Basics of Polarized Beam Acceleration

Protons:

- Transverse beam dynamics.
- Simple model of the proton.
  - Spin dynamics.
  - Depolarizing resonances.
  - Siberian snakes.
- The real machines: RHIC and injectors.

Electrons/Positrons:

- Longitudinal beam dynamics.
  - Synchrotron oscillations and tune.
  - Electrons: Synchrotron radiation
- Radiative polarization.
- Quantum fluctuations ⇒ Spin Diffusion
- Polarization in some real e± machines.
- Measurements with polarized e± beams.
Acceleration with RF cavities

Linac: $\vec{F} = q\vec{E}(t)$.

- Must maintain synchronism of bunch with rf phase.
- Particles oscillate in energy about the stable synchronous phase.
Particle Trajectories in Magnetic Fields

Dipole magnets bend the beam around the ring.

Quadrupole magnets focus the beam for stability.

Charged particles are deflected by magnetic fields. Lorentz Force:

\[ \vec{F} = \frac{q}{\gamma m} \vec{p} \times \vec{B} \]
Hamiltonian without Spin

\[ H(X, P_X, Y, P_Y, Z, P_Z; t) = \sqrt{(\vec{P} - q\vec{A})^2 + m^2c^4} + q\phi \]

After a bunch of canonical transformations and \( \phi = 0, \vec{A} = (0, 0, A_s) \):

\[ \mathcal{H}(x, x', y, y', z, \delta p/p_0; s) \simeq -\frac{q}{p_0}A_s - \left(1 + \frac{x}{\rho}\right) \left(1 + \frac{\delta p}{p_0} - \frac{1}{2}(x'^2 + y'^2) + \cdots \right) \]

\[ \rho = \frac{p}{qB_{\perp}} \]
\[ x' = \frac{dx}{ds} \]
\[ y' = \frac{dy}{ds} \]

Paraxial approx.: \(|x'|, |y'| \ll 1\)

QCDSP: Spin Dynamics in Accelerators
Waldo MacKay 7 June, 2004
Hill’s Equations

\[ x'' + k_x(s)x = \frac{\delta}{\rho(s)}, \]
\[ y'' + k_y(s)y = 0, \]
with \( \delta = \frac{\delta p}{p_0}. \)

For quadrupoles:
\[ k_x = \frac{q}{p} \frac{\partial B_y}{\partial x}, \]
\[ k_y = -\frac{q}{p} \frac{\partial B_y}{\partial x}. \]

Harmonic oscillator with periodic spring constant.

Periodic conditions:
\[ k_j(s + L) = k_j(s), \quad \rho(s + L) = \rho(s) \]
where \( L \) is length of periodic cell.

- Horizontal motion has inhomogeneous dispersion term.
  - Ignore it for now.
Solutions to Hill’s Equation

Use Floquet’s (Block’s) Theorem ⇒
Quasi-periodic solutions of form:

\[ x(s) = \sqrt{\mathcal{W}\beta(s)} \cos(\psi(s)), \text{ with } \]
\[ \psi'(s) = \frac{1}{\beta(s)}. \]

Periodicity of \( \beta \)-function:

\[ \beta(s + L) = \beta(s). \]

Note: In general \( \psi(s + L) \neq \psi(s) + n2\pi \). Resonances are bad!

\[ x'(s) = -\sqrt{\mathcal{W} \over \beta} (\alpha \cos \psi + \sin \psi), \]
with \( \alpha = -{1 \over 2} \beta' \).
Transport and Betatron Oscillations

Alternate focusing and defocusing lenses for stability.

Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.

Vertical Betatron Oscillation
with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.
For a particular trajectory with initial conditions:

- Solve for $\sin \psi$ and $\cos \psi$ from equations for $x$ and $x'$.
- Use $\sin^2 \psi + \cos^2 \psi = 1$ to get an invariant:

$$\mathcal{W} = \frac{1}{\beta} [y^2 + (\alpha y + \beta y')^2]$$  \hspace{1cm} (1)

- Functions of $s$: $y(s)$, $y'(s)$, $\beta(s)$, $\alpha(s)$. ($\beta$ and $\alpha$ are periodic.)
- Eq. (1) is the equation for an ellipse.
  - Area of ellipse = $\pi \mathcal{W}$.
- Beam envelope: $\pm \sqrt{\beta(s)} \epsilon$
  - $\pi \epsilon$ is the rms emittance
Liouville’s Theorem

- Most beams have a low enough density, so that we ignore hard collisions between particles.
  - Thus we can use a 6d phase space rather than a 6N-d phase space.

