A Lattice Field Theory Primer

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Outline

• Introduction
• Lattice Fermions
• Calculating masses and matrix elements
• Accessing the chiral limit
• Renormalization of Operators
• Systematic errors
Physical observables in QFT calculated in path integral formulation. Schematically,

\[ \langle O \rangle = \frac{1}{Z} \int d[U] O(U) \exp \left( -i g S(U) \right) \]

If coupling $g$ is small, expand exponential. $\langle O \rangle$ is calculated to some prescribed order in the coupling, $g^n$. Use Feynman diagrams.
If coupling is not small (low energy QCD) can’t expand exponential. Or if bound states required can’t use PT. Just do the whole integral. Use lattice/numerical monte-carlo techniques.

Either way, integrals are in general divergent: $\infty$ number of degrees of freedom (fields) that can take values from $-\infty$ to $+\infty$.

Must make them finite $\rightarrow$ regularize. Many ways to do this, but must be careful not to destroy symmetries of the original theory (at least they must be recovered when the regulator is removed).
Non-perturbative regularization

Discretize the continuum action on a four-dimensional (Euclidean) space-time lattice with *spacing* \( a \). [K.G. Wilson, 1974]

- Long dist (low energy) physics is insensitive to \( a \) (scaling)
- Path integrals finite: finite number of degrees of freedom (sites)
- Momentum cut-off \( p_{\text{max}} \sim 1/a \)
Do this in a *gauge invariant* way

Replace continuum vector potential (Gluon fields), $A_\mu = A^a_\mu \chi^a$ with

$$A_\mu(x) \rightarrow U_\mu(x) = e^{-igaA_\mu(x)}$$

The “link” $U_\mu(x)$ is an element of the group SU(N), with gauge transformation $g(x)$

$$U_\mu(x) \rightarrow g(x) U_\mu(x) g(x + \hat{\mu})^\dagger \quad U_\mu(x), \; g(x) \in SU(N)$$

So a path-ordered product of link fields transforms like

$$g(x) U_\mu(x) U_\mu(x + \hat{\mu}) \cdots U_\nu(y) g(y + \hat{\nu})^\dagger$$
If the path is a closed loop, e.g.

\[ g(x) U_\mu(x) U_\nu(x + \hat{\mu}) U^\dagger_\mu(x + \hat{\nu}) U^\dagger_\nu(x) g(x)^\dagger \]

And we take the trace, it is gauge invariant. Generally true that the trace of (any) closed path-ordered product of links is gauge invariant.
Treating the fermions is (naively) straightforward. Transcribe the continuum field $\psi(x)$ to the lattice site $x$.

$$\psi(x) \rightarrow \psi_x^\text{latt}$$

Under a gauge transformation,

$$\psi_x^\text{latt} \rightarrow g(x) \psi_x^\text{latt}$$
$$\overline{\psi}_x^\text{latt} \rightarrow \overline{\psi}_x^\text{latt} g^{-1}(x)$$
Construct the action.

Work in Euclidean space: analytically continue $t \rightarrow i\tau$ (Wick rotate) so metric is

$\text{diag}(1,1,1,1)$ and not $\text{diag}(-1,1,1,1)$.

Covariant and contravariant indices mean the same thing.
Fermions first.

For a single flavor

\[ S_f = \int d^4x \overline{\psi}(x)(\not{D} + m)\psi(x) \rightarrow \overline{\psi}_x M_{xy} \psi_y \]

\( \psi \) is now a 12 component vector (3 colors \( \times \) 4 spins) at each site on the lattice.

Fermion matrix is a 12 \( \times \) 12 matrix each pair of n.n. sites

\[ M_{xy} = \sum_\mu \gamma_\mu \frac{U_\mu(x) \delta_{x+a\hat{\mu},y} - U_\mu^\dagger(x - \hat{\mu}) \delta_{x-a\hat{\mu},y}}{2} + am \delta_{x,y} \]

Factors of the links make the lattice action gauge-invariant.

