STRESSES IN MAGNETIC FIELD COILS

W.F. Westendorp and R.W. Kilb
General Electric Company
Research & Development Center
Schenectady, New York

I. INTRODUCTION

The computation of the mechanical stresses in cylindrical magnetic field coils requires a knowledge of the distribution of forces acting upon the conductor, and a theory whereby this force pattern may be converted into a stress pattern.

The methods found in the literature are either accurate and cumbersome[1] or simplifications[2,3] that may lead to serious errors.

II. THEORY OF A SIMPLE COIL

For thick-walled pressure vessels, to which a cylindrical coil may be compared, there exists a stress theory[4] in which it is pointed out that the problem is two-dimensional if pressures are applied inside and outside. In this case the z component of the stress tensor is zero everywhere. If, however, $\overline{J} \times \overline{B}$ body forces are introduced, this will, in general, not be the case for an isotropic material. However, a coil consisting of layers of rectangular wire or of pancakes of metal ribbon is not isotropic, and since the wire or the metal ribbons is provided with a thin coating of plastic insulation, the shear stress between wires or ribbons is negligibly small, and the compressive stress in the z direction will depend only on the boundary conditions. If we assume that the axial forces will be taken up by suitable coil forms repeated at distances small compared to the diameter, we may neglect the entire z component of the stress tensor, and the problem reverts to the simpler two-dimensional case. A differential equation may be derived in a manner analogous to Lamé's original method. We will here follow the standard textbook[5] treatment except for the introduction of $J_\theta B_z$ body forces and the new boundary conditions with zero inside and outside pressures.

Figure 1 shows a view of the coil, with details at "a" and "b". Under the influence of the electromagnetic forces, the circle of radius r in the unstressed state between inner radius r_i and outer radius r_o, becomes a circle of radius $r + u$, where u is the outward displacement of any point, r. Thus u is a function of r, although

---

1. J.D. Cockcroft, Trans. Roy. Soc. London 227, 317 (1928); also W.F. Giauque, Rev. Sci. Instr. 31, 374 (1960).
2. H.P. Furth, M.A. Levine, and R.W. Waniek, Rev. Sci. Instr. 28, 949 (1957).
3. D.B. Montgomery, The Generation of High Magnetic Fields, pp. 70-104 (see p. 80).
4. G. Lamé and B.P.E. Clapeyron, Mémoire sur l'équilibre intérieur des corps solides homogènes, Mémoires présentés par divers savants, Vol. 4 (Académie des Sciences, Paris, 1833).
5. J.P. Den Hartog, Strength of Materials (McGraw-Hill, 1949, and Dover Publ., 1961), pp. 140-145.

numerically very small in comparison to r. All lengths of the circle with original radius r are increased in the ratio $(r + u)/r$ or $1 + (u/r)$. A piece ds of the circle becomes $ds + u\ ds/r$, which means that the tangential strain is u/r.

To find the strain in the radial direction, we look at the detail "b" in Fig. 1 showing an element in the undistorted state in full lines, which, after loading, goes to the dotted-line picture. The location of the inner point is r, and it goes to $r + u$, because of the magnetic pressure. The location of the outer point was $r + dr$ and it goes to $r + dr + u + du$ because u varies with r. Thus the original radial length of the piece was dr, and its distorted radial length is $dr + du = dr(1 + du/dr)$. Hence the radial strain is du/dr.

We will assume that there are no axial forces acting upon the coil and no shear forces between layers of the winding. The two strains can then be expressed in terms of the radial and tangential stresses S_r and S_t as follows[6]:

$$\frac{u}{r} = \frac{1}{E}(S_t - \mu S_r) \quad , \tag{1}$$

$$\frac{du}{dr} = \frac{1}{E}(S_r - \mu S_t) \quad . \tag{2}$$

E is the Young's modulus and μ the Poisson's ratio. These are two equations in the three unknowns u, S_t, and S_r. A third equation is found from the equilibrium of an element $dr \cdot rd\theta$ shown at "a" in Fig. 1 and enlarged in Fig. 2. The radial force 1 is $S_r \cdot rd\theta$ per unit length perpendicular to the paper. The force 2 on top is similar but both S_r and r are slightly different; thus force 2 is $S_r \cdot rd\theta + d(S_r \cdot rd\theta)$ and the resultant of 1 and 2 outward is

$$d(S_r \cdot rd\theta) = \frac{d(rS_r)}{dr} drd\theta \quad .$$

The forces 3 and 4 are numerically equal but not quite opposite since the faces they act upon make an angle $d\theta$. Hence their resultant is directed inward and has the magnitude

$$F_3 d\theta = S_t\ drd\theta \quad .$$

The fifth force that keeps the element in equilibrium is the electromagnetic body force $J_\theta B_z\ rd\theta dr$ acting outward upon the volume element with unit length in the z direction.

