Nucleon (Spin) Structure from the Lattice

Tom Blum
University of Connecticut and RIKEN BNL Research Center

Thanks to Kostas Orginos
Outline

1. Part I: (DIS) Structure Functions
   (a) Introduction
   (b) Lattice operators
   (c) Matrix Elements
   (d) Comparison to experiment

2. Generalized Parton Distributions
   (a) Brief overview
   (b) Lattice results
   (c) Gluon total angular momentum
Part I
\[
\left| \frac{\mathcal{A}}{4\pi} \right|^2 = \frac{\alpha^2}{Q^4} l_{\mu\nu} W_{\mu\nu}
\]
\[
W_{\mu\nu} = W^{[\mu\nu]} + W^{\{\mu\nu\}}
\]

\[
W^{\{\mu\nu\}}(x, Q^2) = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) F_1(x, Q^2) + \left( P^\mu - \frac{\nu}{q^2} q^\mu \right) \left( P^\nu - \frac{\nu}{q^2} q^\nu \right) \frac{F_2(x, Q^2)}{\nu}
\]
\[
W^{[\mu\nu]}(x, Q^2) = i \epsilon^{\mu\nu\rho\sigma} q_\rho \left( \frac{S_\sigma}{\nu} (g_1(x, Q^2) + g_2(x, Q^2)) - \frac{q \cdot S P_\sigma}{\nu^2} g_2(x, Q^2) \right)
\]

with \( \nu = q \cdot P, \ S^2 = -M^2, \ x = Q^2/2\nu \)

Calculate **non-perturbatively** nucleon structure functions:

- **Unpolarized**: \( F_1(x, Q^2), \ F_2(x, Q^2) \)
- **Polarized**: \( g_1(x, Q^2), \ g_2(x, Q^2), \ ( \ h_1(x, Q^2) \ ) \)
Moments of Structure Functions

\[
2 \int_0^1 dxx^{n-1} F_1(x, Q^2) = \sum_{q=u,d} c_{1,n}^{(q)}(\mu^2/Q^2, g(\mu)) \langle x^n \rangle_q(\mu) + \mathcal{O}(1/Q^2),
\]

\[
\int_0^1 dxx^{n-2} F_2(x, Q^2) = \sum_{f=u,d} c_{2,n}^{(q)}(\mu^2/Q^2, g(\mu)) \langle x^n \rangle_q(\mu) + \mathcal{O}(1/Q^2),
\]

\[
2 \int_0^1 dxx^n g_1(x, Q^2) = \sum_{q=u,d} e_{1,n}^{(q)}(\mu^2/Q^2, g(\mu)) \langle x^n \rangle_{\Delta q}(\mu) + \mathcal{O}(1/Q^2),
\]

\[
2 \int_0^1 dxx^n g_2(x, Q^2) = \frac{1}{2n+1} \sum_{q=u,d} [e_{2,n}^{q}(\mu^2/Q^2, g(\mu)) d_n^q(\mu) - 2e_{1,n}^{q}(\mu^2/Q^2, g(\mu)) \langle x^n \rangle_{\Delta q}(\mu)] + \mathcal{O}(1/Q^2)
\]

• \(c_1, c_2\) and \(e_1, e_2\) are the Wilson coefficients (perturbative),

• \(\langle x^n \rangle_q(\mu), \langle x^n \rangle_{\Delta q}(\mu)\) and \(d_n\) are forward nucleon matrix elements of certain local operators \(\mathcal{O}\).
Lattice Operators

Broken Lorentz symmetry: $O(4) \rightarrow H(4)$

Operators in irreducible representations of $O(4)$ transform *reducibly* under the lattice Hyper-cubic group.

Consequence: higher moment operators mix with lower (space-time) dimensional operators.

The lower dimensional operators mix with *power divergent* coefficients.

