

POWER SERIES SOLUTIONS IN NONLINEAR MECHANICS

J.C. Herrera

Brookhaven National Laboratory
Upton, New York 11973

Abstract

We present a new method of solving the nonlinear ordinary differential equations with given initial values, such as arise in nonlinear mechanics. The recurrence relations for the expansion coefficients of the power series solutions are obtained directly from the differential equations by integration in the complex plane. We illustrate the method by applying it to the Van der Pol equation, the Volterra problem of conflicting populations, and the Hénon-Heiles problem.

The solutions of the equations of motion of a classical dynamical system can be expressed as power series expansions of the form

$$f(t) = \sum_{n=0}^{\infty} a_n (t-z_0)^n, \quad (1)$$

where $f(t)$ is a dynamical variable dependent on the time t . The initial values, position and velocity, are given by the coefficients a_0 and a_1 specified at the time $t = z_0$. We use the notation z_0 for the initial time to emphasize that the expansion, Eq. (1), is made about a nonsingular point of the function f . Under these conditions the radius of convergence of the power series is

$$R_a = \lim_{n \rightarrow \infty} |a_n|^{-1/n}. \quad (2)$$

Up to this point we have, to set the stage, repeated the early part of section 32 of Whittaker's book.¹ However, we would now like to show how the recurrence relation, which gives the a_n coefficient in terms of the lower order ones, can be derived directly from the differential equation of motion by integration in the complex t -plane. As a preliminary step, we express the a_n coefficient as a contour integral dependent upon the k^{th} derivative, $f^{(k)}$, of the function $f = f^{(0)}$, thus

$$a_n = \frac{(n-k)!}{n!2\pi i} \oint \frac{dw f^{(k)}(w)}{(w-z_0)^{n-k+1}}. \quad (3)$$

Here $k = 0, 1, 2, 3, \dots$, while $n = k, k+1, k+2, \dots$, and the center of the integration contour is at z_0 with the path within the circle of convergence. It is the application of Eq. (3) that constitutes the kernel of our method. Rather than continue with a general development, we shall, in the interest of clarity, fix our attention on a number of equations of nonlinear mechanics.

The first is the now-classic equation of Van der Pol^{2,3}

$$f^{(2)} - \mu(1 - f^2)f^{(1)} + f = 0 . \quad (4)$$

Dividing this equation by the quantity $(w - z_0)$ and integrating about a contour within the circle of convergence, we have

$$\oint \frac{dw f^{(2)}(w)}{w-z_0} - \mu \oint \frac{dw f^{(1)}}{w-z_0} + \mu \oint \frac{dw f^2 f^{(1)}}{w-z_0} + \oint \frac{dw f}{w-z_0} = 0 : \quad (5)$$

The first, second, and fourth terms are evaluated immediately by applying Eq. (3), while the third term requires Eq. (1) and its derivative. The result is

$$2a_2 = \mu a_1(1-a_0^2) - a_0 , \quad (6)$$

a relationship that we could have written down at once by considering Eq. (4). In order to derive a higher coefficient a_n in terms of the lower order coefficients we generalize the above procedure, dividing the differential equation by the factor $(w - z_0)^{n-1}$. The initial integration yields

$$n(n-1)a_n - \mu(n-1)a_{n-1} + \frac{\mu}{2\pi i} \oint \frac{dw f^2 f^{(1)}}{(w-z_0)^{n-1}} + a_{n-2} = 0 . \quad (7)$$

The contour integral in this equation is found by using Eq. (3) and Eq. (1).

Thus:

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{dw f^2}{(w-z_0)^{n-1}} &= \frac{1}{2\pi i} \oint \frac{dw f}{(w-z_0)^{n-1}} \sum_{\ell=0} a_{\ell} (w-z_0)^{\ell} \\ &= \frac{1}{2\pi i} \sum_{\ell=0} a_{\ell} \oint \frac{dw f}{(w-z_0)^{n-\ell-1}} = \sum_{\ell=0}^{n-2} a_{\ell} a_{n-\ell-2} , \end{aligned} \quad (8)$$

and

$$\frac{1}{2\pi i} \oint \frac{dw f^2 f(1)}{(w-z_0)^{n-1}} = \sum_{k=1}^{n-1} k a_k \sum_{\ell=0}^{n-k-1} a_\ell a_{n-k-\ell-1} \quad (9)$$

The pertinent recurrence relation is then

$$n(n-1)a_n = \mu(n-1)a_{n-1} - \mu \sum_{k=1}^{n-1} k a_k \sum_{\ell=0}^{n-k-1} a_\ell a_{n-k-\ell-1} - a_{n-2}, \quad (10)$$

for all $n \geq 2$.

