# Lecture 2: Scale Evolution and Form Factors

Xiangdong Ji
University of Maryland

# Summary of Lecture I

- One can write down a helicity sum rule in quantum field theory

$$\lambda = \sum_i \lambda_i = \sum_i \langle mp_z s\lambda | J_{zi} | mp_z s\lambda \rangle$$

in which the individual terms,

  - are invariant under a subclass of Lorentz transformations
  - are gauge invariant in QCD
  - depend on the renormalization scale μ.
  - do not obey the simple angular momentum algebra.

Composite operators

- \

# Regularization and Renormalization

- Regularization: In field theory, the formal lagrangian is meaningless. One must introduce "cut-off's" to make the theory well-defined.
  - Two popular regularizations:
    - Dimensional regularization ($d=4-\varepsilon$)
    - Lattice regularization
- Renormalization: Parameters of the theory are fixed according to certain "physical" conditions.

  —>**An Effective Field Theory**: with only the degrees of freedom below momentum scale $\mu$.

# Effective Field Theory

- All degrees of freedom above $\mu$ have been integrated out. The renormalized lagrangian is sensitive to physics above $\mu$ only through parameters in renormalized interactions.
- The quarks and gluons are resolved at the scale $\mu$, or have effective sizes $1/\mu$.
- Effective theory at any $\mu$ is entirely equivalent to the original theory (choice of scale). There are infinitely-many equivalent effective theories (choice of scheme).
- A choice of scheme & scale is like a choice of coordinates. The precise nature of quarks and gluons depends on the choice and therefore has no absolute meaning.

# Renormalization Group Equations

- Auxiliary variables are scale dependent that is calculable in perturbation theoryu: for example

$$\mu \frac{dg(\mu)}{d\mu} = -\beta_0 \frac{g(\mu)^3}{16\pi^2} - \ldots \text{ with } \beta_0 = 11 - 2n_f/3$$

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \log(\mu^2/\Lambda_{QCD}{}^2)} \to 0 \quad \text{as} \quad \mu \to \infty$$

Similarly for parton distributions

- Physical observables, like masses, however, are scale independent:

$dM/d\mu = 0$ $\quad M \sim \mu \exp(2\pi/\beta_0 \alpha_s(\mu))$

Nonperturbative even as $\mu \to \infty$.

# Scale-dependence and experimental probe

- Although scale is $\mu$ arbitrary, however, the best choice in an experimental process involving probing scale Q is $\mu$ = Q.
  - Factorization theorem

$$d\sigma(Q) \sim C(\mu, Q) O(\mu)$$

  - C(Q, $\mu$) is perturbatively calculable, but involves the large loga

ln$Q/\mu$

to accelerate the convergence of the series, let $\mu$ = Q.

-

$$d\sigma(Q) \sim O(Q)$$

Probing scale is roughly the renormalization scale!

# Scale Evolution of the Contributions

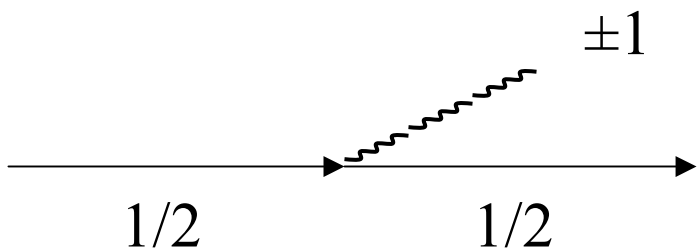
- Proton helicity sum rule,

$$1/2 = J_q(\mu) + J_g(\mu)$$
$$J_q(\mu) = \Delta\Sigma(\mu)/2 + L_q(\mu)$$

- Individual contribution is probing-scale dependent, but the sum is not.
- $\Delta\Sigma$ has been measured in polarized DIS, and is 0.2±0.1 at scale 1 GeV or so.
  - Non-relativistic quark models 1.0.
  - Relativistic quark models 0.6~0.7
- Clearly there are also quark orbital angular momentum and gluon contribution!