Equating forces in the radial direction, we find

$$F_2 - F_1 + J_\theta B_z\ rd\theta dr = F_3 d\theta \quad .$$

After substituting the appropriate values and dividing by $drd\theta$ we get

$$\frac{d(rS_r)}{dr} + rJ_\theta B_z = S_t \quad . \tag{3}$$

---

6. J.P. Den Hartog, Ref. 5, Eq. (8), p. 75.

Solving Eqs. (1) and (2) for S_r and S_t, we find:

$$S_r = \frac{E}{1 - \mu^2} \left(\frac{du}{dr} + \mu \frac{u}{r} \right) \quad , \tag{4}$$

$$S_t = \frac{E}{1 - \mu^2} \left(\frac{u}{r} + \frac{du}{dr} \right) \quad . \tag{5}$$

Substitution of Eqs. (4) and (5) into Eq. (3) then yields the differential equation:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -\frac{1 - \mu^2}{E} J_\theta B_z \quad . \tag{6}$$

The homogeneous solutions of Eq. (6) are r and 1/r. The general solution of Eq. (6) may therefore be written as[7]

$$u = Ar + \frac{B}{r} \quad , \tag{7}$$

where A and B are functions of r that satisfy the first order differential equations:

$$\frac{dA}{dr} = -\frac{1 - \mu^2}{2E} J_\theta B_z \quad , \tag{8}$$

$$\frac{dB}{dr} = -r^2 \frac{dA}{dr} \quad . \tag{9}$$

Integration of Eqs. (8) and (9) gives two integration constants, which are to be determined by the boundary condition that the radial stress S_r is zero at the inside and outside coil radii.

For a uniform current density J_θ, it is usually sufficient to assume that B_z varies linearly with radius from B_1 at the inside radius r_i, to a small negative value B_o at the outside radius r_o. The radial force may then be written:

$$J_\theta B_z = \alpha - \beta r \quad , \tag{10}$$

where α and β are constants.

$$\alpha = \frac{J_\theta(B_i r_o - B_o r_i)}{r_o - r_i} \quad , \tag{11}$$

$$\beta = \frac{J_\theta(B_i - B_o)}{r_o - r_i} \quad . \tag{12}$$

7. D.H. Menzel, Fundamental Formulas of Physics (Dover Publ., 1960), p. 35.

Using Eq (10), we may easily integrate Eqs. (8) and (9) to find the solutions:

$$A = \frac{1-\mu^2}{E}\left(\frac{C_1}{1+\mu} - \frac{1}{2}\alpha r + \frac{1}{4}\beta r^2\right) \tag{13}$$

$$B = \frac{1-\mu^2}{E}\left(\frac{C_2}{1-\mu} + \frac{1}{6}\alpha r^3 - \frac{1}{8}\beta r^4\right) \tag{14}$$

$$u = \frac{1-\mu^2}{E}\left(\frac{C_1 r}{1+\mu} + \frac{C_2}{(1-\mu)r} - \frac{\alpha r^2}{3} + \frac{\beta r^3}{8}\right) \tag{15}$$

$$S_r = C_1 - \frac{C_2}{r^2} - \frac{\alpha(2+\mu)r}{3} + \frac{\beta(3+\mu)r^2}{8} \tag{16}$$

$$S_t = C_1 + \frac{C_2}{r^2} - \frac{\alpha(1+2\mu)r}{3} + \frac{\beta(1+3\mu)r^2}{8} \quad . \tag{17}$$

The constants α and β are found from Eqs. (11) and (12), μ is Poisson's ratio, and E is Young's modulus. Note that the radial and tangential stresses, S_r and S_t, do not depend on Young's modulus. The radial displacement u is, however, a function of Young's modulus.

As mentioned above, the constants C_1 and C_2 are found by setting $S_r = 0$ at the inside and outside coil radii. This yields:

$$C_1 = \frac{\alpha(2+\mu)(r_o^2 + r_o r_i + r_i^2)}{3(r_o + r_i)} - \frac{\beta(3+\mu)(r_o^2 + r_i^2)}{8} \tag{18}$$

$$C_2 = r_i^2 r_o^2\left(\frac{\alpha(2+\mu)}{3(r_o + r_i)} - \frac{\beta(3+\mu)}{8}\right) \quad . \tag{19}$$