A large, sparse matrix: \((L^4 \times 12) \times (L^4 \times 12)\). Can invert it in \(O(12 \times L^4)\) operations, not \(O((12 \times L^4)^2)\).
Gluon action

\[ S_g = \int d^4 x \left( \frac{1}{4} F_{\mu \nu} F_{\mu \nu} \right) \rightarrow S_g^{\text{latt}} = \frac{6}{g^2} \sum_{\text{sites}} \sum_{\mu > \nu} (\mathcal{R} \text{Tr} \Box_{\mu \nu}) \]

Which you can check by expanding

\[ \lim_{a \to 0} U_{\mu}(x) = 1 - i g a A_{\mu}(x) + \cdots \]

and neglecting terms of \( \mathcal{O}(a^2) \) and higher.

At this point, the lattice action, \( S_f + S_g \), has all the symmetries of the continuum, except Euclidean (Lorentz) invariance which is broken down to (invariance under) the Hypercubic group \( \text{H}(4) \).
The continuum limit, $a \to 0$ (remove the regulator).

Adjust bare coupling, $6/g^2$, and quark mass(es) $a m$ to give some observable its physical value, say $M_N/M_\rho$. Move toward $a=0$, $g, m \to 0$ keeping $M_N/M_\rho$ fixed. Predict all other (ratios of) physical observables on this (renormalization “group”) trajectory.
How do we know this works?
Answer: asymptotic freedom of QCD: non-trivial continuum limit

For sufficiently small $g$, solution of the QCD $\beta$ function (physics does not depend on the lattice spacing (regulator)) reads:

$$a \Lambda_{QCD} = (g^2 \gamma_0)^{-\gamma_1/(2\gamma_0^2)} \exp (-1/(2\gamma_0^2 g^2)(1 + O(g^2)))$$

On lattice then, in the asymptotic scaling regime, all observables scale this way, so in particular, ratios of physical observables (e.g. $M_N/M_\rho$, or anything else you can think of) are independent of the lattice spacing→ the renormalization group trajectory.
In practice, the scaling regime is hard to access:

“critical slowing down”: as $a \to 0$ lattice correlation lengths diverge. Physics is scale invariant. Continuum limit is a 2nd order phase transition.

Instead, simulate at several values of $6/g^2$ (modest lattice spacings) and several quark masses at each lattice spacing.

Extrapolate in quark mass to desired physical point, then extrapolate to $a \to 0$ in leading discretization error, i.e. linear or quadratic in $a$. 
Monte Carlo Simulation

Back to the path integral

\[ \langle \mathcal{O} \rangle = \frac{1}{Z} \int d[\bar{\psi}, \psi, U] \mathcal{O}(\bar{\psi}, \psi, U) \exp (-i S(\bar{\psi}, \psi, U)) \]

Analytically continue (Wick rotate) to Euclidean space-time so the integrand behaves sensibly:

\[ \langle \mathcal{O}_E \rangle = \frac{1}{Z_E} \int d[\bar{\psi}, \psi, U] \mathcal{O}_E(\bar{\psi}, \psi, U) \exp (-S_E(\bar{\psi}, \psi, U)) \]

(Now drop all “E” subscripts)

Fermion integrals are Gaussian, do them analytically.

\[ \langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU] \mathcal{O}(U) \det(M(U))^{n_f} e^{-S_g(U)} \]
det(M(U))^{n_f} e^{-S_g} is an ordinary probability weight: do the integral over gauge fields numerically by Monte Carlo simulation (stat. mech. in d+1 dimensions).

Use *importance sampling* to generate an ensemble of gauge field configurations (O(100 – 1000) independent ones):

- 1 configuration = set of link variables over entire lattice
- update algorithm: choose links randomly
- algorithm must satisfy detailed balance and ergodicity
• generate configurations with probability \( \text{det}(M(U))^{n_f e^{-S_g}} \)

• Observables become simple averages over configurations.

