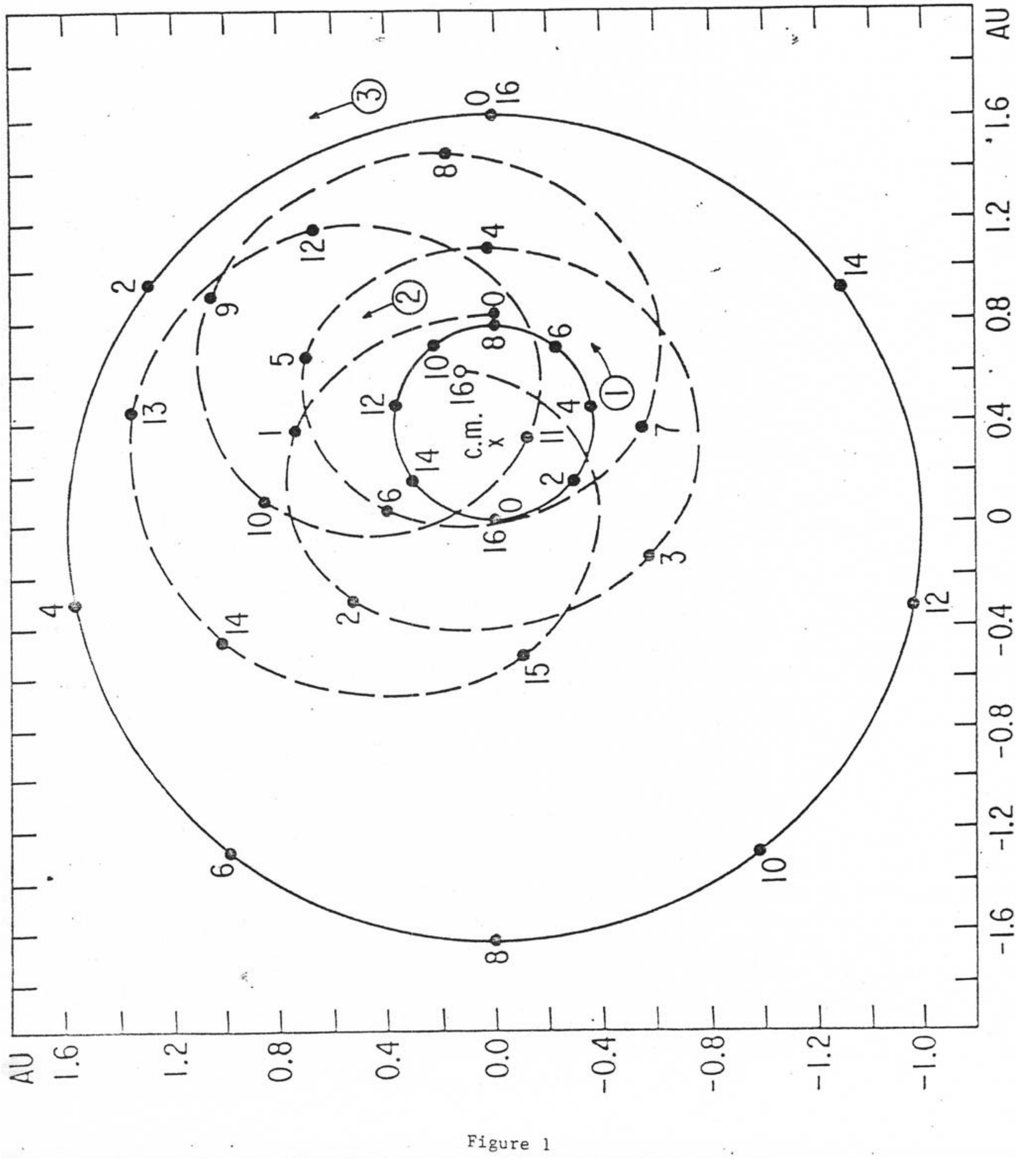

Figure 1

TABLE II. Parameters and Initial Values for the More General Three Body Problem, see Fig. (2).

Masses (Solar Masses)	
m_1; m_2; m_3	1; $(\pi^2/9) - 1$; $(\pi^2/8) - 1$
Two-Body Semi-Major Axes (AU) and Eccentricities	
a_2; a_3; e_2; e_3	1; 2; 0.2; 0.2
Two-Body Sidereal Periods (Gaussian Time Intervals)	
P_2; P_3	6; 16
Initial Values, Positions (AU) and Velocities (AU/Gaussian Interval)	
$a_{1,0}$; $b_{1,0}$; $c_{1,0}$; $a_{1,1}$; $b_{1,1}$; $c_{1,1}$	0; 0; 0; 0; -0.26213395; 0
$a_{2,0}$; $b_{2,0}$; $c_{2,0}$; $a_{2,1}$; $b_{2,1}$; $c_{2,1}$	0.8; 0; 0; 0; 1.02041588; 0
$a_{3,0}$; $b_{3,0}$; $c_{3,0}$; $a_{3,1}$; $b_{3,1}$; $c_{3,1}$	1.6; 0; 0; 0; 0.69977842; 0
Total Energy, (Solar Masses) (AU/Gaussian Interval)^2	
$E = T - U$	-0.15318558
Center of Mass Coordinates (AU)	
$f_{c.m.}$; $g_{c.m.}$; $h_{c.m.}$	0.33918000; 0; 0
Center of Mass Velocity (AU/Gaussian Interval)	
v_f; v_g; v_h	0; 0; 0
Total Angular Momentum, (Solar Masses) (AU)^2 (Gaussian Interval)$^{-1}$	
J_f; J_g; J_h	0; 0; 0.34053804

