Lecture I: SPIN SUM RULE

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Motivation

- Consider a composite system (molecules, atoms, nuclei, nucleon,...) with intrinsic angular momentum \( S \) (spin).

- The spin must arise from the dynamics of the underlying constituents.

- Roughly speaking, one may write
  \[
  S = \sum_i s_i + \sum_i l_i
  \]
  the sum of the spin and orbital angular momentum of the constituents. (non-relativistic)

- How do we write down such expression in quantum field theory and what are the complications?
Non-Relativistic and Relativistic Quantum Mechanics

- In non-relativistic quantum mechanics, spin is introduced by hand, not by theoretical consistency.

- In relativistic quantum mechanics, spin is a consequence of the spacetime symmetry.

  All experimental evidences so far indicate that spacetime is a flat, four-dimensional Minkowski space with signature = (1, -1, -1, -1), when gravity is ignored.

  \[ x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^i) \]

- Symmetry \(\rightarrow\) Conservation Law \(\rightarrow\) Quantum Numbers
Poincare Group

- The spacetime is invariant under the Poincare transformations
  - Translations
    \[ x^\mu \rightarrow x^\mu + a^\mu, \]
  - 4 parameter transformations
  - Lorentz transformations (rotations)
    \[ x^\mu \rightarrow \Lambda^\mu_\nu x^\nu \]
    \[ \Lambda^\mu_\alpha \Lambda^\alpha_\nu = g^\mu_\nu \] (length is invariant)
  - 6 parameters: 3 for spatial rotations in (ij) plane;
    3 for Lorentz boosts in i-direction.

- All transformations form a 10 para. Poincare group
  - Contracted SO(3,2) group
Generators of Transformations

- In quantum mechanical Hilbert space, all Poincare transformations are represented by unitary operators $U(\Lambda, a)$

$$U(\Lambda, a) U^\dagger(\Lambda, a) = U^\dagger (\Lambda, a) U (\Lambda, a) = 1$$

- Finite transformations can be built out of infinitesimal ones with

$$\Lambda^{\mu\nu} = I^{\mu\nu} + \omega^{\mu\nu} \text{ with } \omega^{\mu\nu} = - \omega^{\nu\mu}$$

$$a^\alpha = \varepsilon^\alpha ,$$

$$U(\Lambda, a) = I - i \omega_{\mu\nu} J^{\mu\nu} / 2 + i \varepsilon_\alpha P^\alpha$$

$J^{\mu\nu}$ and $P^\alpha$ are hermitian operators in the Hilbert space and are the generators of the Poincare transformations.
Physical Observables

Thus the spacetime symmetry predicts the existence of the following physical observables for any physical system:

- $p^0 = H$ is the Hamiltonian of the system,
- $p_i$ is the linear momentum
- $j_i = \varepsilon^{ijk} j_{jk}/2$ is the angular momentum
- $k^i = j^{i0}$ generates Lorentz boosts

$k^i$ is not a familiar observable because we usually choose to diagonalize other observables, like ladder operators in angular momentum algebra.

However, in light-cone quantization, states are chosen to have simpler behavior under boosts.
Poincare Algebra

- Poincare generators satisfy the following algebra

\[ [P^\alpha, P^\beta] = 0, \]
\[ [J^{\mu\nu}, P^\alpha] = i(g^{\alpha\nu}P^\mu - g^{\alpha\mu}P^\nu) \]
\[ [J^{\mu\nu}, J^{\alpha\beta}] = i(g^{\nu\alpha}J^{\mu\beta} - g^{\mu\alpha}J^{\nu\beta} + g^{\nu\beta}J^{\alpha\mu} - g^{\mu\beta}J^{\alpha\nu}) \]

- Any physical system represents a realization of this algebra!