# Physics of Angular Momentum Evolution

- Consider a quark of helicity $\frac{1}{2}$ at scale μ, splitting into a quark and a gluon at scale μ+dμ




- Angular momentum is conserved,

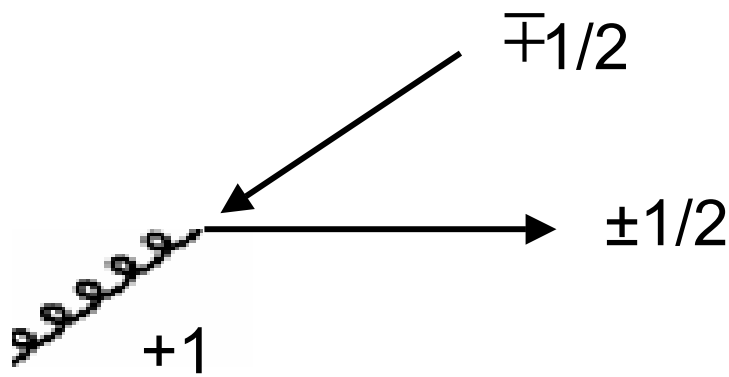
$$\sum J_i^z = const$$

$$1/2 = 1/2 \pm 1 \mp 1$$

$$\Delta g \quad L$$

# Gluon splitting

Consider a gluon of helicity +1 at scale μ,
splitting into a quark and antiquark pair at scale μ+dμ




Δg Δq L

$$1 = \pm\frac{1}{2} \mp \frac{1}{2} + 1$$

Similarly for splitting
Into two gluons

# Evolution

- Helicity generation

$$\frac{d}{dt}\begin{pmatrix}\Delta\Sigma\\ \Delta G\end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix}0 & 0\\ \frac{3}{2}C_F & \frac{\beta_0}{2}\end{pmatrix}\begin{pmatrix}\Delta\Sigma\\ \Delta G\end{pmatrix} ,$$

$$t = \ln \mu^2/\Lambda^2_{\rm QCD},\ C_F = 4/3 \text{ and } \beta_0 = 11 - 2n_f/3$$

- Solution

$$\Delta\Sigma(t) = \text{const} ,$$
$$\Delta G(t) = -\frac{4\Delta\Sigma}{\beta_0} + \frac{t}{t_0}\left(\Delta G_0 + \frac{4\Delta\Sigma}{\beta_0}\right) .$$

# Orbital Evolution

- The splitting process also generates orbital angular momentum

$$\frac{d}{dt}\begin{pmatrix} L_q \\ L_g \end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} -\frac{4}{3}C_F & \frac{n_f}{3} \\ \frac{4}{3}C_F & -\frac{n_f}{3} \end{pmatrix}\begin{pmatrix} L_q \\ L_g \end{pmatrix} + \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} -\frac{2}{3}C_F & \frac{n_f}{3} \\ -\frac{5}{6}C_F & -\frac{11}{2} \end{pmatrix}\begin{pmatrix} \Delta\Sigma \\ \Delta G \end{pmatrix}$$

- Solution

$$L_q(t) = -\frac{1}{2}\Delta\Sigma + \frac{1}{2}\frac{3n_f}{16+3n_f} + (t/t_0)^{-2(16+3n_f)/(9\beta_0)}\left(L_q(0) + \frac{1}{2}\Delta\Sigma - \frac{1}{2}\frac{3n_f}{16+3n_f}\right) ,$$

$$L_g(t) = -\Delta G(t) + \frac{1}{2}\frac{16}{16+3n_f} + (t/t_0)^{-2(16+3n_f)/(9\beta_0)}\left(L_g(0) + \Delta G(0) - \frac{1}{2}\frac{16}{16+3n_f}\right)$$

# RG Equation for Angular Momentum

- Scale evolution equation

$$\frac{d}{d\,ln\,\mu^2}\begin{pmatrix} J_q(\mu^2) \\ J_g(\mu^2) \end{pmatrix} = \frac{\alpha_s}{2\pi\cdot 9}\begin{pmatrix} -16,3n_f \\ 16,-3n_f \end{pmatrix}\begin{pmatrix} J_q(\mu^2) \\ J_g(\mu^2) \end{pmatrix}$$

- Asymptotic solution

$$J_q(\infty) = \frac{1}{2}\frac{3n_f}{16+3n_f}, J_g(\infty) = \frac{1}{2}\frac{16}{16+3n_f}$$

Roughly half of the angular momentum is carried by gluons!

