Basics of QCD perturbation theory

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Abstract

A prediction for experiment based on perturbative QCD combines

- a particular calculation of Feynman diagrams (easy at leading order, not so easy at next-to-leading order).

- use of general features of the theory that allow the Feynman diagrams to be related to experiment:

We will study the general features.
Some general features of QCD

- renormalization group and the running coupling;
- existence of infrared safe observables;
- isolation of hadron structure in parton distribution functions.

I will discuss these structural features of the theory that allow a comparison of theory and experiment. Along the way we will discover something about certain important processes: \( e^+e^- \) annihilation, deeply inelastic scattering, and jet production in hadron-hadron collisions. I will not specifically address spin physics, since that is covered in the rest of the school.
Disclaimer. We will not learn how to do significant calculations in QCD perturbation theory. Three hours is not enough for that.
Electron-positron annihilation and jets

Exploring the QCD final state

A) Kinematics of $e^+e^- \rightarrow 3$ partons.

B) Structure of the cross section.

C) Null plane coordinates.

D) Space-time picture of the singularities.

E) The long time problem and infrared safe observables.

F) Examples of infrared safe observables.
Kinematics of $e^+e^- \rightarrow 3$ partons

Energy fractions:

$$x_i = \frac{E_i}{\sqrt{s}/2} = \frac{2p_i \cdot q}{s}, \quad \Rightarrow \quad 0 < x_i.$$  

Energy conservation:

$$\sum_{i} x_i = \frac{2(\sum p_i) \cdot q}{s} = 2.$$
Angles:

$$2p_1 \cdot p_3 = (p_1 + p_3)^2 = (q - p_2)^2 = s - 2q \cdot p_2.$$  

$$2E_1 E_3 (1 - \cos \theta_{13}) = s (1 - x_2).$$

$$x_1 x_3 (1 - \cos \theta_{13}) = 2 (1 - x_2).$$

So

$$x_i < 1.$$  

$$\theta_{13} \to 0 \iff x_2 \to 1.$$
Regions for the energy fractions

The energy fractions $x_i$ lie within a triangle.
The edges of the allowed region correspond to two partons being collinear.

The corners correspond to one parton being soft.
Structure of the cross section

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}
\]

where \( C_F = 4/3 \) and \( \sigma_0 = (4\pi \alpha_s^2/s) \sum e_q^2 \) is the total cross section for \( e^+e^- \rightarrow \text{hadrons} \) at order \( \alpha_s^0 \).

Collinear singularities:
(1 - \( x_1 \)) \( \rightarrow 0 \) (2&3 collinear).
(1 - \( x_2 \)) \( \rightarrow 0 \) (1&3 collinear).

Soft singularity: (3 soft)
(1 - \( x_1 \)) \( \rightarrow 0 \), (1 - \( x_2 \)) \( \rightarrow 0 \)

\[
\frac{(1-x_1)}{(1-x_2)} \sim \text{const.}
\]
$e^+ e^- \rightarrow 3$ partons in terms of energy and angle

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dE_3 \, d\cos \theta_{13}} = \frac{\alpha_s}{2\pi} C_F \frac{f(E_3, \theta_{13})}{E_3 (1 - \cos \theta_{13})}
\]

where $f(E_3, \theta_{13})$ is finite for $E_3 \rightarrow 0$ and for $\theta_{13} \rightarrow 0$.

Collinear singularities:

$\theta_{13} \rightarrow 0$:

\[
\int_a^1 \, d\cos \theta_{13} \frac{d\sigma}{dE_3 \, d\cos \theta_{13}} = \log(\infty).
\]

Soft singularity: $E_3 \rightarrow 0$:

\[
\int_0^a \, dE_3 \frac{d\sigma}{dE_3 \, d\cos \theta_{13}} = \log(\infty).
\]

That's great, but is there a general reason for it?
**Why is \( e^+ e^- \rightarrow 3 \) partons singular?**

\( \mathcal{M} \) contains a factor \( 1/(p_1 + p_3)^2 \) where

\[
(p_1 + p_3)^2 = 2p_1 \cdot p_3 = 2E_1 E_3 (1 - \cos \theta_{13}).
\]

Also, a numerator factor \( \propto \theta_{13} \) in the collinear limit. So

\[
|\mathcal{M}|^2 \propto \left[ \frac{\theta_{13}}{E_3 \theta_{13}^2} \right]^2
\]

for \( E_3 \rightarrow 0 \) or \( \theta_{13} \rightarrow 0 \).