The states of the system can be classified according to the representations of this algebra, independent of whether it is an elementary particle (quarks) or a composite system (atoms).
Constructing A Representation

- One needs to find a complete set of commuting observables.
  - Start with the Abelian subalgebra \( P^\mu \).
  - **Mass**: Casimir operator \( P^2 = P^\mu P_\mu = m^2 \).
  - **Little group**: for a massive particle, we can choose \( P^\mu_0 = (m, 0, 0, 0) \) particle rest frame
    - The little group is the SO(3) rotation
    - The generators of the little group is the angular momentum operator \( J^i \).
    - label states with different \( J^2 = s(s+1) \) and \( J_z = m_s : s = 0, 1/2, 1, 3/2, ... \) the spin states
Pauli-Lubanski Vector

- Generalizing the angular momentum operator to arbitrary frame (Pauli-Lubanski),
  \[ W^\mu = -\varepsilon^{\mu\alpha\beta\gamma} J_{\alpha\beta} P_\gamma / 2\sqrt{P^2} \]
- In the particle rest frame
  \[ W^\mu = (0, J^i) \quad \text{and} \quad W_\mu P^\mu = 0 \quad \text{(true for any frame)} \]
- \[ [W^\mu, P^\nu] = 0, \]
  \[ W^\mu \] can be diagonalized at the same time as \( P^\nu \)
- \[ [W^\mu, W^\nu] = i \varepsilon^{\mu\nu\alpha\beta} W_\alpha P_\beta / 2\sqrt{P^2} \]
  \( W \)-generators form a group (little group) for a fixed \( p^\mu \).
Frame-Independent Spin Quantum Number

- There must be a Casimir operator which defines the spin as a frame-independent property, just like mass:
  \[ W^2 = W^\mu W_\mu \]
  It has quantum numbers \(-s(s+1)\) with \(s = 0, 1/2, 1, \ldots\)
- For a particle with non-zero momentum, \(W^\mu\) also involves the Lorentz boost generators.
  - In that sense, the content of the spin of a particle seem different in the different frame?
Magnetic Quantum Number

- The magnetic quantum number $m_s$ depends on the quantization axis
  - Choose in the rest frame the 3-vector $s^i$.
    \[ \vec{s} \cdot \vec{J} \left| p_\mu^\mu 0 s m_s \right\rangle = m_s \left| p_\mu^\mu 0 s m_s \right\rangle \]
  - This can be generalized to any momentum
    \[ -s^\mu W_\mu \left| p_\mu^\mu s m_s \right\rangle = m_s \left| p_\mu^\mu s m_s \right\rangle \]
    where $s^\mu$ is a polarization vector reduces to $(0,s^i)$ in the rest frame.
    \[ s^\mu \rho_\mu = 0 \quad \text{true in any frame.} \]
Choices of the Polarization Vector

- Three most popular choices:
  1. \( s^i = (0,0,1) \): \textit{S. Weinberg}
  2. \( s^i \) arbitrary: \textit{Bjorken \& Drell}
    In both cases, \( s^\mu W_\mu \) involves boost operators
  3. \( s^\mu = (|p|/m, p^0 p^i / m |p|) \): \textit{Jacob and Wick}
    magnetic quantum number is called helicity: \( \lambda \)

\[
h = -s^\mu W_\mu = \frac{\vec{J} \cdot \vec{p}}{|\vec{p}|}
\]

the helicity operator is independent of the boost operators and is independent of the velocity of the particle!
Helicity (spin) Sum Rule

- Consider a particle moving in the z-direction with helicity $\lambda$
  \[
  J_z |m p_z s \lambda\rangle = \lambda |m p_z s \lambda\rangle
  \]
  Or we can write
  \[
  \lambda = \langle m p_z s \lambda | J_z | m p_z s \lambda \rangle
  \]

- If the angular momentum operator $J_z$ can be written as a sum of contributions $\Sigma_i J_{zi}$, we have a helicity sum rule
  \[
  \lambda = \sum_i \lambda_i = \sum_i \langle m p_z s \lambda | J_{zi} | m p_z s \lambda \rangle
  \]
Constructing Generators of Poincare Group

- Reps of the Poincare group can be used to classify elementary particles (quarks, leptons, gauge bosons, etc.)
- Particle creation and annihilation operators can be used to construct quantum fields forming representations of the Lorentz group (scalars, spinors, vectors, etc.)
- Quantum fields facilitate constructions of lagragian densities and actions.
- Continuous Spacetime symmetries of the actions lead to conserved charges by Noether’s theorem
A Spin-1/2 Free Particle

- **Lagrangian density**
  \[ L = \overline{\psi} (i\partial - m )\psi \]