In practice, lattice is limited to the lowest few moments if lower dimensional operator mixing is to be avoided. (Non-perturbative lower dimensional operator subtraction is possible, *c.f.* Kaon physics)
Unpolarized ($F_1$ and $F_2$):

$$\frac{1}{2} \sum_s \langle P, S | O^q_{\mu_1 \mu_2 \cdots \mu_n} | P, S \rangle = 2 \langle x^{n-1} \rangle_q (\mu) [P_{\mu_1} P_{\mu_2} \cdots P_{\mu_n} + \cdots - \text{trace}]$$

$$O^q_{\mu_1 \mu_2 \cdots \mu_n} = \bar{q} \left[ \left( \frac{i}{2} \right)^{n-1} \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} - \text{trace} \right] q$$

On the lattice we can measure: $\langle x \rangle_q$, $\langle x^2 \rangle_q$ and $\langle x^3 \rangle_q$

Only $\langle x \rangle_q$ can be measured with $\vec{P} = 0$
Polarized \((g_1 \text{ and } g_2)\):

\[
- \langle P, S | \mathcal{O}^{5q}_{\{\sigma_{\mu_1\mu_2\ldots\mu_n}\}} | P, S \rangle = \frac{2}{n + 1} \langle x^n \rangle_{\Delta q}(\mu) [S_\sigma P_{\mu_1} P_{\mu_2} \cdots P_{\mu_n} + \cdots - \text{traces}] \\

\mathcal{O}^{5q}_{\sigma_{\mu_1\mu_2\ldots\mu_n}} = \overline{q} \left[ \left( \frac{i}{2} \right)^n \gamma_5 \gamma_\sigma \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} - \text{traces} \right] q
\]

\[
\langle P, S | \mathcal{O}^{[5]q}_{\{\sigma\{\mu_1\}_1\mu_2\ldots\mu_n\}} | P, S \rangle = \frac{1}{n + 1} d^q_n(\mu) [(S_\sigma P_{\mu_1} - S_{\mu_1} P_\sigma) P_{\mu_2} \cdots P_{\mu_n} + \cdots - \text{traces}] \\

\mathcal{O}^{[5]q}_{\{\sigma\mu_1\}_1\mu_2\ldots\mu_n} = \overline{q} \left[ \left( \frac{i}{2} \right)^n \gamma_5 \gamma_\sigma \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} - \text{traces} \right] q
\]

On the lattice we can measure: \(\langle 1 \rangle_{\Delta q} (g_A), \langle x \rangle_{\Delta q}, \langle x^2 \rangle_{\Delta q}, d_1, d_2\).

Only \(\langle 1 \rangle_{\Delta q}, \langle x \rangle_{\Delta q}, d_1\) can be measured with \(\vec{P} = 0\)
Transversity \((h_1)\):

\[
\langle P, S | \mathcal{O}_{\rho \nu \{\mu_1 \mu_2 \cdots \mu_n\}}^{\sigma q} | P, S \rangle = \frac{2}{m_N} \langle x^n \rangle_\delta q \left[ (S_\rho P_\nu - S_\nu P_\rho) P_{\mu_1} P_{\mu_2} \cdots P_{\mu_n} + \cdots - \text{traces} \right]
\]

\[
\mathcal{O}_{\rho \nu \mu_1 \mu_2 \cdots \mu_n}^{\sigma q} = q \left[ \left( \frac{i}{2} \right)^n \gamma_5 \sigma_{\rho \nu} \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} - \text{traces} \right] q
\]

On the lattice we can measure \(\langle 1 \rangle_\delta q\) and \(\langle x \rangle_\delta q\)

Only \(\langle 1 \rangle_\delta q\) can be measured with \(\vec{P} = 0\)
Status of lattice calculations

  - Structure func./Generalized PD and Form Factors
  - Wilson fermions improved and unimproved
  - quenched and $N_f = 2$ dynamical (with UKQCD).

- **LHPC - SESAM** [hep-lat/0201021, hep-lat/0312014]
  - Structure func./Generalized PD and Form Factors
  - unimproved Wilson fermions
  - quenched and $N_f = 2$ dynamical.

- **RBC** (preliminary results) [hep-lat/0209137, hep-lat/0309113]
  - Structure func. and Form Factors
  - Domain wall fermions
  - quenched and $N_f = 2$ dynamical
Unpolarized moments for the proton

**QCDSF:**

Quenched results, $2 \leq a^{-1} \leq 4$ GeV. $O(a)$ improved Wilson fermions. Volume $\sim (1.5-3 \text{ fm})^3$, $m_\pi > 600$ MeV. Nonperturbative renormalization
LHPC-SESAM:
diamonds - quenched,
squares - dynamical

Roughly the same masses, as QCDSF
\(a^{-1} \sim 2 \text{ GeV}\)
Volume only \((1.5 \text{ fm})^3\)
Perturbative renormalization

QCDSF:
quenched - triangles

[hep-lat/0201021]
What happens at the chiral limit?