# How to measure $J_q$ and $J_g$

- Observe that the angular momentum is related to the moment of the momentum density

$$\vec{J} = \int \vec{r} \times \vec{T} d^3 \vec{r}$$

- Therefore, the matrix element of J is related to the form factor of the momentum density.
- Similar example exists in QED.

# Magnetic Moment and E&M Form Factors

- The magnetic moment is a spatial moment of the current density

$$\vec{\mu} = \frac{1}{2}\int \vec{r} \times \vec{j} d^3\vec{r}$$

- It can be extracted from the form factors of the electromagnetic current

$$\boxed{\langle p' | j^\mu | p \rangle = U(p')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]U(p)}$$

- In the Breit frame, the spin-flip form factor $qG_M(q^2) = q(F_1(q^2)+F_2(q^2))$ yields the electric current distribution in the NR limit. With this, we have

$$\boxed{\mu_p = G_M(0) = F_1(0) + F_2(0)}$$

# Derivation

- Consider off forward matrix element

$$\langle p' | \mu | p \rangle$$

- Using Heisenberg equations of motion to get rid of the coordinate dependence in the current
- Turn r factor into a derivative on momentum, and one is left with a matrix element

$$\langle p' | j | p \rangle$$

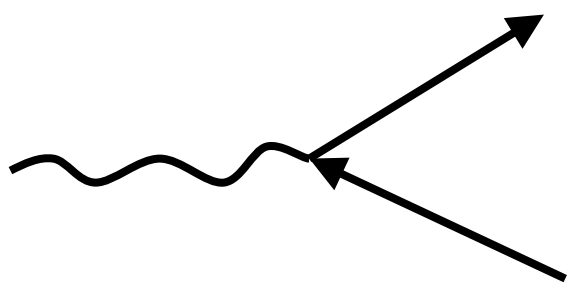
- Calculate the derivative of the matrix element

# Form Factors of the Stress-Energy Tensor

- Angular momentum may be obtained from the form factors of the energy-momentum tensor

$$\langle P'|T^{\mu\nu}(\mu^2)|P\rangle = \overline{U}(p')\Big[A(q^2)\gamma^\mu \overline{P}^\nu + B(q^2)\frac{i\sigma^{\mu\alpha}q_\alpha}{2M}\overline{P}^\nu + C(q^2)(g^{\mu\nu}q^2 - q^\mu q^\nu)/4M\Big]U(p)$$

- These form factors can be measured, in principle, through gravitational scattering,



# Gravitational Helicity Form Factors

- Consider the scattering in the Breit frame in which the graviton and the nucleon momenta are collinear and the graviton has zero-energy.

  Helicity conservation limits the off-shell graviton helicity to 0, ±1.

What are the physics of these helicity amplitudes?

- Helicity-0 gravition scattering yields the matrix element of mass density $\langle p'|T^{00}|p\rangle$. One has then the mass form factor

$$G_{Mass}\left(q^2\right) = A(q) + \frac{q^2}{4M^2}\left(B(q) + C(q)\right)$$

- Its Fourier transformation yields the mass distribution in the NR limit.