III. APPLICATION TO SIMPLE COIL

The computation derived above has been applied to the case of a "superconducting coil." In this coil the copper is employed primarily for purposes of thermal stability, and it has a cross section large compared to that of the actual superconductor, a niobium-tin alloy. The ribbon[8] of which the coil is wound consists of two strips of copper a few thousandths of an inch thick with the niobium-tin of only a few ten-thousandths of an inch sandwiched between. Therefore, in the computation the stress contribution of the niobium-tin may be entirely ignored even though both its Young's modulus and its ultimate strength are higher than those of copper. For this reason we shall refer to this type of superconducting coil as the "copper" coil. The coil is supported by a metal spool 20-in. i.d. so that the copper has an i.d. equal to 21 in. (r_i = 26.67 cm). The outside diameter of this particular coil was dictated by the size of a Dewar vessel, 38-in. i.d., and was consequently assumed to be 36-in. o.d. (r_o = 45.72 cm). The axial dimension of the coil was 4 in., but it formed part of a coaxial system of coils, and the field values and stresses computed here are all

---

8. M.G. Benz, G.E. Research & Development Center Report No. 66-C-044 (1966).

for this coil as part of the coaxial fully-energized system. The coil was known as a large size "end" coil, there being one smaller "end" coil beyond it. In all cases considered here, the central field at the center of the assembly and on the axis was 75 kG. Digital computer field determinations under these conditions gave for the "copper" coil: B_i = 91 kG, B_o = -18 kG, and J = 9.88 kA/cm^2. Taking the Young's modulus of copper as 1.03×10^{12} dyn/cm^2 (14.93×10^6 psi) and the Poisson's ratio of copper as $\mu = 0.33$, we found for the "copper" coil the tangential tensile stress curve, so marked in Fig. 3, going from 17.2×10^8 dyn/cm^2 (24 000 psi) on the inside of the coil to 8.04×10^8 dyn/cm^2 (11 600 psi) on the outside. The radial compressive stresses are plotted as a negative stress on the left in Fig. 3 and the radial motion on the right in Fig. 3. The maximum compressive stress (1.6×10^8 dyn/cm^2 or 2320 psi) although low for copper, should be considered with regard to the insulation used on the metal ribbon. The radial motion is greatest on the inside (0.0445 cm or 0.0175 in.) and also the largest of the three systems considered.

IV. APPLICATION TO REINFORCED COIL

Because of the high tangential stress on the inside (24 900 psi) it was decided to consider other configurations than the "copper" coil.

The second design considered is referred to as the "two band" coil. It has an inner "band", consisting of the "copper" winding, of 21-in. i.d. and 31-in. o.d. (r_o = 39.37 cm), which carries the current, and a stainless-steel band of 31-in. i.d. and 36-in. o.d. (r_x = 45.72 cm), which carries no current but serves strictly as reinforcement. Because of the different location of the magnetizing ampere turns, the digital computer was again used to determine the field, and the result was B_i = 99 kG and B_o = -20 kG and J = 14.7 kA/cm^2. In the mathematical stress-strain treatment of this combination, the "copper" is handled as before with the exception that at the outside boundary of the copper the radial stress must equal the radial stress of the stainless steel. For the stainless steel, Eqs. (11) through (17) may be used provided we set $J_\theta = 0$ so $\alpha = 0$ and $\beta = 0$. The ratio E_{ss}/E_{cu} was taken as 2 and $\mu_{ss} = 0.30$. Another condition is that the external radial displacement u_{cu} of the copper must equal the internal radial motion u_{ss} of the stainless-steel band. The four unknowns C_{1cu}, C_{2cu}, C_{1ss} and C_{2ss} may therefore be found from the four boundary conditions: $S_r = 0$ at r_i and r_x; $u_{cu} = u_{ss}$ and $S_{rcu} = S_{rss}$ at r_o.

Carrying through this procedure leads to the curves marked "two band coil" in Figs. 3a and 3b. The main surprise is that the improvement is so small, the new maximum tensile stress in the copper being 14.9×10^8 dyn/cm^2 or 21 600 psi. The maximum compressive stress has more than doubled, 3.6×10^8 dyn/cm^2 or 5210 psi, and the radial motion has been reduced to a maximum at the inside of the coil of 0.0385 cm or 0.0151 in.

V. COPPER-STEEL RIBBON COIL

A third attempt was made to reduce the maximum tensile stress in the copper, while keeping the over-all dimensions the same. This time the coil was considered to be wound of a combination of copper and stainless-steel ribbons so that in the compound ribbon the area of the copper was twice that of the steel.