Existing calculations (quenched and dynamical) at relatively heavy quark masses seem to disagree with experiment.

Possible resolution(?):
- Finite lattice spacing
- Finite volume(?) (see $g_A$)
- Chiral logs

\[
V(m_\pi^2) = V_c \left[ 1 + C_\chi m_\pi^2 \ln \frac{m_\pi^2}{\mu^2} \right]
\]

Assume:

\[
V(m_\pi^2) = V_c \left[ 1 + C_\chi m_\pi^2 \ln \frac{m_\pi^2}{m_\pi^2 + \mu^2} \right] + B m_\pi^2
\]

$C_\chi$ calculated in $\chi$-PT, depends of $f_\pi$ and $g_A$ only
$\mu^2$ phenomenological parameter $\mu \sim 550$ MeV.

Detmold et.al.
Arndt&Savage
Chen&Ji, Chen&Savage
Simulation parameters (quenched)

- **Gauge Action:** DBW2

\[ S_g = \frac{\beta}{3} \text{Re} \text{Tr} \left[ (1 - 8c_1) \left< 1- \right> + 2c_1 \left< 1- \right> \right] \]

With \( c_1 = -1.4067 \) computed by non-perturbative RG blocking. [Takaishi Phys. Rev. D54 (1996)]

- \( \beta = 0.870 \) or \( a^{-1} = 1.3 \text{GeV} \), Volume: \( 16^3 \times 32 \sim 2.4^3 \text{fm}^3 \) box.

- **Fermion Action:** Domain wall fermions \( L_s = 16 \rightarrow m_{\text{res}} \sim 0.7 \text{MeV} \)

- quark mass: as light as \( 1/4 \times m_{\text{strange}} \)

- **Statistics:** 416 Lattices QCDSP 300Gflops for 4 months
Simulation parameters ($n_f = 2$ dynamical)

- **Gauge Action:** DBW2
  \[ S_g = \frac{\beta}{3} \text{Re} \text{Tr} \left[ (1 - 8c_1) \left\langle 1 - \begin{array}{cc} \cdot & \end{array} \right\rangle + 2c_1 \left\langle 1 - \begin{array}{cc} \cdot & \end{array} \right\rangle \right] \]
  With \( c_1 = -1.4067 \) computed by non-perturbative RG blocking. [Takaishi Phys. Rev. D54 (1996)]

- \( \beta = 0.800 \) or \( a^{-1} = 1.7 \text{GeV} \), Volume: \( 16^3 \times 32 \sim 2.0^3 \text{fm}^3 \) box.

- **Fermion Action:** Domain wall fermions \( L_s = 12 \rightarrow m_{\text{res}} \sim 2.5 \text{MeV} \)

- **quark mass:** as light as \( 1/2 \times m_{\text{strange}} \)

- **Statistics:** about 50 Lattices

- **Status:** ON GOING
Flavor non-singlet

\[ \langle x \rangle_q = q_{\gamma 4} \mapsto D_{4\gamma} - \frac{1}{3} \sum_{k=1}^{3} \gamma_k \mapsto D_k \]

Hypercubic group rep.

Momentum: \( \vec{P} = 0 \)

Renormalization: Multiplicative

Quark density

\[ \mathcal{O}_{44}^q = \bar{q} \left[ \gamma_4 \mapsto D_{4\gamma} - \frac{1}{3} \sum_{k=1}^{3} \gamma_k \mapsto D_k \right] q \]

- Hypercubic group rep. \( 3_1^+ \)
- Momentum: \( \vec{P} = 0 \)
- Renormalization: Multiplicative

Note:
- Renormalization:
  \[ Z = 1.02(10) \]
  \[ \overline{MS} = 2 \text{GeV} \text{ 2-loop running} \]
- No Curvature in the chiral limit

\[ \frac{\langle x \rangle_u}{\langle x \rangle_d} = 2.41(4) \text{ at the chiral limit} \]
Flavor non-singlet