Integration:

\[
d\sigma \sim \int \frac{E_3^2 dE_3 d \cos \theta_{13} d\phi}{E_3} \left[ \frac{\theta_{13}}{E_3 \theta_{13}^2} \right]^2 \sim \int \frac{dE_3}{E_3} \frac{d\theta_{13}^2}{\theta_{13}^2} d\phi.
\]
Null plane coordinates

Use $p^\mu = (p^+, p^-, p^1, p^2)$ where

$$p^\pm = (p^0 \pm p^3)/\sqrt{2}.$$  

• For a particle with large momentum in the $+z$ direction and limited transverse momentum, $p^+$ is large and $p^-$ is small.

• Often one chooses the $+$ axis so that a particle or group of particles of interest have large $p^+$ and small $p^-$ and $p_T$. 
\[ \frac{p^\pm}{p_0^\pm + p^3}/\sqrt{2}. \]

- Covariant square:
  \[ p^2 = 2p^+p^- - p^2_T. \]
- \( p^- \) for a particle on its mass shell:
  \[ p^+ > 0, \quad p^- > 0. \]
  \[ p^- = \frac{p^2_T + m^2}{2p^+}. \]
• Integration over the mass shell:

$$(2\pi)^{-3} \int \frac{d^3\vec{p}}{2\sqrt{p^2 + m^2}} \cdots = (2\pi)^{-3} \int d^2p_T \int_0^\infty \frac{dp^+}{2p^+} \cdots.$$ 

• Fourier transform:

$$p \cdot x = p^+ x^- + p^- x^+ - p_T \cdot x_T.$$ 

So $x^-$ is conjugate to $p^+$ and $x^+$ is conjugate to $p^-$. (Sorry.)
Define $p_1^\mu + p_3^\mu = k^\mu$.

Choose null-plane coordinates with $k^+$ large and $k_1^T = 0$.

Then $k^2 = 2k^+k^-$ becomes small when

$$k^- = \frac{p_3^2}{2p_1^+} + \frac{p_3^2}{2p_3^+}$$

becomes small. (Collinear or soft singularity.)
Singularity is from large $k^+$ and small $k^-$. 

Consider the Fourier transform.

$$S_F(k) = \int dx^+ dx^- dx \exp(i[k^+x^- + k^-x^+ - k \cdot x]) S_F(x).$$

Contributing values of $x$ have small $x^-$ large $x^+$. 
Long time picture suggested by perturbation theory

Thus QCD suggests a jet structure of final state hadrons.

"Summed" perturbation theory suggests this is OK. But beware of "nonperturbative" effects!

That's a qualitative success. But can you predict reliable numbers?
The long time problem

Perturbation theory not effective for long time physics. But the detector is a long distance away!
Answer

Use measurements that are not sensitive to long time physics.

Example: the $e^+e^-$ annihilation total cross section.

Effects from $\Delta x \gg 1/\sqrt{s}$ cancel because of unitarity:

$$\langle 0| J(y') U(y', \infty) U(\infty, y) J(y) |0 \rangle = \langle 0| J(y') U(y', y) J(y) |0 \rangle$$

At order $\alpha_s$, this works by a cancellation between real gluon emission graphs and virtual gluon graphs.

If the total cross section is all you can look at, QCD physics will be a little boring!
Infrared safe quantities

Some quantities are not sensitive to infrared effects.

\[ I = \frac{1}{2!} \int d\Omega_2 \frac{d\sigma[2]}{d\Omega_2} S_2(p_1^\mu, p_2^\mu) \]
\[ + \frac{1}{3!} \int d\Omega_2 dE_3 d\Omega_3 \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} S_3(p_1^\mu, p_2^\mu, p_3^\mu) \]
\[ + \frac{1}{4!} \int d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4 \]
\[ \times \frac{d\sigma[4]}{d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4} S_4(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) \]
\[ + \cdots. \]