Consider this compound or sandwich coil as consisting of a "new" material with a mean Young's modulus expressed as follows:

$$E_m = f_c E_c + f_s E_s$$

and a Poisson's ratio

$$\mu_m = f_c\mu_c + f_s\mu_s$$

where f_c is the fraction of the total cross section that is copper, f_s is the fractional area for the stainless steel, and E_c and E_s the corresponding moduli, etc. This value of the mean Young's modulus may be rigorously derived for the case where the compound ribbon hangs vertically and is loaded by a weight causing a strain, δ, in both copper and stainless steel and tensile stresses, respectively S_c and S_s. The material may even be subjected to a stress, S_r, perpendicular to the ribbon.

According to Eq. (1) with some rearranging:

$$S_c = \delta E_c + \mu_c S_r \quad \text{and} \quad S_s = \delta E_s + \mu_s S_r \quad .$$

The mean stress may be written as

$$S_m = f_c S_c + f_s S_s \quad ,$$

or by substitution

$$S_m = \delta f_c E_c + f_c\mu_c S_r + \delta f_s E_s + f_s\mu_s S_r \quad ,$$

or rearranged

$$S_m = \delta(f_c E_c + f_s E_s) + (f_c\mu_c + f_s\mu_s)\ S_r \quad ,$$

and since for the "new" material we must have

$$S_m = \delta E_m + \mu_m S_r \quad ,$$

we conclude

$$E_m = f_c E_c + f_s E_s \quad \text{and} \quad \mu_m = f_c\mu_c + f_s\mu_s \quad .$$

The results are shown in the curves marked "sandwich coil". The mean Young's modulus became 1.37×10^{12} dyn/cm^2 and the mean Poisson's ratio 0.32, when $f_c = 2/3$ and $f_s = 1/3$. It can be seen in Fig. 3 that the maximum tangential stress in the copper for the sandwich coil has dropped to 13×10^8 dyn/cm^2, which is 3/4 of the 17.2×10^8 dyn/cm^2 for the simple copper coil.

Note that the maximum tangential stress in the steel of 26×10^8 dyn/cm^2 is twice as great as the 13×10^8 dyn/cm^2 value for the copper, which is the same ratio of two as for their Young's moduli. This ratio follows from Eq. (1) if we recall that the maximum stress point occurs at the inside surface of the coil where $S_r = 0$. Since the radial motion u is essentially the same for the copper and steel in the ribbon, we then have from Eq. (1) that $S_{tcu}/E_c = S_{tss}/E_s$, which shows that the ratio of the maximum tangential stresses in the two materials is in the ratio of their Young's moduli.

A further point is that the <u>total</u> tangential stress S_t at a given radius is essentially the same whether the coil is a "simple copper" coil or a "sandwich coil". For the sandwich coil we have

$$S_t = f_c S_{tcu} + f_s S_{tss}$$

where S_{tcu} and S_{tss} are the tangential stresses in the copper and steel, respectively, and f_c and f_s are the fractional area cross sections of the copper and steel in the sandwich ribbon. At the inside coil surface we may substitute $S_{tss} = E_s S_{tcu}/E_c$ in the above equation, and solve it for S_{tcu}:

$$S_{tcu} = \frac{1}{f_c + f_s E_s/E_c} S_t \quad .$$

Since S_t is the same for the "simple copper" and "sandwich coil", the factor $1/(f_c + f_s E_s/E_c)$ gives the reduction in maximum stress in the copper of the "sandwich coil" as compared to the "simple copper" coil. For $f_c = 2/3$, $f_s = 1/3$ and $E_s/E_c = 2$, this reduction factor is 3/4, just as the numerical values in Fig. 3 show. Also note that the maximum reduction occurs in the limit of very little copper ($f_c \rightarrow 0$) and nearly all steel ($f_s \rightarrow 1$), in which case the reduction factor becomes E_c/E_s. Thus the greatest possible reduction in the copper tangential stress is 50%, which occurs when the ribbon is nearly all steel. Note that one-third copper and two-thirds steel would give a reduction factor of 60%.

VI. CHARTS AND GRAPHS

For the simple coil we have set up a computer program that determines the field in the center, in the center plane on the inside wall and on the outside, and in various places inside the winding at the mean radius. From these values the tangential tensile and the radial compressive stresses are computed. The radial components at the mean radius allow us to compute the maximum axial compressive stress in the center. We are not considering the increased strain caused by the superposition of tangential tension and axial compression nor what theory of strength to use. Figure 4 in the original report was a "fold out and fold up" chart in which the central field may be found from length/i.d. and o.d./i.d. on the left and to the right by means of nomograms from current density and i.d. and by a final reading on the extreme right. Figure 5 shows other field values expressed in the one read from Fig. 4.