Equation (1), with this relation defining the coefficient a_n , is the general solution to the nonlinear Van der Pol equation, valid within the circle of convergence specified by Eq. (2). Since the expansion coefficients can be evaluated to any desired order, a good numerical estimate of the radius can also be calculated by going to large values of n . Successive values of the dynamical variable are then obtainable by the process of analytically continuing³ this solution for a sequence of new initial points, each within the previous circle of convergence. We would like to mention that the general solution found is not a perturbation expansion in powers of the nonlinearity parameter μ of the Van der Pol equation, and it is not limited to small values of μ . The complexity of the nonlinear behavior manifest itself in the involved dependence (Eq. (10)) of successive expansion coefficients on the initial values and the nonlinearity parameter, or equivalently on the location of the nearest singularity in the complex t -plane which determines the radius of convergence. As the strength of the nonlinearity becomes smaller ($\mu \rightarrow 0$), the singularity moves toward infinity and the dynamical behavior of the simple harmonic oscillator appears.

Another illustrative example which we would like to present is that of the growth in two populations conflicting with one another. This problem of

Volterra⁴ is clearly formulated in the book by Davis,³ and the two appropriate first order nonlinear coupled differential equations are written, in our notation, as

$$f^{(1)} = \alpha(f - fg) , \quad (11a)$$

and

$$g^{(1)} = -\beta(g - fg) . \quad (11b)$$

The multiplicative factors, α and β , are the growth constants. The two power series for the independent variable are assumed to be

$$f(t) = \sum_{n=0}^{\infty} a_n (t - z_0)^n , \quad (12a)$$

and

$$g(t) = \sum_{n=0}^{\infty} b_n (t - z_0)^n . \quad (12b)$$

When Eqs. (11a) and (11b) are divided by $(w - z_0)^n$ and then integrated in the complex plane about the nonsingular point z_0 , one arrives at the two coupled recurrence relations

$$na_n = \alpha a_{n-1} - \alpha \sum_{\ell=0}^{n-1} b_{\ell} a_{n-\ell-1} , \quad (13a)$$

and

$$nb_n = -\beta b_{n-1} + \beta \sum_{\ell=0}^{n-1} a_{\ell} b_{n-\ell-1} . \quad (13b)$$

Corresponding to the sets of coefficients, a_n and b_n , each power series has its own radius of convergence, and in any numerical calculation it is necessary to work within the smaller of these two radii, where both series are convergent.

As a third application we consider the two dimensional model that Henon and Heiles⁵ have studied relative to the existence of a third integral of motion. The dynamical system has the invariant Hamiltonian

$$H(f, f^{(1)}, g, g^{(1)}) = \frac{1}{2} [(f^{(1)})^2 + (g^{(1)})^2] + \frac{1}{2} [f^2 + g^2] + f^2 g - \frac{1}{3} g^3, \quad (14)$$

and the accompanying equations of motion

$$f^{(2)} = -f - 2fg, \quad (15a)$$

and

$$g^{(2)} = -g + g^2 - f^2. \quad (15b)$$

Once again we assume the dynamical variables, f and g , to be power series, Eqs. (12a-12b), and perform the necessary integrations in the complex plane. The outcome is the pair of coupled recurrence relations

$$2a_2 = -a_0 - 2b_0 a_0, \quad (16a)$$

$$2b_2 = -b_0 + b_0^2 - a_0^2, \quad (16b)$$

and for $n \geq 3$,

$$n(n-1)a_n = -a_{n-2} - 2 \sum_{\ell=0}^{n-2} b_\ell a_{n-2-\ell}, \quad (17a)$$

$$n(n-1)b_n = -b_{n-2} + \sum_{l=0}^{n-2} (b_l b_{n-2-l} - a_l a_{n-2-l}) . \quad (17b)$$

In summary, the method of solving a set of nonlinear ordinary differential equations, as demonstrated in this paper, converts these equations into a set of coupled recurrence relations. Associated with each sequence of expansion coefficients there is a circle of convergence. Within the smallest of these circles, all the power series are convergent. To follow the evolution of a dynamical variable in time beyond this circle, it is necessary to analytically continue the solution, that is, to evaluate the expansion coefficients with the same recurrence relations but at a new origin and determine a new circle of convergence. This step-by-step feature of using the general series solutions is a burdensome aspect of the process of analytic continuation. However, we may infer that this piecemeal characteristic is inherent in the nature of the singularities³ (fixed and movable) which exist in the complex t -plane for nonlinear differential equations. Numerical methods which use less than the general solutions, essentially ignoring the nearby singularities, can give rise to hidden errors in the time evolution of a dynamical variable, particularly in coupled nonlinear systems.

To further test our method we have also found the series solutions of the Duffing³ problem, the Lorenz⁶ problem, as well as the simple pendulum³ problem. In an accompanying note,⁷ we consider the classical three body problem.

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References

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