Simulation with \( \text{det}(M(U)) \) (dynamical fermions) is costly. \( \text{det}(M(U)) = 1 \) is the quenched approximation, i.e., no virtual quark loops in the vacuum \( (m_q \rightarrow \infty) \).
Fermion discretizations
(why not naive fermions?)

\[ \bar{\psi} \not{D} \psi \rightarrow \bar{\psi} \gamma_\mu (\psi(x + \hat{\mu}) - \psi(x - \hat{\mu}))/2a \]

\[ G_{latt}(p) = \frac{i \gamma_\mu \sin(\alpha p_\mu)}{\sum_\mu \sin^2(\alpha p_\mu)} \rightarrow \frac{i \gamma_\mu \alpha p_\mu}{\sum_\mu (\alpha p_\mu)^2}. \]

\( G_{latt}(p) \) has a pole at each corner of the Brilloiun zone:
\( p^\mu = (\pi/a,0,0,0), (0,\pi/a,0,0), \ldots, (\pi/a,\pi/a,\pi/a,\pi/a) \)

Lattice theory corresponds to \( 2^d \) fermion flavors instead of one.
These extra fermions are called **doublers**. Appeared because of the inherent periodicity of the lattice.

Minkowski space dispersion relation ($E = |p|$)

Even worse for the Standard model, the doublers appear in pairs with opposite chirality—theory is **vector-like** (*Nielsen-Niyomiyia No-Go theorem*). Deep connection to gauge-invariance.
Must get rid of the doublers.

1. **Wilson fermions.** Add an irrelevant term to the action

\[
S_W = \frac{a}{2} \bar{\psi} \partial^2 \psi \\
\sim \frac{1}{a} \sum_\mu 1 - \cos(p_\mu)
\]

Like a mass term. Doubler mass \( \sim 1/a \), and they decouple.
Problems with Wilson Fermions:

- Chiral symmetry (of QCD) is explicitly broken, badly broken. (flavor symmetry is still exact, as in the continuum)

- Chiral limit $\neq m_q \to 0$.

- Complicated **fine tuning** (operator mixing) of observables required for correct chiral behavior.

- Errors are $O(a)$: slow approach to the continuum (can be improved to $O(a^n)$ $n = 2$ now, big job)

All problems solved as $a \to 0$
2. Kogut-Susskind.

Spin diagonalization. Throw away 3/4 of components: 16 Dirac fermions = 64 components → 16. One component “spinor” on a lattice site.

Exact remnant chiral symmetry, so \( m_q \to 0 \) is the chiral limit.

Can reconstruct 4 Dirac fermions from components in \( 2^4 \) hypercube. In the continuum limit this is a theory of 4 degenerate quarks. For \( a \neq 0 \) flavor, spin, and space-time symmetries are mixed.

Take fractional power of fermion determinant to simulate real QCD (2+1 flavor).
Problems with Kogut-Susskind fermions

- Have to take fractional powers of the determinant!

- Flavor symmetry is broken: one light pion instead of $16 - 1 = 15$

- Relation to continuum operators can be very difficult to work out

- Errors are $O(a^2)$ but are unusually large because of flavor symmetry violation. Again, slow approach to continuum. Can be improved: now the state-of-the-art for dynamical fermion simulations ($a^2$-tad).

Discovered in 1987 (then forgotten) the most chiral symmetry that a lattice theory can have

\[ \gamma_5 D + D \gamma_5 = a DR \gamma_5 D \]
\[ \gamma_5 D^{-1} + D^{-1} \gamma_5 = a R \gamma_5 \]

Meanwhile, domain wall fermions (DWF) (Kaplan 1992) and later overlap fermions (Neuberger 1997) were discovered. Worked for vector gauge theories. Hasenfratz rediscovered the G-W relation, and it was soon realized that DWF and overlap are examples (with \( R = 1/2 \)).
G-W fermions Remove the doublers while (essentially) preserving full $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry of the continuum at non-zero lattice spacing.

We (RBC) use domain wall fermions (Shamir 1993)

Errors are $O(a^2)$
Problems with Ginsparg-Wilson fermions

- Expensive!

- 1st large-scale dynamical fermion simulation done here at BNL (and Columbia University). Light (up and down) quark mass 1/2 to 1 times $m_{strange}$ (need to reduce by 10). Volume is not large ($\sim (2\text{fm})^3$), and only one lattice spacing.

- Took almost 2 years on our own supercomputer (QCDSP)!