- In the phase space of coordinates and their corresponding canonical momenta, the phase flow of the particle trajectories evolves so that the volumes of differential volume elements are preserved.
  - In other words, the Jacobian determinant is 1.

- Emittance is the area of the projection of the beam’s phase-space volume onto a particular \((x_i, P_i)\) plane.
Horizontal Betatron Oscillation with tune: $Q_x = 3.28$, tracked through 10 turns with 8 periodic cells.

Poincaré plot of proton on successive turns for one location in the ring.
Torque on Classical Magnetic Moment

Semiclassical model:
- The spin $\vec{S}$ has a constant magnitude in the rest frame.
- The magnetic moment $\vec{\mu} \propto \vec{S}$.
  - $\vec{\mu}$ has a constant magnitude in the rest frame.
    (Sort of like a loop of infinite inductance.)
Simple Model of Proton

Gyroscope + Bar magnet + Charge = "proton"

Magnetic Spin Dipole Moment

Polarization: Average spin of the ensemble of protons.

\[ \vec{P} = \frac{1}{N} \sum_{j=1}^{N} \frac{\vec{S}}{|S|} \]
Relativistic Angular Momentum

Energy-momentum tensor (à la Weinberg)

\[ T^{\alpha \beta}(x) = T^{\beta \alpha}(x) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(x - x_n(t)) \]

For isolated system

\[ \frac{\partial}{\partial x^\alpha} T^{\alpha \beta} = 0. \]

Define 4d analogue of \( \vec{r} \times \vec{p} \):

\[ M^{\alpha \beta \gamma} = x^{\alpha} T^{\beta \gamma} - x^{\beta} T^{\alpha \gamma} \]

\[ J^{\alpha \beta} = \int M^{0 \alpha \beta} d^3x = \int x^{\alpha} T^{\beta 0} - x^{\beta} T^{\alpha 0} d^3x \]

Spin (intrinsic angular momentum):

\[ S_\alpha = \frac{1}{2c} \epsilon_{\alpha \beta \gamma \delta} J^{\beta \gamma} u^\delta, \quad \text{proper velocity: } u^\delta = \frac{dx^\delta}{d\tau}. \]
For a particle at rest with CM at rest at the origin:

\[ J^{\mu \nu} : \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_z^{\circ} & -S_y^{\circ} \\ 0 & -S_z^{\circ} & 0 & S_x^{\circ} \\ 0 & S_y^{\circ} & -S_x^{\circ} & 0 \end{pmatrix}, \quad (\vec{J}^{\circ} = \vec{S}^{\circ}) \]

Boost along \( z \):

\[ J^{\mu \nu} : \begin{pmatrix} 0 & \gamma \beta S_y^{\circ} & -\gamma \beta S_x^{\circ} & 0 \\ -\gamma \beta S_y^{\circ} & 0 & S_z^{\circ} & -\gamma S_y^{\circ} \\ \gamma \beta S_x^{\circ} & -S_z^{\circ} & 0 & \gamma S_x^{\circ} \\ 0 & \gamma S_y^{\circ} & -\gamma S_x^{\circ} & 0 \end{pmatrix}, \quad \Rightarrow \quad \vec{J} = \begin{pmatrix} \gamma S_x^{\circ} \\ \gamma S_y^{\circ} \\ \gamma S_z^{\circ} \end{pmatrix} \]

\[ S^{\mu} : \begin{pmatrix} \gamma \beta S_z^{\circ} \\ S_x^{\circ} \\ S_y^{\circ} \\ \gamma S_z^{\circ} \end{pmatrix}, \quad \Rightarrow \quad \vec{S} = \begin{pmatrix} S_x^{\circ} \\ S_y^{\circ} \\ \gamma S_z^{\circ} \end{pmatrix}, \quad S^0 = \vec{\beta} \cdot \vec{S} \]