- **Under infinitesimal translation** \( x^\mu \rightarrow x^\mu + a^\mu \)
  \[ \psi(x) \rightarrow (1-a^\mu \partial_\mu ) \psi(x) \]
  The action \( S = \int Ld^4x \) is invariant. The conserved current is the energy-momentum (stress-energy) tensor

\[
T^{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \partial_\nu \phi_\alpha - g^{\mu\nu} L \\
= \frac{1}{2} \left( \overline{\psi} \gamma^\mu i\partial_\nu \psi + \overline{\psi} \gamma^\mu i\partial_\nu \psi \right)
\]

- The conserved charge is \( P^\mu = \int d^3 x T^{0\mu} \)
Angular Momentum Density

- Under Infinitesimal rotation $x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$, the action is also invariant,
  \[
  \psi(x) \rightarrow S(\Lambda)\psi(\Lambda^{-1}x)
  \]
  \[
  = \left(1 - \frac{i}{2} \omega_{\mu\nu} j^{\mu\nu}\right)\psi(x)
  \]
  \[
  j^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2} + (x^\mu \partial^\nu - x^\nu \partial^\mu)
  \]

- The symmetry leads to the following conserved current (angular momentum density)
  \[
  M_{\alpha\mu\nu} = x^\mu T^\alpha_{\mu\nu} - x^\nu T^\alpha_{\mu\nu} + \frac{\partial L}{\partial \partial_{\alpha} \phi_\gamma} (-i \Sigma^{\mu\nu})_{\gamma\beta} \phi^\beta
  \]
  \[
  = \overline{\psi} \gamma^\alpha \left[\sigma^{\mu\nu} / 2 + (x^\mu \partial^\nu - x^\nu \partial^\mu)\right]\psi
  \]

- The conserved charges are
  \[
  J^{\mu\nu} = \int d^3x M^{0\mu\nu}
  \]
Gauge Theories

- The *canonical* energy-momentum and angular-momentum densities are not gauge-invariant under gauge transformations.

\[ T^{\mu \nu} = \frac{1}{4} g^{\mu \nu} F^2 - F^{\mu \alpha} \partial^\nu A_\alpha \]

- Although Noether’s theorem guarantees the existence of a conserved charge, conserved current is not unique,

\[ j^\mu = j^\mu_c + \partial^\nu X^{[\mu \nu]} \]

X is called a *superpotential*.

- One can use this freedom to obtain an energy-momentum tensor which is symmetric in their indices and gauge invariant (*couple to gravity*).
An “improved” Energy-Momentum Tensor

**Belinfante Improvement:** make the energy-momentum tensor symmetric in the two indices. It can be shown

\[
\frac{\partial L}{\partial \partial^\mu \phi^\alpha} \partial^\nu \phi^\alpha - \frac{\partial L}{\partial \partial^\nu \phi^\alpha} \partial^\mu \phi^\alpha = \partial_\beta \lambda^{\beta\mu\nu}
\]

\[
\lambda^{\beta\mu\nu} = \frac{\partial L}{\partial \partial_\beta \phi^\alpha} (\Sigma^{\mu\nu})_{\alpha\alpha'} \phi^{\alpha'}
\]

Thus,

\[
T^{\mu\nu}_B = T^{\mu\nu}_C + \partial_\beta X^{\beta\mu\nu}
\]

\[
X^{\beta\mu\nu} = \frac{1}{2} \left[ -\lambda^{\beta\mu\nu} + \lambda^{\mu\beta\nu} + \lambda^{\nu\beta\mu} \right]
\]

- Antisymmetric in beta and mu
- Adding a symmetric piece
- Cancel out the antisymmetric part
Quantum Chromodynamics

- A fundamental theory of strong interactions
- Building blocks
  - Spin-1/2 quarks $\Psi_{\alpha if}$
    of three colors $i=1,2,3$ &
    of six different flavors: $f=u,d,s,c,b,t$
  - Spin-1 massless gluons $A_{\mu a}$
    of eight colors $a=1,...,8$
- An SU(3) gauge theory

\[
L_{\text{QCD}} = \overline{\Psi}_{if} (i\partial - m_{qf}) \Psi_{if} - \frac{1}{4} F_{\mu \nu a} F^{\mu \nu a} - g_s \overline{\Psi}_{if} A_{ij} \Psi_{jf} \\
F_{\mu \nu a} = \partial_{\mu} A_{\nu a} - \partial_{\nu} A_{\mu a} - g_s f_{abc} A_{b\mu} A_{c\nu}
\]
Improved Energy-Momentum Tensor and Angular Momentum Denstty of QCD