Quark density: (dynamical)

\[ \mathcal{O}_{44}^q = \bar{q} \left[ \gamma_4 \hat{D}_4 - \frac{1}{3} \sum_{k=1}^{3} \gamma_k \hat{D}_k \right] q \]

- Hypercubic group rep. \(3_1^+\)
- Momentum: \(\vec{P} = 0\)
- Renormalization: Multiplicative

Note:
- Unrenormalized
Axial Charge

$\langle 1 \rangle_{\Delta q} \ (g_A : \text{Axial charge})$ Plot: Horsley review, Lattice 2002, Boston
Axial Charge

- Renormalization:

\[ \langle A^{\text{cons}} \bar{q} \gamma_5 q \rangle = Z_A \langle A^{\text{loc}} \bar{q} \gamma_5 q \rangle \]

[Y. Aoki LAT01, hep-lat/0201021]

\[ Z_A = 0.77759(45) \]

- Chiral limit (Linear fit):

\[ g_A = 1.212 \pm 0.027_{\text{stat}} \pm 0.024_{\text{norm}} \]

See Sasaki, Orginos, Ohta, Blum
Finite volume effect for $g_A$

Previous RBC study: [Blum, Ohta, Sasaki] 1.6fm box
New RBC study: 2.4fm and 1.2fm box
Clear finite volume effect

For $\text{dwf (chiral symmetry)}$: $Z_A = Z_V = 1/g_V$
Axial Charge (dynamical)

- Renormalization:
  \[ \langle A^\text{cons} \bar{q} \gamma_5 q \rangle = Z_A \langle A^\text{loc} \bar{q} \gamma_5 q \rangle \]

\[ Z_A = 0.75765(45) \]

See Orginos, Ohta Lattice 2003
Measure:

\[ O_{34}^{5q} = \frac{1}{4} \bar{q} \gamma_5 \left[ \gamma_3 \vec{D}_4 + \gamma_4 \vec{D}_3 \right] q \rightarrow \langle x \rangle_{\Delta} \]

- Hypercubic group rep.: \(6^-\)
- Momentum: \( \vec{P} = 0 \)
- Renorm.: Multiplicative

Note:
- Renormalization:
  \[ Z = 1.02(9) \]
  \[ \overline{MS} = 2\text{GeV} \] 2-loop running
- No curvature in the chiral limit
- Light mass needs more statistics
Large discrepancy here as well
Transversity:
\[ \mathcal{O}^{\sigma q}_{34} = \bar{q}\gamma_5\sigma_{34}q \rightarrow \langle 1 \rangle \delta q \]
- Hyper-cubic group representation: \( 6_1^+ \)
- Momentum: \( \vec{P} = 0 \)
- Renorm.: Multiplicative

Note:
- Renormalization (NPR)

\[ \langle 1 \rangle_{\delta u - \delta d} = 1.193(30) \]
\[ \overline{MS}(2 GeV) \) 2-loop running

QCDSF (quenched continuum):
\[ \langle 1 \rangle_{\delta u - \delta d} = 1.214(40) \]
\[ \overline{MS}(2 GeV) \) 1-loop perturbative
Transversity (dynamical):

$$\mathcal{O}_{34}^{\sigma q} = \bar{q}\gamma_5\sigma_{34}q \rightarrow \langle 1 \rangle_{\delta q}$$

- Hyper-cubic group representation: $6^+_1$
- Momentum: $\vec{P} = 0$
- Renorm.: Multiplicative

Note:
- Unrenormalized (will do NPR)
The $g_2$ structure function

$$2 \int_0^1 dx x^n g_2(x, Q^2) = \frac{1}{2n+1} \sum_{q=u,d} [e_{2,n}^q (\mu^2/Q^2, g(\mu)) d_{n}^q (\mu) - 2e_{1,n}^q (\mu^2/Q^2, g(\mu)) \langle x^n \rangle_{\Delta_q (\mu)}],$$

\[
\langle x^n \rangle_{\Delta_q (\mu)} \rightarrow \text{T\lowercase{wist} 2} \\
\]

\[
d_{n}^q (\mu) \rightarrow \text{T\lowercase{wist} 3}
\]