\[ \vec{J} - \vec{S} = \begin{pmatrix} (\gamma - 1)S_x^{\circ} \\ (\gamma - 1)S_y^{\circ} \\ (1 - \gamma)S_z^{\circ} \end{pmatrix} \]
Center-of-Mass shift

\[ \vec{r}_{\text{CM}} \times \vec{p}_{\text{CM}} = (\vec{J} - \vec{S})_{\perp} \]

\[ \gamma \beta mc(-x_{\text{CM}} \hat{y} + y_{\text{CM}} \hat{x}) = (\gamma - 1)\vec{S}_{\perp} \]

\[ \gamma \beta mc(x_{\text{CM}} + y_{\text{CM}}) = (\gamma - 1) \hat{z} \times \vec{S}_{\perp} \]

\[ \vec{r}_{\perp \text{CM}} = \frac{\gamma}{\gamma + 1} \frac{\vec{\beta} \times \vec{S}}{mc} \]

CM at rest.

Boost into screen

Center of charge wobbles: classical “Zitterbewegung”

Brookhaven National Laboratory

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Waldo MacKay 7 June, 2004
Thomas Precession

1. Boost observer to left.
2. Boost observer downward.
3. Boost back to rest.
   - Net rotation of rest frame.
Thomas—Frenkel (BMT) Equation

In the local rest frame of the proton, the spin precession of the proton obeys the Thomas-Frenkel equation:

\[
\frac{d\vec{S}^\circ}{dt} = \frac{q}{\gamma m} \vec{S}^\circ \times \left[ (1 + G\gamma)\vec{B}_\perp + (1 + G)\vec{B}_\parallel + \left( G\gamma + \frac{\gamma}{\gamma + 1} \right) \frac{\vec{E} \times \vec{v}}{c^2} \right].
\]

This is a mixed description: \( t, \vec{B}, \) and \( \vec{E} \) in the lab frame, but spin \( \vec{S}^\circ \) in local rest frame of the particle:

Proton: \( G = \frac{g - 2}{2} = 1.792847, \quad 523.34 \) MeV/unit \( G\gamma \)

Electron: \( a = G = \frac{g - 2}{2} = 0.001159652, \quad 440.65 \) MeV/unit \( a\gamma \)

\[
\gamma = \frac{\text{Energy}}{mc^2}.
\]
In the local rest frame of the proton, the spin precession of the proton obeys the Thomas-Frenkel equation:

\[
\begin{align*}
\text{Torque:} & \quad \frac{d\vec{S}}{dt} = \frac{q}{\gamma m} \vec{S} \times \left[ (1 + G\gamma) \vec{B}_\perp + (1 + G) \vec{B}_\parallel \right] \quad \text{TF} \\
\text{Force:} & \quad \frac{d\vec{p}}{dt} = \frac{q}{\gamma m} \vec{p} \times \vec{B}_\perp \quad \text{Lorentz}
\end{align*}
\]

(This is a mixed description: \( t \), and \( \vec{B} \) in the lab frame, but spin \( \vec{S} \) in local rest frame of the proton.)

\[
G = \frac{g - 2}{2} = 1.7928, \quad \gamma = \frac{\text{Energy}}{mc^2}.
\]
Example with 6 precessions of spin in one turn:

\[ G\gamma + 1 = 6. \]

Spin tune: number of precessions per turn relative to beam’s direction. So we subtract one:

\[ \nu_{\text{spin}} = G\gamma \propto \text{energy}, \]

i.e., 5 in this example.
Misalignments or Imperfections

- A misaligned quadrupole creates a steering error which propagates through the lattice.
- For an accelerator ring, this shifts the closed orbit away from the design trajectory.
- If the misalignment is vertical, then the design trajectory will have a periodic set of small vertical bends interspersed with the normal horizontal bends of the bending magnets.
- This leads to an integer resonance condition for the spin tune.
In general, rotations don’t commute.
Depolarizing Resonances

Simple Resonance Condition:

\[ \nu_{\text{spin}} = N + N_v Q_v, \]

(imperfection) (intrinsic)

where \( N \) and \( N_v \) are integers.
AGS Intrinsic Resonances

![Graph showing AGS Intrinsic Resonances with various Qy labels and resonance strengths plotted against Gγ.](image-url)
Adding a partial snake opens up stop bands around the integer imperfection resonances.