Figure 6 is a chart similar to Fig. 4 showing at the extreme right the maximum tangential tensile stress based on, from left to right, length/i.d., o.d./i.d., current density, and i.d.

Figure 7 shows other stresses expressed in the tensile stress read from the chart.

If coils are loosely wound or have very compressible insulation the tensile stress becomes higher than read in the chart. Figure 8 shows the maximum factor of increase for the "loose" coil.

In all cases the space factor has been assumed to be unity. Actual current densities, tensile and compressive stresses are higher and may be found by dividing by the appropriate space factors.

For certain coil shapes of o.d./i.d. ratio larger than two the maximum tensile stress may occur at a radius larger than the inside and the compressive stress at smaller radii disappears. The computer program and the charts take this into account.

VII. CONCLUSION[9,10]

We have derived an accurate theory for the tangential and radial stresses in simple cylindrical coils. To apply this theory, the maximum and minimum values of B_z should be found by means of a computer program. From there on the computation involves only the determination of two integration constants in the expressions for the two stresses and the tangential strain.

The theory may also be applied to a coil surrounded by a reinforcing band, but in this case four constants have to be determined from the boundary conditions.

For a third situation, where the coil is wound of a compound ribbon made up of two different materials, our theory is adequate provided we use specified mean values of Young's modulus and Poisson's ratio.

By means of a computer program charts and graphs were plotted of magnetic fields in various places and of the three maximum stresses for a large range of shapes and sizes of coils of the simplest type.

---

9. Since the original publication date of this report (August 1966, Report No. 68-C-255), the following paper was presented on this subject, which contains further references to recent literature: A.J. Middleton and C.W. Trowbridge, Proc. 2nd Intern. Conf. Magnet Technology, Oxford, 1967, p. 140.

10. R.W. Kilb and W.F. Westendorp, G.E. Research & Development Center Report No. 67-C-440 (1967).

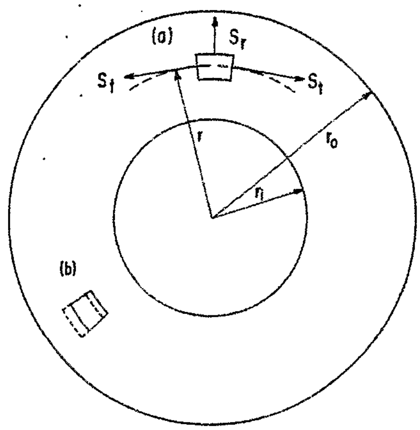

Fig. 1. Cross section of coil considered as a thick-walled pressure vessel showing stresses (a) and deformation (b).

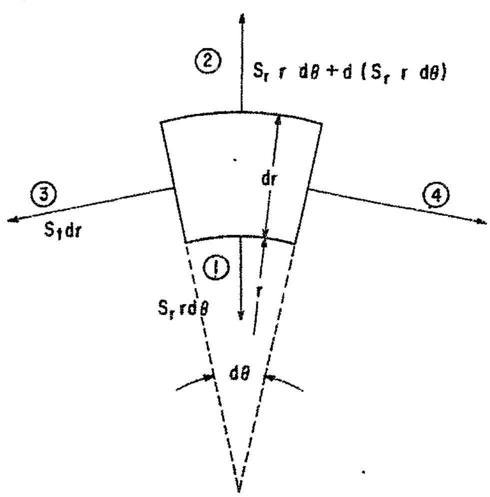

Fig. 2. Forces acting on an element. Not shown is the electromagnetic force $J_\theta B_z r d\theta dr$, directed radially outward.