- Helicity-1 gravition scattering yields the ME of momentum density $\langle p'|T^{0i}|p\rangle$. One has then the linear momentum form factor

$$G_{mom}\left(q^2\right) = q\left(A(q) + B(q)\right)$$

- Its Fourier transformation yields the momentum density distribution in the NR limit

From which, we find the proton spin

$$\boxed{J_p = \left(A(0) + B(0)\right)/2}$$

- Dilaton scattering yields the ME of the scalar density $\langle p'|T^{\mu}{}_{\mu}|p\rangle$. One has the dilaton form factor

$$G_{dilaton}\left(q^2\right) = A(q) + \frac{q^2}{4M^2}\left(B(q) - 3C(q)\right)$$

# Conservation Law and charge

- Conservation laws lead to to constant charge
- Energy momentum conservation leads to

  $A(0) = 1$
- Conservation of the angular momentum leads to

  $A(0) + B(0) = 1$

  Therefore $B(0) = 0$

  Anomalous gravitational magnetic moment is zero! (known for a long time)

# Quark and gluon parts of angular momentum and gravitational form factors

- Define the gravitational form factors of the quark and gluon parts of the energy-momentum tensor

  $A_{q,g}(q^2)$ and $B_{q,g}(q^2)$

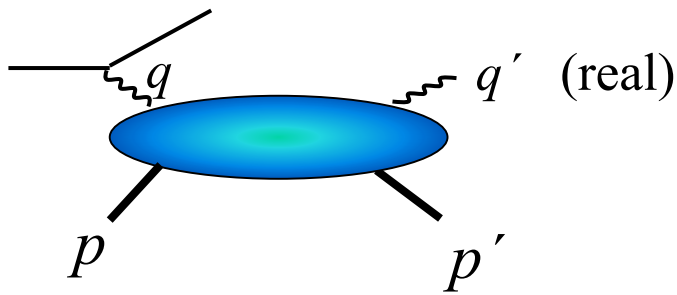
- The quark & gluon contributions to the spin of the nucleon can be obtained from the corresponding form factors

$$J_{q,g} = \frac{1}{2}\left(A_{q,g}(0) + B_{q,g}(0)\right)$$

But how to measure this form factors??

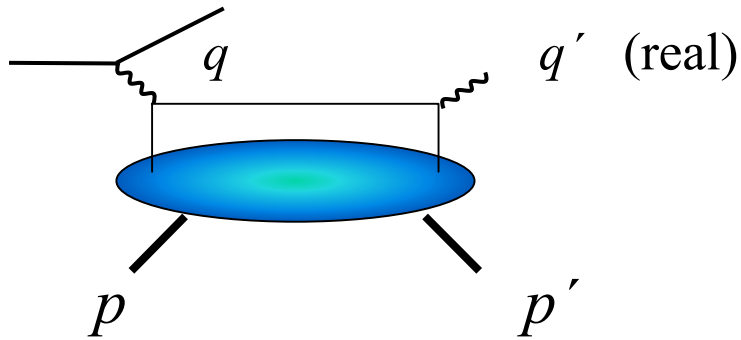
# Measuring gravitational form factors without graviton scattering

- Generic virtual Compton scattering




  - Scattering mechanism complicated

- When the virtual photon is in the Bjorken limit, the scattering process simplifies enormously




  - Single-quark scattering dominates!
    Compton scattering on a single quark
  - Non-invasive surgery
  - Deeply-virtual Compton scattering (X. Ji, '96)

# Scaling Analysis

- In the scaling limit, one can use the operator production expansion,

$$TJ^{\mu}(\xi)J^{\nu}(0) = \ldots + C^{\mu\nu\alpha\beta}(\xi)\overline{\psi}\gamma_{\alpha} iD_{\beta}\psi + \ldots$$

The DVCS amplitude then contains the off-forward matrix elements of the energy-momentum tensor.

But it contains much more

- Generalized parton distribution
- Quantum phase-space Wigner type of distribution

Lectures by M. Diehl

# Connection to RHIC spin

- One of the goals at RHIC is to measure $\Delta g$ which is gauge invariant, but only has simple interpretation as the gluon helicity in light-cone gauge $A^+=0$
- Total gluon contribution to the spin of the nucleon $J_g$ cannot be decomposed in a gauge invariant way into spin and orbital contribution. It may be measured through DVCS type of experiment with heavy-quarkonium production.
- Result on $\Delta g$ provides a way to understand $J_g$ in a specific gauge. However, it is not known how to measure the complementary part.