$d_{n}^q (\mu)$ estimations:

- Negligible $\Rightarrow$ Wandzura - Wilczek relation of $g_1$ and $g_2$
- Need not be small in a confining theory [Jaffe and Ji Phys.Rev.D43,91].
Twist Three

\[
\langle P, S \mid \mathcal{O}[\sigma \{\mu_1 \mu_2 \cdots \mu_n\}] \mid P, S \rangle = \frac{1}{n + 1} d_n^q(\mu)(S_\sigma P_{\mu_1} - S_{\mu_1} P_\sigma) P_{\mu_2} \cdots P_{\mu_n} + \cdots - \text{traces}
\]

\[
\mathcal{O}_{[\sigma \mu_1 \mu_2 \cdots \mu_n]}[5]^q = \bar{q} \left[ \left( \frac{i}{2} \right)^n \gamma_5 \gamma_{\{\sigma D_{\mu_1} \cdots D_{\mu_n} \} - \text{traces} \} \right] q
\]

Measure:

\[
\mathcal{O}_{34}[5]^q = \frac{1}{4} \bar{q} \gamma_5 \left[ \gamma_3 \leftrightarrow D_4 - \gamma_4 \leftrightarrow D_3 \right] q \rightarrow d_1^q
\]

- Hyper-cubic group representation: \( 6_1^+ \)
- Momentum: \( \vec{P} = 0 \)
- Renormalization: Multiplicative (DWF Chiral symmetry)

Chiral symmetry breaking causes mixing with

\[
\mathcal{O}_{34}^{\sigma q} = \bar{q} \gamma_5 \sigma_{34} q
\]
Note:

- Unrenormalized
- Disagreement with the Wilson results
  Power divergent mixing
  [LHPC-SESAM: hep-lat/0201021]
- Small at chiral limit
Dynamical

- Unrenormalized
- Chiral symmetry is good
- Small at chiral limit
• Lattice QCD can compute non-perturbatively certain low moments of structure functions.

• Several systematic errors still need careful study:
  • Finite lattice spacing, Finite volume
  • Chiral limit
  • Quenching

• Started the calculation of Nucleon matrix elements with Domain wall fermions ... improved chirality, scaling

• Preliminary results look promising
  Simulation at light quark masses possible

• \( g_A : 1.212 \pm 0.027_{\text{stat}} \pm 0.024_{\text{norm}} \) ... finite volume (quenched)
  dynamical : consistent/need better statistics

• \( \langle x \rangle_{u-d} \) and \( \langle x \rangle_{\Delta u-\Delta d} \) : No curvature down to 390MeV pion mass

• \( d_1 \) : Absence of power divergent mixing (chiral symmetry)
  Small at the chiral limit
Part II: Generalized parton distributions
Start with the light-cone operator $O_q(x)$ that describes parton-parton correlations:

$$O_q(x) = \int \frac{d\lambda}{4\pi} e^{i\lambda x} \bar{q}(-n) \mathcal{P} e^{-ig \int_{-\lambda/2}^{\lambda/2} d\alpha n \cdot A(\alpha n)} q(\frac{\lambda}{2}n).$$

Its forward matrix element is the usual parton distribution:

$$q(x) = \langle P | O_q(x) | P \rangle$$

Expand in terms of local operators (OPE)

$$O_q^{\{\mu_1 \mu_2 \ldots \mu_n\}} = \bar{q}^{\{\mu_1 \vec{D} \mu_2 \ldots \vec{D} \mu_n\}} q,$$

and, as before

$$\langle x^n \rangle_q(\mu) = \int dxdx^{n-1} q(x)$$
Off-forward matrix element:

\[ \langle P' | \mathcal{O}(x) | P \rangle = \bar{u}(p') \left( n H(x, \xi, t) + \frac{i \Delta^\nu}{2m} \sigma^{\mu\nu} n_\mu E(x, \xi, t) \right) u(p) \]

defines GPD's: \( H(x, \xi, t), E(x, \xi, t) \)

\( t = \Delta^2, \xi = -n \cdot \Delta/2, \Delta = P' - P \)

Insert \( \gamma_5 \rightarrow \) spin dependent GPD's: \( \tilde{H}(x, \xi, t), \tilde{E}(x, \xi, t) \)
Generalized Form Factors

The moments of the GPD’s give (generalized) form factors.