At the snake the stable spin direction points along the snake’s rotation axis when $G\gamma = \text{integer}$.

Partial snake strength: $\frac{\mu}{\pi}$

$$\cos \pi \nu_s = \cos(G\gamma \pi) \cos \frac{\mu}{2}$$
Crossing an Isolated Spin Resonance

Froissart—Stora Formula:

\[ \frac{P_f}{P_i} = 2\exp\left(-\frac{\pi|\epsilon|^2}{2\alpha}\right) - 1. \]

Ramp rate: \( \alpha = \frac{dG_\gamma}{d\theta} \), \( \theta : 2\pi/\text{turn.} \)

Resonance strength: \( \epsilon = \text{Fourier amplitude.} \)
Resonance Crossing in AGS

AC dipole used to increase strength of \( \nu_s = Q_y \) resonance.

(Simulations by Mei Bai)
AGS has 12 superperiods.
Vertical betatron tune: 8.7
Snake strength: 5%

(From Jeff Woods)

AC dipole pulses at resonances:
- $0 + Q_y$
- $12 + Q_y$
- $36 - Q_y$
- $36 + Q_y$

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pC CNI Asymmetry during AGS Ramp

![Graph showing pC CNI Asymmetry during AGS Ramp with data points and labels for 2004 and 2003 data sets.]

- 2004 data, P = 42% at Gy = 46.5
- 2003 data, P = 28% at Gy = 46.5
**Depolarizing Resonances in RHIC**

\[ Q_x = 28.236 \]
\[ Q_y = 29.219 \]
\[ \pi \epsilon_y = 10\pi \text{ \( \mu \)m} \]

Will depolarize beam during acceleration.

**Solution: Snakes**
• 2 snakes: spin is up in one half of the ring, and down in the other half.

• Spin tune: $\nu_{\text{spin}} = \frac{1}{2}$  
  (It’s energy independent.)

• “The unwanted precession which happens to the spin in one half of the ring is unwound in the other half.”
Hamiltonian with Spin

(Here I drop the “≈” superscript on $\vec{S}$.)

$$\frac{d\vec{S}}{dt} = \vec{W} \times \vec{S}$$

$$H(q, \vec{P}, \vec{S}; s) = H_{\text{orb}} + H_{\text{spin}}$$

$$= H_{\text{orb}} + \vec{W} \cdot \vec{S} + O(\hbar^2)$$

Group symmetries:

- Orbital motion without spin: $\text{Sp}(6, r)$.
- Spin by itself: $\text{SU}(2, c) \cong \text{SO}(3, r)$ (homomorphic).
- Full blown symmetry: $\text{Sp}(6, r) \oplus \text{SU}(2, c)$.
  - Spin dependence on orbit (Thomas-Frenkel).
  - Orbit dependence on spin (Stern-Gerlach Force)—Usually ignored.
SU(2) with usual spinor notation:

Pauli matrices: \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

\[
R_{\hat{n}}(\theta) = e^{i \hat{n} \cdot \vec{\sigma}/2} = \begin{pmatrix}
\cos \frac{\theta}{2} + i n_z \sin \frac{\theta}{2} & (n_y + i n_x) \sin \frac{\theta}{2} \\
-(n_y + i n_x) \sin \frac{\theta}{2} & \cos \frac{\theta}{2} - i n_z \sin \frac{\theta}{2}
\end{pmatrix}.
\]

SO(3):

\[
R_{\hat{n}}(\theta) = \mathbf{I} \cos \theta + \begin{pmatrix}
0 & n_z & -n_y \\
n_z & 0 & n_x \\
n_y & -n_x & 0
\end{pmatrix} \sin \theta
\[
+ \begin{pmatrix}
n_x^2 & n_x n_y & n_x n_z \\
n_x n_y & n_y^2 & n_y n_z \\
n_x n_z & n_y n_z & n_z^2
\end{pmatrix} (1 - \cos \theta).
\]
For the closed orbit:  \( \vec{n}_0(s) = \vec{n}_0(s + L) \),
with  \( \vec{q}_0(s) = \vec{q}_0(s + L) \) and  \( \vec{P}_0(s) = \vec{P}_0(s + L) \).