- The QCD energy-momentum tensor can be decomposed into quark and gluon contributions

\[ T^{\mu\nu}_B = T^{\mu\nu}_q + T^{\mu\nu}_g \]
\[ T^{\mu\nu}_q = \sum_f \bar{\psi}_f \gamma^\mu iD^\nu \psi_f, \]
\[ T^{\mu\nu}_g = \left( \frac{1}{4} \right) g^{\mu\nu} F^2 - F^{\mu\alpha} F_{\alpha}^\nu \]

- Improved angular momentum density

\[ M^{\alpha\mu\nu}_B = T^{\alpha\mu}_B \chi^\nu - T^{\alpha\nu}_B \chi^\mu \]
Angular momentum Operator of QCD

- Angular momentum is a spatial moment of the momentum density \( T^{0i} \)

\[
\vec{J} = \int r \times \vec{T} \, d^3 r
\]

thus it depends on the stress-energy tensor!

- Correspondingly the QCD angular momentum can be decomposed as

\[
\vec{J} = \vec{J}_q + \vec{J}_g
\]

\[
\vec{J}_q = \int \psi^+ \Sigma / 2 \psi + \int \psi^+ r \times iD\psi
\]

\[
\vec{J}_g = \int r \times (E \times B)
\]

However, the individual terms do not transform like the spatial components of four-vectors!
Nucleon Helicity Sum Rule

- Consider a proton in its helicity eigenstate. If it is moving along the z-direction

\[ 1/2 = \langle p | J_z | p \rangle \]

- Inserting the QCD expression for the angular momentum operator, we have

\[ 1/2 = J_q(\mu) + J_g(\mu) \]

\[ J_q(\mu) = \Delta \Sigma(\mu) / 2 + L_q(\mu) \]

- \( \Delta \Sigma \) is measurable through polarized deep-inelastic scattering.

But, how to get \( J_q \) and \( J_g \)?
Lorentz Transformation

- The individual contributions are invariant under a subclass of Lorentz transformations
  - Boost along the z-direction
  - Rotation around the z-direction

Consider the matrix element

\[ \langle p \bigg| \int d^4 x M^{\mu\alpha\beta} \bigg| p \rangle = \tilde{U}(p) \left[ \frac{i}{2} A \left( p^\mu [\gamma^\alpha, \gamma^\beta] + \gamma^\mu [\gamma^\beta p^\alpha - \gamma^\alpha p^\beta] \right) ight. \\
\left. + B \left( g^\alpha^\gamma \gamma^\beta - g^\alpha^\beta \gamma^\gamma \right) \right] U(p) (2\pi)^4 \delta^4(0) + \ldots \]

Taking \( \mu = 0, \alpha = 1, \) and \( \beta = 2, \) we have,

\[ \langle p + |J^z| p+ \rangle / \langle p + |p+ \rangle = A. \]

Thus \( A \) must be \( 1/2. \)
Lorentz Invariance

Now suppose $M^{\mu\alpha\beta} = \sum_i M_i^{\mu\alpha\beta}$. The Lorentz symmetry yields,
\[
\langle p \left| \int d^4 x M_i^{\mu\alpha\beta} \right| p \rangle = U(p) \left[ \frac{i}{2} A_i(\mu) p^\mu [\gamma^\alpha, \gamma^\beta] + B_i(\mu) \gamma^\mu \left( \gamma^\beta p^\alpha - \gamma^\alpha p^\beta \right) \\
+ C_i(\mu) \left( g^{\mu\alpha} \gamma^\beta - g^{\mu\beta} \gamma^\alpha \right) \right] U(p)(2\pi)^4 \delta^4(0) + ...
\]

where $A_i(\mu)$, $B_i(\mu)$ and $C_i(\mu)$ are scalar constants. Then the consistency between Eqs. (1) and (20) yields,
\[
\sum_i A_i(\mu) = A = \frac{1}{2}.
\]

From this, one has
\[
\langle p + |J_i^z|p+\rangle / \langle p + |p+\rangle = A_i(\mu),
\]

Independent of the momentum $p$!

The sum rule is valid in the rest frame and infinite momentum frame.