The lowest give Dirac and Pauli form factors:

\[
\int_{-1}^{1} \, dx \, H_q(x, \xi, \Delta^2) = F_1^q(\Delta^2),
\]
\[
\int_{-1}^{1} \, dx \, E_q(x, \xi, \Delta^2) = F_2^q(\Delta^2).
\]

\[
\int_{-1}^{1} \, dx \, x \, H_q(x, \xi, \Delta^2) = A_2^q(\Delta^2) + \xi^2 C_2^q(\Delta^2),
\]
\[
\int_{-1}^{1} \, dx \, x \, E_q(x, \xi, \Delta^2) = B_2^q(\Delta^2) - \xi^2 C_2^q(\Delta^2),
\]
nucleon matrix elements of the energy-momentum tensor (EMT):

\[ \langle p'|O^q_{\{\mu\nu\}}|p \rangle \equiv \frac{i}{2} \langle p'|\bar{q}\gamma_{\{\mu}D_{\nu\}}q|p \rangle \]

\[ = A_2^q(\Delta^2) \bar{u}(p')\gamma_{\{\mu}\bar{p}_{\nu\}}u(p) \]

\[ - B_2^q(\Delta^2) \frac{i}{2m_N} \bar{u}(p')\Delta^{\alpha}\sigma_{\alpha\{\mu}\bar{p}_{\nu\}}u(p) \]

\[ + C_2^q(\Delta^2) \frac{1}{m_N} \bar{u}(p')u(p)\Delta_{\{\mu}\Delta\nu\}}. \]

In the forward limit, \( \Delta^2 \to 0 \),

\[ A_2^q(0) = \langle x_q \rangle \]

and

\[ \frac{1}{2}(A_2^q(0) + B_2^q(0)) = J_q, \]

The angular momentum is given by the gauge invariant pieces (Ji 1997)

\[ J_q = L_q + S_q \]

One calculates the spin \( S_q \) separately, then determines the orbital piece \( L_q \).
Lattice results


- quenched
- non-pert improved Wilson fermions
- Wilson gauge action, $a^{-1} \sim 2$ GeV
- $m_\pi \sim 600 - 1000$ MeV
- $L \sim 1.6$ fm ($16^3 \times 32$)
For the EMT, use two sets of operators

\[ \frac{1}{\sqrt{2}} (O_{\mu\nu} + O_{\nu\mu}), \quad 1 \leq \mu < \nu \leq 4 \]

and

\[ \frac{1}{2} (O_{11} + O_{22} - O_{33} - O_{44}), \]

\[ \frac{1}{\sqrt{2}} (O_{33} - O_{44}), \quad \frac{1}{\sqrt{2}} (O_{11} - O_{22}). \]

Each set transforms irreducibly under the hypercubic group.

The operators are renormalized at 2 GeV in the $\overline{MS}$ scheme:

$Z^{\overline{MS}}_{\nu_{2a}} (2 \text{ GeV}) = 1.10$ and $Z^{\overline{MS}}_{\nu_{2b}} (2 \text{ GeV}) = 1.09$. 
\( \kappa = 0.1333 \) results, \( m_\pi \sim 800 \text{ MeV} \)
Dipole Fit

\[ A_2^q(\Delta^2) = \frac{A_2^q(0)}{1 - \frac{\Delta^2}{M^2}} \]
\[ M = 1.1(2) \text{ (GeV)} \]

Close to \( f_2 \) and \( a_2 \) meson masses (lattice systematics ?)
Assuming dipole (Regge) behavior holds for higher moments,

$$\int_{-1}^{1} dx \, x^n \, H_q(x, 0, \Delta^2) = \langle x_q^n \rangle / (1 - \Delta^2/M_{n+1}^2)^2,$$