For other locations in phase space:  \( \vec{n}(\vec{q}, \vec{P}, s) = \vec{n}(\vec{q}, \vec{P}, s + L) \),
even though in general  \( q(s + L) \neq q(s) \) and  \( P(s + L) \neq P(s) \).
\( \hat{n} \)-vector at 1\( \sigma \) and 800 GeV

- Simulation with only vertical betatron motion.
- 802 GeV is closer to a resonance spin resonance than 800 GeV.

Des Barber et al.
HERA-p: Invariant Spin Field

\[ \hat{n} \text{-vector at } 4\sigma \text{ and } 800 \text{ GeV} \]

- Larger amplitude oscillations have a larger tune shift due to nonlinear elements.
- 802 GeV is closer to a resonance spin resonance than 800 GeV.

Des Barber et al.

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Particles with larger amplitude betatron oscillations may experience more precession away from the stable spin direction of the center of the beam.

(Alfredo Luccio)

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Accelerators with Polarized Protons

LINAC: Linear Accelerator
AGS: Alternating Gradient Synchrotron
RHIC: Relativistic Heavy Ion Collider
High Intensity Polarized H\textsuperscript{−} Source

KEK OPPIS*
upgraded at TRIUMF
70 → 80\% Polarization
15 \times 10^{11} \text{protons/pulse}
at source
6 \times 10^{11} \text{protons/pulse}
at end of LINAC

*Optically Pumped Polarized Ion Source
Optically Pumped Polarized Ion Source

Fig. 1. 1) ECR Proton Source, 2) Superconducting Solenoid, 3) Optically-Pumped Rb Cell, 4) Deflection Plates, 5) Sona Transition Region, 6) Ionizer Cell, 7) Ionizer Solenoid, 8) Quartz Tube, 9) ECR Cavity, 10) Three Grid Extraction System, 11) Boron-Nitride End Cups, 12) Indium Seals.

- Generate protons
- Capture polarized electron from optically pumped alkali atom
- Transfer of electron-spin polarization to nuclear-spin polarization (Sona transition)
- Ionization

H⁺  →  H⁰ (e↑)  →  H⁰ (p↑)  →  H⁻ (p↑)

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Trajectory and Spin through Snakes

Fields through first Snake
100 GeV proton

Orbit Trajectories through Snake
100 GeV proton

Spin motion through one Snake
100 GeV proton

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The rotation axis of the snake is $\phi$, and $\mu$ is the rotation angle.

Note that the $\phi$ contours shift slightly from injection (red) at 25 GeV to storage (pink) at 250 GeV.
RHIC Beam Polarization

![Graphs showing beam polarization over time.](image)
Snake Resonances

\[ \epsilon_{\text{int}} = 0.5, \quad \epsilon_{\text{imp}} = 0.05, \quad 2 \text{ Snakes}, \quad \text{spin tune} = 0.5 \]

Vertical Betatron Tune

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Helical Spin Rotators

Magnetic Field

24.3 GeV proton

Beam Trajectory

24.3 GeV proton

Spin Components

24.3 GeV proton

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Compensation for D0-DX Bends

\[ E = 25 \text{ GeV} \]

\[ E = 250 \text{ GeV} \]

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The rotation axis of the spin rotator is in the \( x-y \) plane at an angle \( \theta \) from the vertical. The spin is rotated by the angle \( \mu \) around the rotation axis.