Need (for \( \lambda = 0 \) or \( 0 < \lambda < 1 \))

\[ S_{n+1}(p_1^\mu, \ldots, (1 - \lambda)p_n^\mu, \lambda p_n^\mu) = S_n(p_1^\mu, \ldots, p_n^\mu). \]
What does infrared safety mean?

\[ S_{n+1}(p_1^\mu, \ldots, (1 - \lambda)p_n^\mu, \lambda p_n^\mu) = S_n(p_1^\mu, \ldots, p_n^\mu). \]

The physical meaning is that for an IR-safe quantity a physical event with hadron jets should give approximately the same measurement as a parton event:
What does infrared safety mean?

\[ S_{n+1}(p_1^\mu, \ldots, (1 - \lambda)p_n^\mu, \lambda p_n^\mu) = S_n(p_1^\mu, \ldots, p_n^\mu). \]

The calculational meaning is that infinities cancel.
Examples of infrared safe observables

\[ \mathcal{I} = \frac{1}{2!} \int d\Omega_2 \frac{d\sigma[2]}{d\Omega_2} S_2(p_1^\mu, p_2^\mu) \]
\[ + \frac{1}{3!} \int d\Omega_2 dE_3 d\Omega_3 \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} S_3(p_1^\mu, p_2^\mu, p_3^\mu) \]
\[ + \ldots , \]

with

\[ S_{n+1}(p_1^\mu, \ldots, (1 - \lambda)p_n^\mu, \lambda p_n^\mu) = S_n(p_1^\mu, \ldots, p_n^\mu). \]

The simplest example is the total cross section \( \sigma_T \):

\[ S_n(p_1^\mu, \ldots, p_n^\mu) = 1. \]
Thrust Distribution

An example is the thrust distribution $d\sigma/dT$:

$$S_n(p^\mu_1, \ldots, p^\mu_n) = \delta \left( T - T_n(p^\mu_1, \ldots, p^\mu_n) \right).$$

$$T_n(p^\mu_1, \ldots, p^\mu_n) = \max_{\vec{u}} \frac{\sum_{i=1}^n |\vec{p}_i \cdot \vec{u}|}{\sum_{i=1}^n |\vec{p}_i|}.$$
\( \mathcal{T}_n(p_1^\mu, \ldots, p_n^\mu) = \max_{\vec{u}} \frac{\sum_{i=1}^n |\vec{p}_i \cdot \vec{u}|}{\sum_{i=1}^n |\vec{p}_i|} \).

- Contribution from a particle with \( \vec{p} \to 0 \) drops out.

- Replacing one particle by two collinear particles doesn’t change the thrust:

  \[ |(1 - \lambda) \vec{p}_n \cdot \vec{u}| + |\lambda \vec{p}_n \cdot \vec{u}| = |\vec{p}_n \cdot \vec{u}|, \]

  and

  \[ |(1 - \lambda) \vec{p}_n| + |\lambda \vec{p}_n| = |\vec{p}_n|. \]
Energy-energy correlation function

Another example is the energy-energy correlation function $d\Sigma/d\cos(\theta)$:

$$S_n(p_1^\mu, \ldots, p_n^\mu) = \sum_{ij} \frac{E_i E_j}{s} \delta \left( \cos(\theta_{ij}) - \cos(\theta) \right).$$

- Contribution from a particle with $E_i \to 0$ drops out.
- Replacing one particle by two collinear particles doesn’t change the result:

$$(1 - \lambda) E_n E_j + \lambda E_n E_j = E_n E_j,$$
Jet cross sections

Another example is the $n$ jet cross section $\sigma_n$.

There are several algorithms to choose from. Here is the simplest (but not the best) one.

- Start with a list of momenta $p_{1}^{\mu}, p_{2}^{\mu}, \ldots, p_{N}^{\mu}$. At the start, these represent the momenta of particles. (In a perturbative calculation, they are the momenta of partons.)
• Choose a parameter $y_{\text{cut}}$.

1. Find the pair $(i, j)$ such that $(p_i + p_j)^2$ is the smallest.
2. If $(p_i + p_j)^2 > y_{\text{cut}}$, exit. Else continue.
3. Replace the two momenta $p_i$ and $p_j$ in the list by their sum $p_k^\mu = p_i^\mu + p_j^\mu$.
4. Go to 1.

