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Title: The Solution of Euler's Equation and the Value of Pi

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The Solution of Euler's Equation and the Value of Pi

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1. Introduction:

The classic Euler's relation, $\exp(i\pi) = -1$, may be looked at as defining the constant π in terms of the numbers (–1) and e. To emphasize this, the relation can be represented as a function of the complex variable, z

$$f(z) = \exp(z) + 1 \tag{1}$$

Then a root of Eq.(1) will equal the complex value:

$$z = 0 + i\pi \quad . \tag{2}$$

It is our purpose in this paper to solve for this root and thereby obtain a series expansion for the numerical value of π. The method we shall follow is that developed in reference [1] for finding the roots of a general polynomial of arbitrary degree.

2. Taylor Series for the Euler Equation:

Consistent with reference [1], we now expand Eq.(1) into a Taylor series about an arbitrary non-singular offset point, z_0. Thus, we have

$$f(z) = 1 + e^{z_0} e^{z - z_0} = 1 + \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!} (z - z_0)^n \tag{3}$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n(z_0)(z - z_0)^n \tag{4}$$

with the infinite set of Taylor coefficients:

$$a_0(z_0) = 1 + e^{z_0}; \qquad a_n(z_0) = \frac{e^{z_0}}{n!}, \quad n \geq 1 \quad . \tag{5}$$

We note that the Taylor series, Eq.(4) has an infinite radius of convergence about the point z_0.

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3. Reversion Series of Euler's Equation:

The reversion series for the Taylor expansion, Eq.(4), is written as:

$$z(f) = \sum_{n=0}^{\infty} b_n(a_0)[f - a_0(z_0)]^n \tag{6}$$

with the infinite set of reversion coefficients:

$$b_0(a_0) = z_0; \qquad b_n(a_0) = (-1)^{n+1}\left(\frac{e^{-nz_0}}{n}\right), \quad n \geq 1 \quad . \tag{7}$$

Eqs.(6) and (7) now allow us to write the reversion series expanded about the offset $a_0(z_0)$ in the complex f-plane as:

$$z(f) = z_0 + \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{e^{-nz_0}}{n}\right)[f - a_0(z_0)]^n \quad . \tag{8}$$

This Taylor series has a radius of convergence in the f-plane expressible as:

$$R_b = \lim_{n \to \infty} \left|b_n(a_0)\right|^{-1/n} = \left|e^{z_0}\right| \quad . \tag{9}$$

Therefore, in order that the reversion series, Eq.(6), converges, it is necessary that

$$\left|f - a_0(z_0)\right| < \left|e^{z_0}\right| \tag{10}$$

or

$$\left|f - 1 - e^{z_0}\right| < \left|e^{z_0}\right| \tag{11}$$

4. The root of Euler's Equation:

If we insert into Eq.(6) the value of f equal to zero, we obtain a series expression for the root of Euler's equation, that is

$$z(0) = z_0 - \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(e^{-z_0} + 1\right)^n \tag{12}$$

where the initial offset, z_0, of Eq.(3) is now required, according to Eq.(11), to satisfy the condition

$$\left|1 + e^{z_0}\right| < \left|e^{z_0}\right| \quad . \tag{13}$$

At this point in our development we choose the initial offset z_0 equal to (it_0) with the result that Eq.(12) becomes

$$z(0) = it_0 - \sum_{n=1}^{\infty}\left(\frac{2^n}{n}\right)\exp\left(-in\frac{t_0}{2}\right)\left(\cos\frac{t_0}{2}\right)^n = 0 + i\pi \tag{14}$$

while t_0 is restricted, according to Eq.(11), to the range

$$\pi - \arccos\left(\frac{1}{2}\right) < t_0 < \pi + \arccos\left(\frac{1}{2}\right) \quad . \tag{15}$$

When we separate Eq.(14) into its real and imaginary parts, we arrive at

$$-\sum_{n=1}^{\infty}\left(\frac{2^n}{n}\right)\cos\left(\frac{nt_0}{2}\right)\left(\cos\frac{t_0}{2}\right)^n = 0 \tag{16}$$

and

$$t_0 + \sum_{n=1}^{\infty}\left(\frac{2^n}{n}\right)\sin\left(\frac{nt_0}{2}\right)\left(\cos\frac{t_0}{2}\right)^n = \pi \quad . \tag{17}$$

Of course, the numerical value chosen for t_0 must be within the continous linear range specified by Eq.(15). As a consequence, it may be concluded that this condition is equivalent to saying that the entire set of series represented by Eq.(17), each of value π, has the cardinality of Cantor's continuum [2]. Interestingly, though we have searched the literature on the calculation of π, including the recently published text [3] and the encyclopedic work of Berggren, Borwin, and Borwein [4], we have not found any series which would be a member of this noncountable set.

5. Numerical Value of π by Series:

Eq.(17), when evaluated for any value of t_0 within the specified range, Eq.(15), will yield the numerical value of π. Thus, for example, if we let $t_0 = 3$, we have

$$\pi = 3 + \sum_{n=1}^{\infty}\left(\frac{2^n}{n}\right)\sin\left(\frac{3n}{2}\right)\left(\cos\frac{3}{2}\right)^n \tag{18}$$

while from Eq.(16)

$$0 = \sum_{n=1}^{\infty}\left(\frac{2^n}{n}\right)\cos\left(\frac{3n}{2}\right)\left(\cos\frac{3}{2}\right)^n \tag{19}$$

When we evaluate Eqs.(18) and (19) over a finite number of terms, the vanishing of the value of the finite sum in Eq.(19) serves as an indication of the correctness of the value of π up to the same number of terms. However, we should emphasize that the true check of using Eq.(17) is that it must give the identical numerical value of π for different initial values of t_0.

6. Iterative Evaluation of π:

Eq.(17), as such, does not converge rapidly (except, of course, if we insert the sought after value of π). This suggests that in an actual calculation of π, we should employ the following iterative procedure based on the first two terms of Eq.(17). Hence, upon choosing an arbitrary value of t_0, we calculate successive values of t_n according to the iterative sequence:

$$\begin{aligned} t_1 &= t_0 + \sin t_0 \\ t_2 &= t_1 + \sin t_1 \\ &\vdots \\ t_n &= t_{n-1} + \sin t_{n-1} \quad . \end{aligned} \tag{20}$$

Similar to the exact series solution, each step of this sequence must bring the first term of Eq.(16) closer to zero. Again, we should emphasize that the true check of the above algorithm is that it gives the identical value of π for different starting values of t_0. Because of the freedom in choosing a specific value of t_n at every step, the overall algorithm is self-correcting.

A good way of visualizing what one is really doing when we calculate π using Eq.(17) is to imagine the whole noncountable set of series, each having its own starting value. One then proceeds by summing an arbitrary number of terms of the t_0 series, and having arrived at some t_1, one then "jumps" over to the series which starts with t_1. One then sums some terms of this series and then "jumps" to a new series with t_2 as its starting value, and so on with a succession of starting and "jump" values. Had we chosen the jumping value when two terms of each series had been summed, we would have been following the iterative sequence of Eq.(20). Such a sequence has been suggested by Donald Shanks for finding an improved approximation for π from a given number of n accurate digits [5].

7. Conclusion:

We have verified Eq.(17) and (16) as well as the iterative sequence, Eq.(20) with 30-digit and 90-digit arithmetic precision. Finally we can state that our method of solving Euler's equation for the numerical value of π is a new approach to investigating this fundamental number.

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