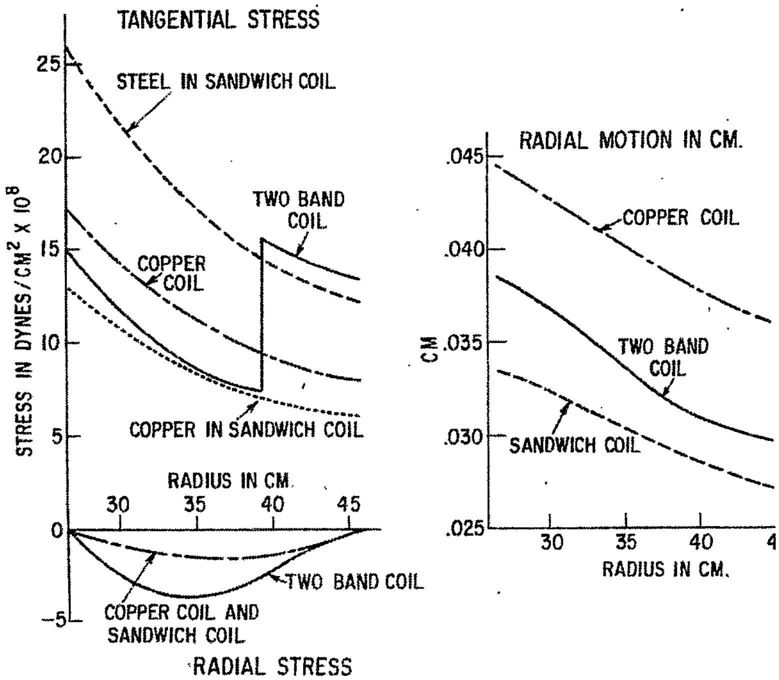

Fig. 3. Stresses and radial motion shown as a function of radius. For the "copper coil" r_i = 26.67 cm, r_{o_2} = 45.72 cm, B_i = 91 kG, B_o = -18 kG, J = 9.88 kA/cm^2; "sandwich coil" same as copper coil; "two-band coil" r_i = 26.67 cm, r_o = 39.37 cm, r_x = 45.72 cm, B_i = 99 kG, B_o = -20 kG, J = 14.7 kA/cm^2, E_{ss}/E_{cu} = 0.33, μ_s = 0.3.

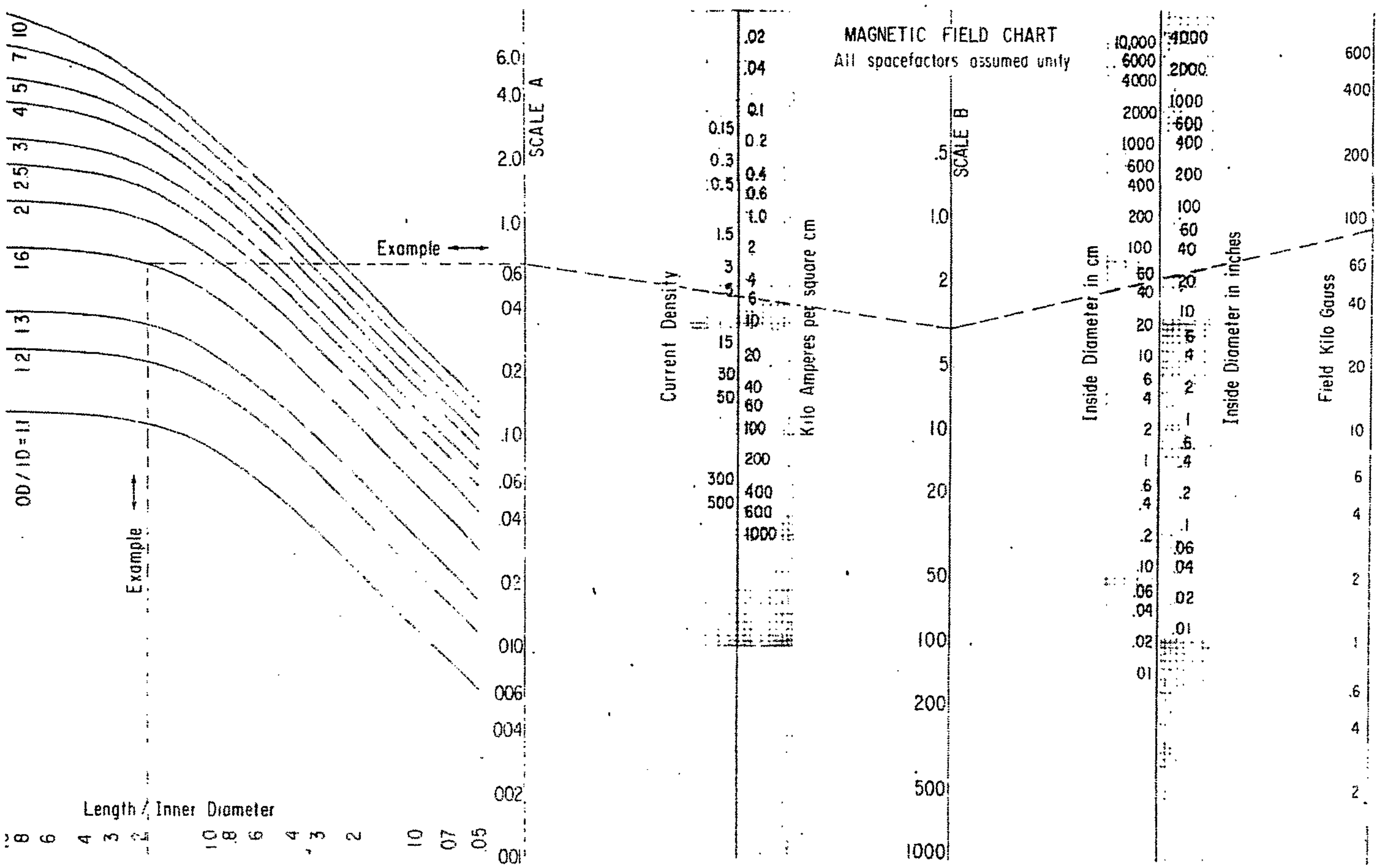

Fig. 4. Magnetic field chart.