Continuum-like properties $\rightarrow$ approach to continuum is faster

New computer(s) coming: QCDOC ($\times20$ faster, 5 TFlops/sustained)
Masses and Matrix elements from Euclidean space correlation functions.

Consider the pseudo-scalar meson (pion) 2-point correlation function

$$J_5(t) = \sum \bar{\psi}(x, t) \gamma_5 \psi(x, t) e^{\vec{p} \cdot \vec{x}}$$

Sum over $\vec{x}$ projects onto the state with momentum $\vec{p}$

The zero momentum correlation function reads

$$C(t) = \sum \langle 0 | \bar{\psi}(x, t) \gamma_5 \psi(x, t) \bar{\psi}(0, 0) \gamma_5 \psi(0, 0) | 0 \rangle$$

Wick contract fields into quark propagators

$$C(t) = \sum \text{Tr} \left[ M_{0; x, t}^{-1} \gamma_5 M_{x, t; 0}^{-1} \gamma_5 \right]$$
What's it good for?

Use time-translation operator \( U = \exp(-Ht) \) and insert a complete set of states (\( H \) is the QCD Hamiltonian, and the states are eigenstates of \( H \)) (in Euclidean space there is no \( i \) in \( U \))

\[
C(t) = \sum_x \langle 0| e^{Ht} \bar{\psi}(x) \gamma_5 \psi(x) e^{-Ht} \bar{\psi}(0) \gamma_5 \psi(0) |0 \rangle = \sum_x \langle 0| e^{Ht} \bar{\psi}(x) \gamma_5 \psi(x) e^{-Ht} \sum_n \frac{|n\rangle\langle n|}{2E_nV} \bar{\psi}(0) \gamma_5 \psi(0) |0 \rangle = \sum_n \langle 0| \bar{\psi} \gamma_5 \psi |n \rangle\langle n| \bar{\psi} \gamma_5 \psi |0 \rangle \frac{e^{-E_n t}}{2E_nV} = \lim_{t \to \infty} \frac{|\langle 0| \bar{\psi} \gamma_5 \psi |\pi \rangle|^2}{2m_\pi} e^{-m_\pi t}
\]

Fit yields physical particle mass and matrix element.
or the nucleon 3 point correlation function,

\[ \langle \chi_N(p', t') \sum_x e^{i \vec{q} \cdot \vec{x}} [\bar{\psi}_q(x, t) \Gamma_\mu \psi_q(x, t)] \chi^\dagger_N(p, 0) \rangle \rightarrow \sum_{s,s'} \langle 0 | \chi_N(p', s') | p', s' \rangle \langle p', s' | \Gamma_\mu(q) | p, s \rangle \langle p, s | \chi^\dagger_N(p, s) | 0 \rangle \times e^{-E t - E' (t' - t)} \frac{2 E 2 E'}{2 E 2 E'} \]

where \( t' \gg t \gg 0 \), \( \vec{q} = \vec{p'} - \vec{p} \), and \( \chi_N \) is the nucleon interpolating operator.

Euclidean space continued LSZ reduction formula that relates (the Fourier transform of) Minkowski space Greens functions to S-matrix elements. Exponentials pick them out instead of poles.
This always works for single-particle states (like nucleon matrix elements).

For multi-particle states (i.e. non-leptonic decays) this is much more difficult.
Accessing the chiral limit, $m_q \to 0$

Ideally, adjust the quark masses in our simulations until observables (masses, decay constants, ...) match their physical values. For example, adjust $m_u$ and $m_d$ until the pseudo-scalar meson mass is $m_{\pi} = 135$ MeV. Knowing the value of the light quark masses, we can predict the proton mass, neutron mass, $f_\pi$, etc.
Not so simple. The chiral limit, $m \to 0$ is difficult.