This produces a list of momenta $p_i$ of jets. $\sigma_n$ is the cross section to have $n$ jets.

Variations on this theme: change the resolution condition or the combination prescription.
Deeply inelastic scattering

*The effect of partons*

A) Kinematics of deeply inelastic scattering.

B) Structure functions for DIS.

C) Space-time structure of DIS.

D) Factored cross section.

E) The hard scattering cross section.
Kinematics of deeply inelastic lepton scattering

\[ \ell(k) + h(p) \rightarrow \ell'(k') + X. \]

\[ q^\mu = k^\mu - k'^\mu \]

\[ Q^2 = -q^2 \]

\[ x_{bj} = \frac{Q^2}{2p \cdot q} \]

or \[ x_{bj} = A \frac{Q^2}{2p \cdot q} \]

- “Deeply inelastic” \( \Rightarrow Q^2 \rightarrow \infty \), \( x \) fixed.
\( Q^2 \rightarrow \infty, \ x \) fixed also implies

\[
W^2 = (p + q)^2 = m_h^2 + \frac{1-x}{x}Q^2 \rightarrow \infty.
\]

\( \bullet \) Lepton variables related to hadron variables by

\[
y = \frac{p \cdot q}{p \cdot k}.
\]
Structure functions for DIS

$$d\sigma = \frac{4\alpha^2}{s} \frac{d^3k'}{2|k'|} \frac{1}{(q^2 - M^2)^2} L^{\mu\nu}(k, q) W_{\mu\nu}(p, q).$$

Analysis does not require QCD, just electroweak theory.
\[ d\sigma = \frac{4\alpha^2 d^3k'}{s} \frac{1}{2|k'| (q^2 - M^2)^2} L^{\mu\nu}(k, q) W_{\mu\nu}(p, q). \]

\[ L^{\mu\nu} = \frac{1}{2} Tr \left( k \cdot \gamma \Gamma^\mu k' \cdot \gamma \Gamma^\nu \right). \]

\[ W_{\mu\nu} = - \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) \]
\[ + \left( p_\mu - q_\mu \frac{p \cdot q}{q^2} \right) \left( p_\nu - q_\nu \frac{p \cdot q}{q^2} \right) \frac{1}{p \cdot q} F_2(x, Q^2) \]
\[ - i\epsilon_{\mu\nu\lambda\sigma} p^\lambda q^\sigma \frac{1}{p \cdot q} F_3(x, Q^2). \]
A convenient frame is
\[ (q^+, q^-, q) = \frac{1}{\sqrt{2}} (-Q, Q, 0) \]
\[ (p^+, p^-, p) \approx \frac{1}{\sqrt{2}} \left( \frac{Q}{x}, \frac{x m_h^2}{Q}, 0 \right) \]

- Hadron momentum is big.
- Momentum transfer is big.
Lorentz transformation spreads out interactions. Hadron at rest has separation between interactions

\[ \Delta x^+ \sim \Delta x^- \sim \frac{1}{m}. \]

Moving hadron has

\[ \Delta x^+ \sim \frac{1}{m} \times \frac{Q}{m} = \frac{Q}{m^2}, \]
\[ \Delta x^- \sim \frac{1}{m} \times \frac{m}{Q} = \frac{1}{Q}. \]
The virtual photon meets the fast moving hadron

Moving hadron has

$$\Delta x^+ \sim Q/m^2.$$  

Interaction with photon with $$q^- \sim Q$$ is localized to within

$$\Delta x^+ \sim 1/Q.$$  

Thus quarks and gluons = “partons” are effectively free.
At a given $x^+$, find partons with an amplitude

$$\psi(p_1^+, p_1; p_2^+, p_2; \cdots), \quad 0 < p_i^+.$$ 

The $p_i$ are negligible. For $p_i^+$, use momentum fractions

$$\xi_i = p_i^+/p^+, \quad 0 < \xi_i < 1.$$ 

$\Rightarrow$ Hadron is like a collection of free massless partons with $v = 1$, parallel momenta.
Factored cross section

Treat hadron as a collection of free massless partons with parallel momenta.