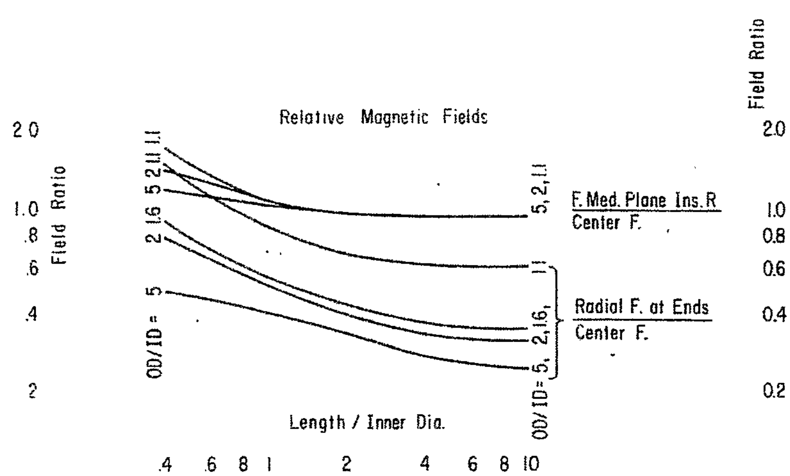

Fig. 5. Ratios of magnetic fields in conductors to field in center.