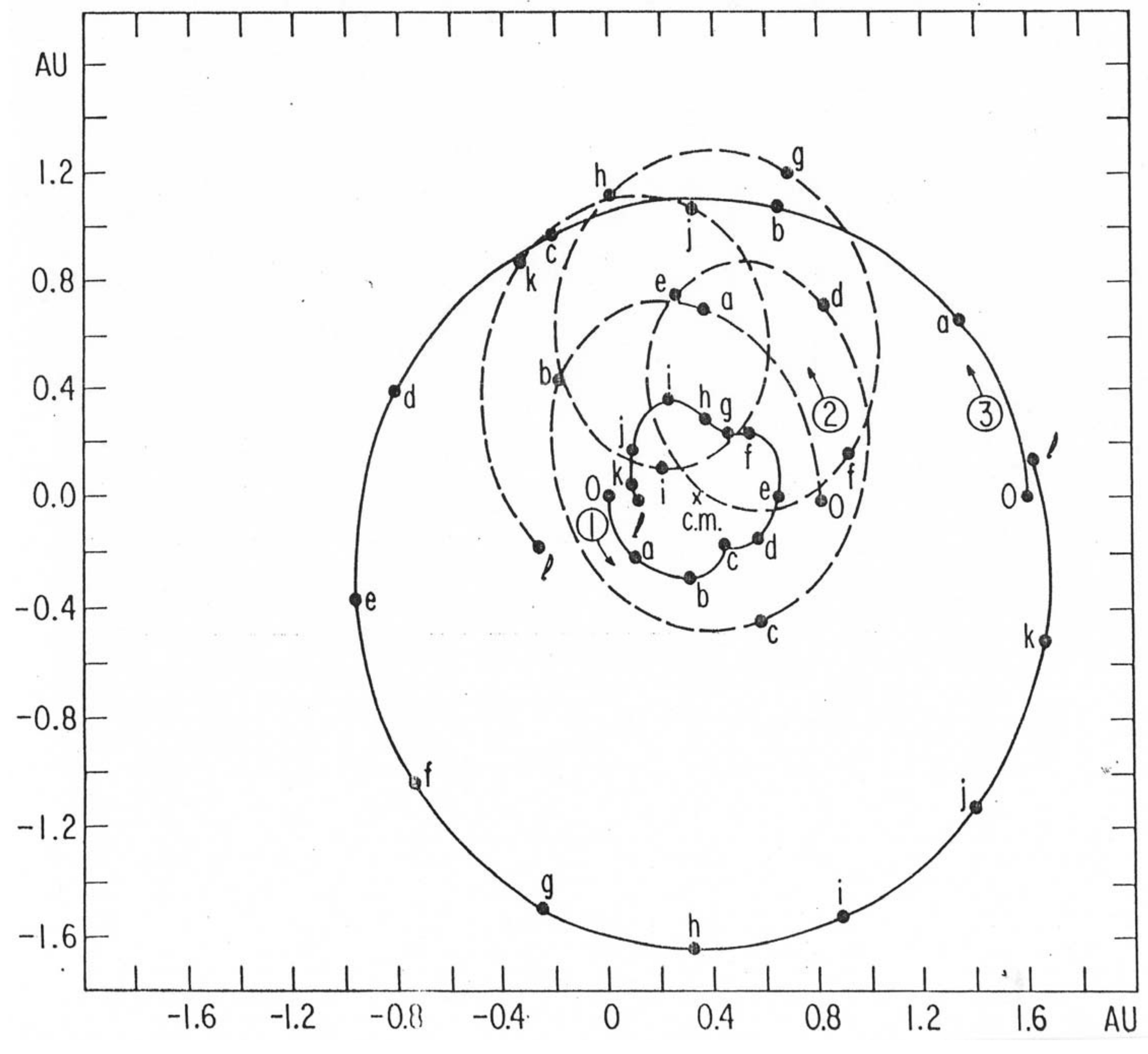
AU
1.2
0.8
0.4
0.0
-0.4
-0.8
-1.2
-1.6
-1.6
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
1.6
AU
c.m.

Figure Captions

Fig. 1. Trajectories for a restricted problem of three bodies, see Table I.

Fig. 2. Trajectories for a general problem of three bodies, see Table II. The lower case letters near the points on the trajectories indicate the elapsed time interval from the start of the motion. Corresponding to the sequence, a through ℓ, these times are 1, 2, 3, 4, 5, 6, 7.05, 8.05, 9, 10.05, 11.05, and 11.95 Gaussian time intervals.