\[
\frac{d\sigma}{dE' d\omega'} \sim \int_0^1 d\xi \sum_a \frac{f_a(h(\xi, \mu))}{dE' d\omega'} \frac{d\tilde{\sigma}_a(\mu)}{dE' d\omega'} + O(m/Q).
\]
\[
\frac{d\sigma}{dE'd\omega'} \sim \int_0^1 d\xi \sum_a \frac{f_a/h(\xi, \mu)}{\hat{\sigma}_a(\mu)} \frac{d\tilde{\sigma}_a(\mu)}{dE'd\omega'} + O(m/Q).
\]

\(f_{a/h}(\xi, \mu)\ d\xi = \) probability to find a parton

with flavor \(a = g, u, \bar{u}, d, \ldots,\)

in hadron \(h,\)

carrying momentum fraction \(\xi = p_i^+/p^+.\)

\(d\hat{\sigma}_a/dE'd\omega' = \) cross section for scattering that parton.

We delay discussion of the \(\mu\) dependence.
The hard scattering cross section

To calculate $d\tilde{\sigma}_a(\mu)/dE' d\omega'$ use diagrams like

- Lowest order.
- Higher order.
Kinematics of lowest order diagram:

\[ \xi p^+ + q^+ = 0 \]

\[ p^+ = \frac{Q}{x\sqrt{2}} \]

\[ q^+ = -\frac{Q}{\sqrt{2}} \]

So

\[ \xi = x. \]
Summary so far

- Singularities for collinear parton splitting and joining and for soft gluon emission and absorption suggest a jet structure of final states.
- The singularities reflect long time physics.
- For short time physics, use infrared safe observables.
- In DIS, collinear splitting and joining in initial hadron
- This long distance physics $\rightarrow$ parton distributions.
- At lowest order, the hard scattering part of DIS is trivial, so measured structure functions $\approx$ parton distributions.
Renormalization and partons

What QCD looks like depends on the scale at which you look.

A) What renormalization does.

B) The running coupling.

C) The choice of renormalization scale.

D) The factorization scale.

E) Definition of the parton distributions.

F) Evolution of the parton distributions.

G) Discussion of the parton distributions.
What renormalization does

Use $\overline{\text{MS}}$ renormalization with renormalization scale $\mu$:

- Physics of time scales $|t| \ll 1/\mu$ removed from perturbative calculation.

- Effect of small time physics accounted for by adjusting value of the coupling*: $\alpha_s = \alpha_s(\mu)$.

*There are also other adjustments. In addition, renormalization by dimensional regularization and minimal subtraction is not exactly the same as imposing a cut-off $|\Delta x| > 1/\mu$. 
The running coupling

We account for time scales much smaller than $1/\mu$ (but bigger than a cutoff $M$ at the “GUT scale”) by using the running coupling.

\[ \log(1/M) + \log(1/\mu) + \log(\Delta t) \]

This sums the effects of short time fluctuations of the fields.
Result of one-loop renormalization group equation:

\[ \alpha_s(\mu) \sim \alpha_s(M) - \left(\frac{\beta_0}{4\pi}\right) \log(\mu^2/M^2) \alpha_s^2(M) \]
\[ + \left(\frac{\beta_0}{4\pi}\right)^2 \log^2(\mu^2/M^2) \alpha_s^3(M) + \cdots \]
\[ = \frac{\alpha_s(M)}{1 + \left(\frac{\beta_0}{4\pi}\right) \alpha_s(M) \log(\mu^2/M^2)} \]
\[ = \frac{4\pi}{\beta_0 \log(\mu^2/\Lambda^2)}. \quad (1) \]

- \(\alpha_s(\mu)\) decreases as \(\mu\) increases.

But what should the scale \(\mu\) be?
The choice of scale

Example: Cross section for $e^+ e^- \rightarrow$ hadrons via virtual photon:

$$\sigma_{\text{tot}} = \frac{12\pi\alpha^2}{s} \left( \sum_f Q_f^2 \right) [1 + \Delta]$$

$$\Delta(\mu) = \frac{\alpha_s(\mu)}{\pi} + \left[ 1.4092 + 1.9167 \log \left( \frac{\mu^2}{s} \right) \right] \left( \frac{\alpha_s(\mu)}{\pi} \right)^2$$

$$+ \left[ -12.805 + 7.8179 \log \left( \frac{\mu^2}{s} \right) + 3.674 \log^2 \left( \frac{\mu^2}{s} \right) \right] \left( \frac{\alpha_s(\mu)}{\pi} \right)^3$$