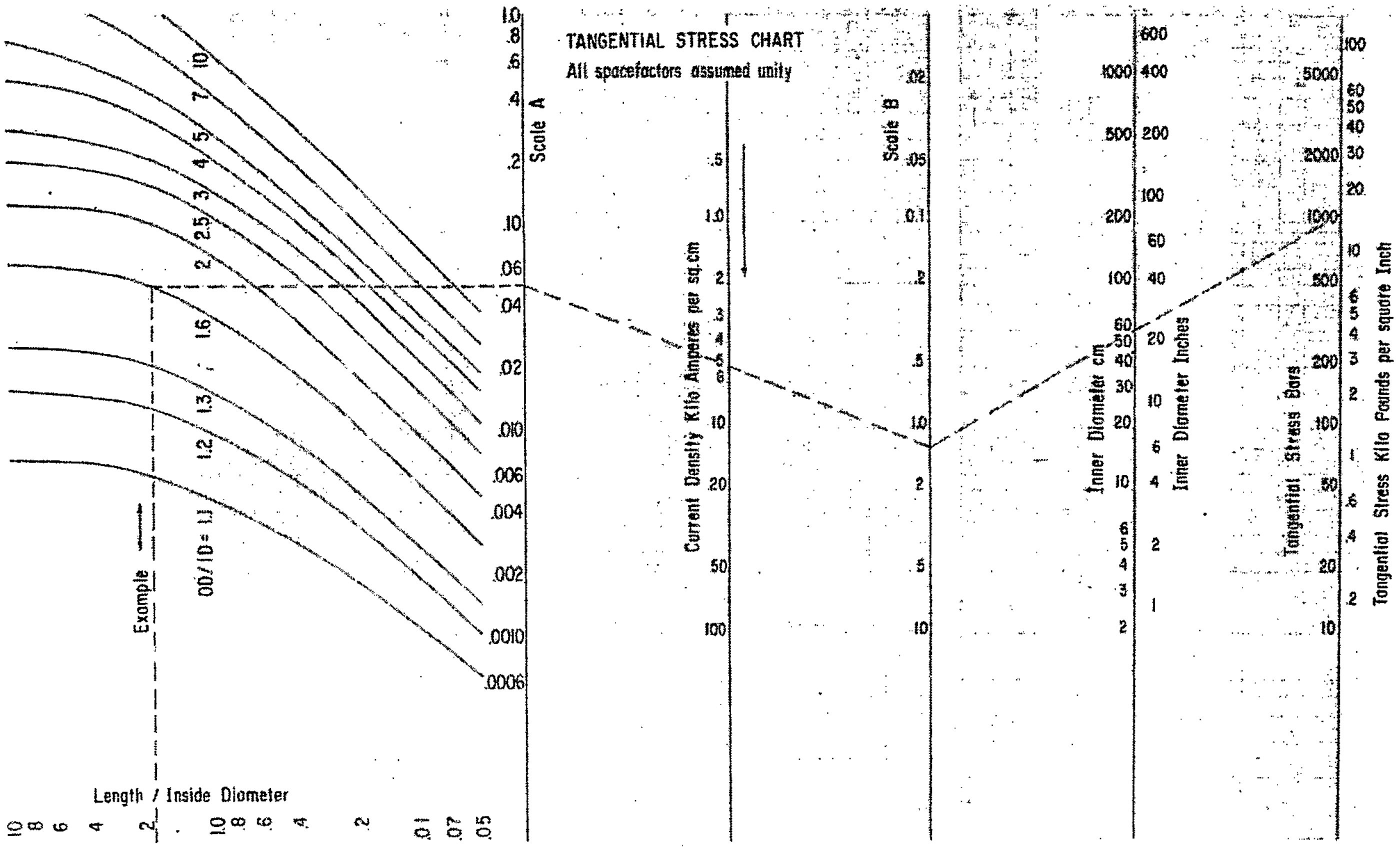

Fig. 6. Tangential, tensile stress chart.

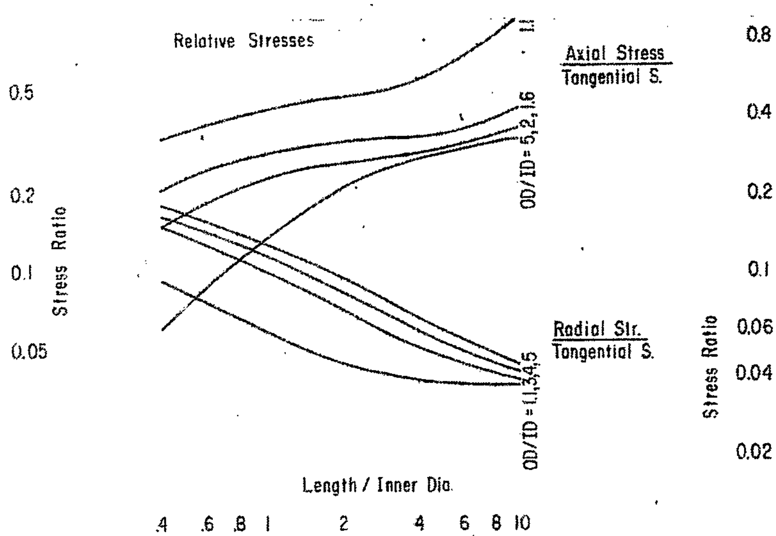

Fig. 7. Ratios of axial and radial compression stresses to maximum tensile stress.

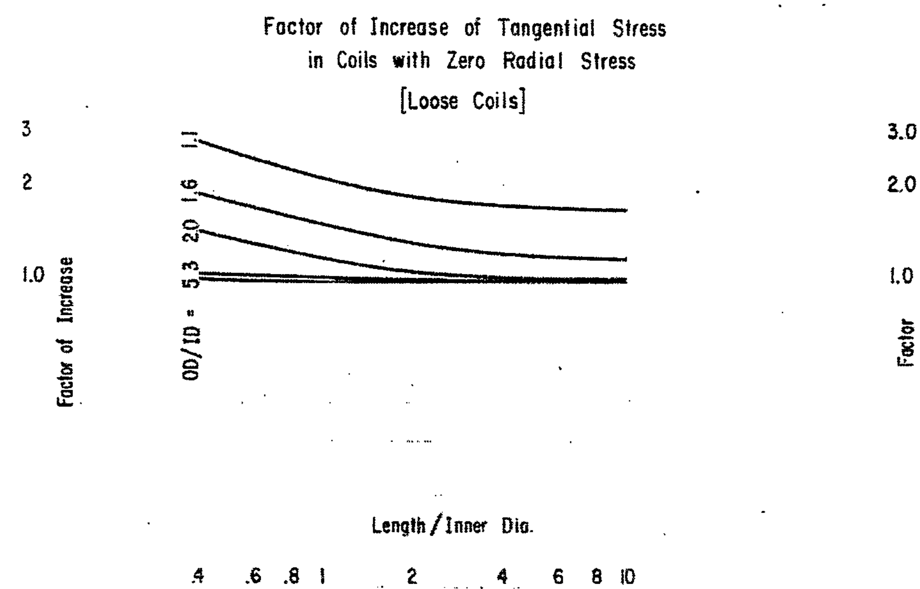

Fig. 8. Factor of increase of tangential stress for a "loose" coil.