- “cost” of quark propagator $M^{-1}$: #iterations $\sim \frac{1}{m}$

- Compton wavelength of the pion $\frac{1}{m_\pi} \to \infty$ as $m_q \to 0$, so must take $V \to \infty$ to avoid finite volume effects

- Instead, work at unphysical (larger) $m_q$ and extrapolate to the physical regime (chiral limit). Use Chiral Perturbation Theory as a guide.
Chiral Perturbation Theory (S. Weinberg)

Low energy effective field theory of QCD. Systematic expansion in $p^2$, around $p^2 = 0$ (chiral limit). (Pseudo-) Goldstone bosons are the only degrees of freedom left.

$$\mathcal{L}^{(2)}_{QCD} = \frac{f^2}{8} \text{tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + \frac{f^2 B_0}{4} \text{tr}[\chi^\dagger \Sigma + \Sigma^\dagger \chi]$$

$$\Sigma = \exp \left[ \frac{2i \phi^a \lambda^a}{f} \right]$$

$$\Sigma \rightarrow V_L \Sigma V_R^\dagger \quad \text{(under a chiral transformation)}$$

\(\Sigma\) is the unitary chiral matrix field \((V_L, R \in SU(N_f))\), \(\lambda^a\) are proportional to the Gell-Mann matrices with \(\text{tr}(\lambda_a \lambda_b) = \delta_{ab}\), \(\phi^a\) are the real pseudoscalar-meson fields, and \(f\) is the meson decay constant in the chiral limit. \(\chi = (m_u, m_d, m_s)_{\text{diag}}\)
To lowest order

\[ m_\pi^2 = B_0 (m_u + m_d) \]
\[ m_K^2 = B_0 (m_d + m_s) \]

\ldots

At this order, we can work with mesons made from *degenerate* quarks, so the quark masses corresponding to the physical mesons are

\[ m_l = \frac{m_u + m_d}{2} \]
\[ m_s/2 = \frac{m_d + m_s}{2} \]
Can go to higher order in $\chi$PT ($\mathcal{O}(p^4)$)

RBC $n_f = 2$ dynamical quark simulation:

\[ f_K/f_\pi = 1.194(12) \text{ (statistical error only)} \]
Operator Renormalization

In lattice QCD calculations, we often calculate matrix elements of local operators generated by an Operator Product Expansion (OPE) of a non-local operator (usually a product of two currents). e.g. DIS, or non-leptonic Weak decay of hadrons.

We do this out of necessity since the physical processes can not be calculated purely perturbatively or non-perturbatively.

\[ A^{\text{phys}} = \sum_n C_n(\mu) \langle f | O_n(\mu) | i \rangle \]

\( A^{\text{phys}} \) and states do not depend on scale \( \mu \).
Define finite, renormalized operator at scale $\mu$

$$\mathcal{O}(\mu) = Z_{\mathcal{O}}(a\mu)\mathcal{O}(a)$$

$Z_{\mathcal{O}}(a\mu)$ can be computed:

- In lattice perturbation theory
- Non-perturbatively (RI-MOM) (mimic perturbation theory $\sim$ very high order perturbative calculation)
- Perturbative matching to $\overline{\text{MS}}$, or whatever scheme is used to compute $C_n(\mu)$
Lattice complications:

Broken symmetries (Lorentz, chiral symmetry, flavor, ...) \(\Rightarrow\) operator mixing

Non-perturbative renormalization (NPR) \textit{required} when mixing with lower dimensional operators occurs. These are power divergent in the lattice spacing \(a^{-(d-d')}\) instead of the usual logarithmic divergence \(\log(a\mu)\) (... domain wall fermions)
To calculate $Z_O$ compute Landau gauge off-shell matrix elements of $O(a)$ between quark and/or gluon states

$$\text{Tr} V_O(p^2) \Gamma \bigg|_{p^2 = \mu^2} \frac{Z_O}{Z_q} = 1$$

- $V_O(p^2)$ the amputated vertex constructed from the full non-pert quark propagator
- $\Gamma$ a projector

This defines the MOM scheme. Extrapolate to $m_f \rightarrow 0$ and we have the RI scheme (Regularization Independent).

\( Z_s(\mu^2) (\bar{\psi} \psi) \) renormalization factor, and divided by 3-loop perturbative running.

RBC (2001).
Statistical and Systematic errors

- Finite sample of configurations: statistical errors
- Finite volume
- Non-zero lattice spacing
- Chiral limit
- Quenched approximation

Lattice Gauge Theory provides a first principles framework to solve QCD, with (in principal) arbitrary precision.