Note: Purple contour for rotation into horizontal plane. Black dots show settings for RHIC energies in increments of 25 GeV from 25 to 250 GeV.
Rotator Spin Precession

Rotator’s spin vector at injection energy

Rotator’s spin vector at 250 GeV

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"Left–Right" Asymmetry (Tilted at 45°)

The PHENIX Local Polarimeter measures an asymmetry in small angle scattered neutrons which is proportional to transverse polarization.

\[ A_{LR} = \frac{\sqrt{L^+R^-} - \sqrt{L^-R^+}}{\sqrt{L^+R^-} + \sqrt{L^-R^+}} \propto P_y \]
Vertical polarization with rotators off.
Spin is down.

Rotators on
Spin is radially inwards!
OOPS!

Reverse all rotator power supplies and try again.
YES!
AC Dipole pulse at $G\gamma = 0 + Q_y$

Top: AC dipole pulse amplitude (current)
Bottom: Beam current.
(Just scrapes the beam pipe.)

Top: Beam coherence
Bottom: Tune spectrum

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Properties of synchrotron radiation

- Radiated power:

\[ P_\gamma = \frac{2}{3} r_e m c^3 \frac{\gamma^4 \beta^4}{\rho^2}, \quad r_e = \frac{e^2}{4\pi \varepsilon_0 m c^2}. \]

Radiation in forward direction with opening angle \( \propto \gamma^{-1} \)

- Energy loss per turn:

\[ U_\gamma = \oint P_\gamma \frac{d\rho}{c} \]

- Critical energy: half the power is radiated by photons less than the critical energy, and the other half, above.

\[ u_c = \hbar \omega_c = \frac{3\hbar c}{2\rho} \gamma^3 \]
• Number of photons per second:

\[ N_\gamma = \int_0^{U_{\text{max}}} n_\gamma(u_\gamma) \, du_\gamma = \frac{5}{2\sqrt{3}} \frac{\alpha c}{\rho} \gamma \]

here: \( \alpha = 1/137 \)

• Number of photons per radian:

\[ N_r = \frac{5\alpha}{2\sqrt{3}} \gamma \]

• Average photon energy and 2\textsuperscript{nd} moment:

\[ \langle u_\gamma \rangle = \frac{1}{N_\gamma} \int_0^{U_{\text{max}}} u \, n_\gamma(u) \, du = \frac{8}{15\sqrt{3}} u_c \simeq 0.32u_c \]

\[ \langle u^2_\gamma \rangle = \frac{1}{N_\gamma} \int_0^{U_{\text{max}}} u^2 \, n_\gamma(u) \, du = \frac{11}{27\sqrt{3}} u^2_c \simeq 0.41u^2_c \]
- Energy spread: 
  \[ \sigma_u = \sqrt{\frac{C_q}{J_s \rho}} \gamma^2 m c^2 \]
  with \( C_q = 3.8 \times 10^{-8} \text{ m} \) and \( J_s \sim 2 + D \).

<table>
<thead>
<tr>
<th>Ring</th>
<th>Energy ([\text{GeV}])</th>
<th>( \sigma_u ) ([\text{MeV}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>CESR</td>
<td>5.5</td>
<td>3</td>
</tr>
<tr>
<td>HERAe</td>
<td>27.5</td>
<td>3</td>
</tr>
<tr>
<td>LEP</td>
<td>45</td>
<td>30</td>
</tr>
<tr>
<td>LEP</td>
<td>60</td>
<td>53</td>
</tr>
<tr>
<td>LEP</td>
<td>100</td>
<td>150</td>
</tr>
</tbody>
</table>

Remember: Integer resonances separated by only 440 MeV.

The polarization in LEP dropped down to nothing just above 60 GeV.
Longitudinal Synchrotron Oscillations

\[
\omega_{\text{rf}} = h\omega_{\text{rev}}
\]

\[
W = -\frac{U - U_s}{\omega_{\text{rf}}}
\]

\[
\frac{dW}{dt} = \frac{qV}{2\pi h} (\sin \phi_s - \sin \phi)
\]

\[
\frac{d\phi}{dt} \approx \frac{\omega_{\text{rf}}^2 \eta_{\text{ph}}}{\beta^2 U_s} W
\]

\[
\frac{d\omega_{\text{rev}}}{\omega_{\text{rev}}} = \frac{d\beta}{\beta} - \frac{dL}{L} = \eta_{\text{ph}} \frac{dp}{p}
\]

\[\eta_{\text{ph}} < 0 \text{ above transition energy.}\]