+ …

Clearly, $\log \left( \frac{\mu^2}{s} \right)$ should not be big.
Choosing $\mu$ in

$$\Delta_N = \sum_{n=1}^{N} c_n(\mu) \alpha_s(\mu)^n$$

- $\alpha_s$ depends on $\mu$.
- Coefficients depend on $\mu$.
- Physical cross section does not depend on $\mu$.
- The harder we work, the less the calculated cross section depends on $\mu$:

$$\frac{d}{d \log\mu} \sum_{n=1}^{N} c_n(\mu) \alpha_s(\mu)^n \sim \mathcal{O}(\alpha_s(\mu)^{N+1})$$
Take $\alpha_s(M_Z) \approx 0.117$, $\sqrt{s} = 34\text{GeV}$, 5 flavors. I plot $\Delta(\mu)$ versus $p$ defined by $\mu = 2^p \sqrt{s}$.

First curve:
$\Delta_1(\mu) = \alpha_s(\mu)/\pi$.

Second curve:
$\Delta_2(\mu)$ including $\alpha_s^2$ term.

- Possible choice:

$$\Delta_{PMS} = \Delta(\hat{\mu}), \quad \left[ \frac{d\Delta(\mu)}{d \log \mu} \right]_{\mu = \hat{\mu}} = 0.$$ 

Error band: estimated using $\mu = 2\hat{\mu}$ or $\mu = (1/2)\hat{\mu}$. 
One more order

I plot again $\Delta(\mu)$ versus $p$ ($\mu = 2^p \sqrt{s}$).

Three curves: $\Delta_1(\mu), \Delta_2(\mu), \Delta_3(\mu)$. 
Magnified view (including our $\Delta_2(\mu)$ error band):

Was the error estimate valid?
Recall the expression for the factored cross section in DIS:

\[
\frac{d\sigma}{dE'd\omega'} \sim \int_0^1 d\xi \sum_a \frac{f_a/h(\xi, \mu)}{dE'd\omega'} d\hat{\sigma}_a(\mu) + O(m/Q).
\]

What about the $\mu$ dependence?
**$\mu_F$ dependence**

$\Delta x^+$'s cover a range from $Q/m^2$ to $1/Q$.

- A gluon emission with $k^2 \sim m^2$ is part of $f(\xi)$.

- A gluon emission with $k^2 \sim Q^2$ is part of $d\hat{\sigma}$.

When calculating $\hat{\sigma}$, we (roughly speaking) count it as part of $\phi(\xi)$ for $k^2 < \mu_F^2$, and as part of $d\hat{\sigma}$ for $\mu_F^2 < k^2$.

\[
\begin{array}{c|c}
\text{hard scattering} & \text{parton distributions} \\
\hline
\log(1/\mu_F) & \log(\Delta t)
\end{array}
\]
\[ \frac{d\hat{\sigma}_a(\mu_F)}{dE'} d\omega' \] and \( f_{a/h}(\xi, \mu_F) \) depend on \( \mu_F \).

- \( \mu_F \) in \( f_{f/h}(\xi, \mu_F) = \) “factorization scale.”
- \( \mu \) in \( \alpha_s(\mu) = \) “renormalization scale.”
- Usually one wants \( \mu_F \approx Q \).
- As with \( \mu \), the higher order calculation you use, the less dependence on \( \mu_F \) there is.
- Often one sets \( \mu_F = \mu \).
Contour graphs of scale dependence

As an example, look at the one jet inclusive cross section in proton antiproton collisions at $\sqrt{s} = 1800$ GeV.

How does it depend on $\mu$ in $\alpha_s(\mu)$ and $\mu_F$ in $f_{a/p}(x, \mu_F)$?

$$\mu = \left(\frac{E_T}{2}\right) \times 2^{N_{uv}}, \quad \mu_F = \left(\frac{E_T}{2}\right) \times 2^{N_{CO}}$$

Plot $d\sigma/dE_T d\eta$ at rapidity $\eta = 0$ with arbitrary normalization
Contour plots with 5% contour lines.