Then from inverse Mellin transform (using previous $\langle x_q^n \rangle$, $0 \leq n \leq 3$)

$$H_q(x, 0, b_{\perp}^2) = \frac{1}{(2\pi)^2} \int d^2 \Delta_{\perp} \, e^{ib_{\perp} \Delta_{\perp}} H_q(x, 0, \Delta_{\perp}^2)$$  [M.Burkhardt (2000)]

probability of quark with momentum x at transverse impact parameter (Schierholz, et al., hep-ph/0312104)
LHPC-SESAM (hep-lat/0312014)

- $N_f = 2$

- unimproved Wilson fermions

- Wilson gauge action, $a^{-1} \sim 2$ GeV

- $\kappa = 0.1570, 0.1560$: $m_\pi \sim 750, 900$ MeV

- $L \sim 1.6$ fm ($16^3 \times 32$)
Dipole fits

$n = 2$ moments consistent with QCDSF (quenching effects?)

Slope at origin $\sim$ transverse size, decreases as $n$ increases

Weaker dependence at lighter mass

Flavor dependence

Spin dependent case shows markedly weaker dependence on $n$
Nucleon spin carried by the quarks

Ji (1997): Total angular momentum from EMT:
\[
J = \frac{1}{2} \sum_q \left( A^q_2(0) + B^q_2(0) \right)
= \sum_q S_q + \sum L_q
= S + L
\]

Spin from usual moments $\Delta^q$

<table>
<thead>
<tr>
<th></th>
<th>$J$</th>
<th>$J_u$</th>
<th>$J_d$</th>
<th>$S_u$</th>
<th>$S_d$</th>
<th>$S$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QCDSF(2003)</td>
<td>0.33(7)</td>
<td>0.37(6)</td>
<td>-0.04(4)</td>
<td>0.42(1)</td>
<td>-0.12(1)</td>
<td></td>
<td>0.03(7)</td>
</tr>
<tr>
<td>Kentucky(2000)</td>
<td>0.30(7)</td>
<td></td>
<td></td>
<td>0.13(6)</td>
<td>0.17(6)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

30-40% (?) Carried by the gluons
The gluon total angular momentum (Ji 1997)

Need nucleon matrix element of $T_{\mu\nu}^g$ (same as before)

$$\langle p'|T_{\mu\nu}^g|p\rangle \equiv \langle p'| \frac{1}{2} \left( \frac{1}{4} \delta_{\mu\nu} F^2 + F_{\mu\sigma} F_{\sigma\nu} \right) |p\rangle$$

$$= A_2^q (\Delta^2) \bar{u}(p') \gamma_{\{\mu} \bar{p}_{\nu\}} u(p)$$

$$- B_2^q (\Delta^2) \frac{i}{2m_N} \bar{u}(p') \Delta^\alpha \sigma_{\alpha\{\mu} \bar{p}_{\nu\}} u(p)$$

$$+ C_2^q (\Delta^2) \frac{1}{m_N} \bar{u}(p') u(p) \Delta_{\{\mu} \Delta_{\nu\}}.$$ 

On the lattice, $T_{\mu\nu}$ is given by the plaquette $U(x)_{\mu\nu}$ (Karsch and Wyld (1987)):

$$T_{\mu\nu} = \frac{2}{g^2} \left( - \sum_{\nu \neq \mu} \text{Tr} U(x)_{\mu\nu} + \sum_{\sigma \neq \mu, \sigma > \nu} \text{Tr} U(x)_{\mu\nu} \right) + O(g^2)$$

$$T_{\mu\nu} = \frac{2}{g^2} \text{Tr} \bar{U}(x)_{\mu\sigma} U(x)_{\nu\sigma}$$

$$\bar{U}(x)_{\mu\nu} \equiv - \frac{i}{2} (U_{\mu\nu} - U_{\nu\mu})_{\text{traceless}}$$
The above yields $J^g$.

Likely very noisy: non-zero momentum nucleons, gluonic operators make life difficult.

Worth a try?

Can we get $S^g$ and $L^g$ separately? Probably not.

non-local operator $\mathcal{O}$ matrix element yields (lowest moment of) $\Delta g$.

$\mathcal{O}$ corresponds to $S^g$ in the $A^+ = 0$ gauge in the infinite momentum frame. Not accessible to a Euclidean lattice calculation.
Summary

- Lattice calculation of GPD’s just getting started
- interesting qualitative results
- Systematics not yet under control