Add in synchrotron oscillations to resonance condition:

\[\nu_{\text{spin}} = N + N_v Q_v + N_h Q_h + N_{\text{sy}} Q_{\text{sy}}\]
Longitudinal Phase Space

Canonical coordinate: $\varphi$ and conjugate momentum: $W$

a) $\eta_{tr} > 0$

b) $\eta_{tr} < 0$
In a homogeneous magnetic field the transition rates are

\[
W_{\uparrow\downarrow} = \frac{5\sqrt{3}}{16} \frac{e^2 \gamma^5 \hbar}{4\pi \epsilon_0 m_e c^2 |\rho|^3} \left( 1 + \frac{8}{5\sqrt{3}} \right)
\]

\[
W_{\uparrow\uparrow} = \frac{5\sqrt{3}}{16} \frac{e^2 \gamma^5 \hbar}{4\pi \epsilon_0 m_e c^2 |\rho|^3} \left( 1 - \frac{8}{5\sqrt{3}} \right).
\]

Evaluating the equilibrium polarization have (Sokolov Ternov)

\[
P_{ST} = \frac{W_{\uparrow\downarrow} - W_{\downarrow\uparrow}}{W_{\uparrow\downarrow} + W_{\downarrow\uparrow}} = \frac{8}{5\sqrt{3}} = 0.9238.
\]

An unpolarized beam polarizes:

\[
P(t) = P_{ST} \left[ 1 - \exp(-t/\tau_{ST}) \right],
\]

where the polarization rate is given by

\[
\tau_{ST}^{-1} = \frac{5\sqrt{3}}{8} \frac{e^2 \gamma^5 \hbar}{4\pi \epsilon_0 m_e c^2} \frac{1}{L} \int \frac{ds}{|\rho|^3}.
\]
## Typical Sokolov-Ternov Rates

<table>
<thead>
<tr>
<th>Ring</th>
<th>Particle</th>
<th>Energy [GeV]</th>
<th>$N_\gamma$ /turn</th>
<th>$\Delta U$ [MeV]</th>
<th>$\tau_{ST}$ [min]</th>
<th>$\frac{W_{\uparrow\downarrow}}{f_{rev}N_\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CESR</td>
<td>e$^\pm$</td>
<td>5.5</td>
<td>700</td>
<td>$-1$ MeV</td>
<td>167</td>
<td>$1 \times 10^{-13}$</td>
</tr>
<tr>
<td>HERAe</td>
<td>e$^\pm$</td>
<td>27.5</td>
<td>3600</td>
<td>$-83$ MeV</td>
<td>23</td>
<td>$1 \times 10^{-12}$</td>
</tr>
<tr>
<td>LEP</td>
<td>e$^\pm$</td>
<td>45</td>
<td>5800</td>
<td>$-120$ MeV</td>
<td>300</td>
<td>$2 \times 10^{-13}$</td>
</tr>
<tr>
<td>LEP</td>
<td>e$^\pm$</td>
<td>60</td>
<td>7800</td>
<td>$-380$ MeV</td>
<td>81</td>
<td>$8 \times 10^{-13}$</td>
</tr>
<tr>
<td>RHIC</td>
<td>p</td>
<td>100</td>
<td>7</td>
<td>$-3$ meV</td>
<td>$3 \times 10^{14}$ yr</td>
<td>$6 \times 10^{-29}$</td>
</tr>
<tr>
<td>RHIC</td>
<td>p</td>
<td>250</td>
<td>18</td>
<td>$-0.13$ eV</td>
<td>$3 \times 10^{12}$ yr</td>
<td>$2 \times 10^{-27}$</td>
</tr>
<tr>
<td>HERAp</td>
<td>p</td>
<td>920</td>
<td>65</td>
<td>$-8.5$ eV</td>
<td>$1 \times 10^{11}$ yr</td>
<td>$3 \times 10^{-26}$</td>
</tr>
<tr>
<td>Tevatron</td>
<td>p</td>
<td>1000</td>
<td>70</td>
<td>$-8.5$ eV</td>
<td>$2 \times 10^{11}$ yr</td>
<td>$2 \times 10^{-26}$</td>
</tr>
<tr>
<td>SSC</td>
<td>p</td>
<td>20000</td>
<td>1400</td>
<td>$-0.12$ MeV</td>
<td>$7 \times 10^7$ yr</td>
<td>$3 \times 10^{-23}$</td>
</tr>
</tbody>
</table>
In phase space quantum fluctuations cause instantaneous hops of momentum from one ellipse to another. (Hops in the Action.)
For the closed orbit: \( \vec{n}_0(s) = \vec{n}_0(s + L) \),
with \( \vec{q}_0(s) = \vec{q}_0(s + L) \) and \( \vec{P}_0(s) = \vec{P}_0(s + L) \).

For other locations in phase space: \( \vec{n}(\vec{q}, \vec{P}, s) = \vec{n}(\vec{q}, \vec{P}, s + L) \),
even though in general \( q(s + L) \neq q(s) \) and \( P(s + L) \neq P(s) \).
Equilibrium with Real Lattice

Derbenev–Kondratenko formula for equilibrium polarization:

\[
P_{DK} = \frac{8}{5\sqrt{3}} \frac{\int \left\langle \frac{1}{|\rho|^3} \hat{b} \cdot \left( \hat{n} - \frac{\partial \hat{n}}{\partial \delta} \right) \right\rangle_s ds}{\int \left\langle \frac{1}{|\rho|^3} \left[ 1 - \frac{2}{9} (\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left( \frac{\partial \hat{n}}{\partial \delta} \right)^2 \right] \right\rangle_s ds}
\]

\[
\frac{1}{\tau_{DK}} = \frac{5\sqrt{3}}{8} \frac{r_e \gamma^5 \hbar}{m_e L} \frac{1}{L} \int \left\langle \frac{1}{|\rho|^3} \left[ 1 - \frac{2}{9} (\hat{n} \cdot \hat{s}) + \frac{11}{18} \left( \frac{\partial \hat{n}}{\partial \delta} \right)^2 \right] \right\rangle_s ds
\]

averaged over phase space at azimuth \( s \).

- \( \delta = \Delta p/p \) is the fractional momentum deviation from design.
- \( \hat{n} \) is the invariant spin field.
- \( \hat{b} = \frac{\hat{s} \times \hat{s}}{|\hat{s}|} \) is the direction of magnetic field if \( \vec{E} = 0 \).
- \( \rho \) is the cyclotron radius of the trajectory.
- \( L \) is circumference of synchrotron.
Spin Resonances of SPEAR

\[ P/P_{\text{max}} \]
\[ \nu = 8 \]
\[ \nu - \nu_x + \nu_y = 8 \]
\[ \nu - \nu_x + 2\nu_y = 3 \]
\[ \nu - \nu_x - \nu_y = -2 \]

- 3.52
- 3.56
- 3.60
- 3.64
- 3.68
- 3.72
- 3.76

E (GeV)
\[ \frac{1}{\tau_{\text{dep}}} \approx \frac{5\sqrt{3}}{8} \frac{r_e \gamma^5 \hbar}{m_e} \frac{1}{L} \int \left\langle \frac{1}{|\rho|^3} \left( \frac{\partial \hat{n}}{\partial \delta} \right)^2 \right\rangle_s \, ds \]

\[ \frac{1}{\tau_{\text{pol}}} \approx \frac{1}{\tau_{\text{ST}}} + \frac{1}{\tau_{\text{dep}}} \]
As an example from CESR (CUSB): $M_Y = 9459.97 \pm 0.11 \pm 0.07\text{ MeV}$

Tidal Effects at LEP

From Angelika Drees’ Thesis

QCDSP: Spin Dynamics in Accelerators
Waldo MacKay       7 June, 2004
The solenoid spin rotators

(Des Barber in "eRHIC Zero\textsuperscript{th}-Order Design Report)
Polarization in eRHIC-e

Equilibrium polarizations with misalignments

Polarization times with misalignments

QCDSP: Spin Dynamics in Accelerators
Waldo MacKay 7 June, 2004
Some References (by no means all!)