$E_T = 100 \text{ GeV}$

$E_T = 500 \text{ GeV}$

- Variation with scale is roughly $\pm 10\%$ both for medium and large $E_T$. 
**$\overline{\text{MS}}$ definition of parton distribution functions**

Quarks:

\[
f_{i/h}(\xi, \mu_F) = \frac{1}{2} \int \frac{dy^-}{2\pi} e^{\im \xi p^+ y^-} \langle p | \overline{\psi}_i(0, y^-, 0) \gamma^+ F \psi_i(0) | p \rangle.
\]

\[
F = \mathcal{P} \exp \left( -i g \int_0^{y^-} dz^- A^+_a(0, z^-, 0) t_a \right).
\]

This is renormalized ($\overline{\text{MS}}$) with scale $\mu_F$: $k^2 < \mu^2_F$ included in $f$.

Gluons: similar definition using gluon field.
The physical picture for parton distributions
Evolution of the parton distributions

\[
\frac{d}{d \log \mu_F} f_{a/h}(x, \mu_F) = \sum_b \int_x^1 \frac{d \xi}{\xi} \; P_{ab}(x/\xi, \alpha_s(\mu_F)) \; f_{b/h}(\xi, \mu_F).
\]

\[P_{ab}(x/\xi, \alpha_s(\mu_F)) = P^{(1)}_{ab}(x/\xi) \frac{\alpha_s(\mu_F)}{\pi} + P^{(2)}_{ab}(x/\xi) \left( \frac{\alpha_s(\mu_F)}{\pi} \right)^2 + \ldots.\]
Evolution equation as a summation of perturbative effects

The differential equation

\[
\frac{d}{d \log \mu_F} f_{a/h}(x, \mu_F) = \sum_b \int_x^1 \frac{d \xi}{\xi} P_{ab}(x/\xi, \alpha_s(\mu_F)) f_{b/h}(\xi, \mu_F).
\]

accomplishes a summation.

The physical effect that we account for here is fluctuations within fluctuations within . . . .
Discussion of the parton distributions

- The $\overline{MS}$ definition in terms of operators is process independent.

- Sum rules are automatic. Eg.

$$\sum_{a} \int_{0}^{1} d\xi \, \xi \, f_{a/h}(\xi, \mu) = 1.$$  

- We don’t calculate $f$, but the definition adopted determines how $d\sigma$ is calculated.

- The parton distributions appear in the QCD formula for any process with one or two hadrons in the initial state.
Fitting the parton distributions

- Comparison of theory with experiment allows one to fit the parton distributions.

- The evolution is predicted, so one has only to fit the parton distributions at a starting scale $\mu_0$.

- There are lots of experiments, so this program won’t work unless QCD is right.
**QCD in hadron-hadron collisions**

Initial state, hard scattering, final state.

A) Kinematics: rapidity.

B) Production of $\gamma^*$, $W$, $Z$.

C) Discussion of factorization.

D) Jet production and jet definitions.
Kinematics: rapidity

Rapidity $y$ (or $\eta$) is useful for hadron-hadron collisions.

Consider production of a massive particle like a Z boson. Choose c.m. frame with z-axis along the beam direction.

- Z-boson momentum $q^\mu = (q^+, q^-, q)$. Define

$$ y = \frac{1}{2} \log \left( \frac{q^+}{q^-} \right). $$

Thus

$$ q^\mu = (e^y \sqrt{(q^2 + M^2)/2}, e^{-y} \sqrt{(q^2 + M^2)/2}, q). $$
Transformation property of rapidity under a boost along z-axis:

\[ q^+ \rightarrow e^{\omega} q^+ \], \quad q^- \rightarrow e^{-\omega} q^-, \quad q \rightarrow q.

\[ y \rightarrow y + \omega. \]

Good because the c.m. frame isn’t so special.
Pseudorapidity

Recall the definition of rapidity:

\[ y = \frac{1}{2} \log \left( \frac{q^+}{q^-} \right). \]

For a massless particle this is

\[ y = -\log (\tan(\Theta/2)) \]

If the particle isn’t quite massless, \( -\log (\tan(\Theta/2)) \) is the “pseudorapidity.”
Consider the process ("Drell-Yan")

\[ A + B \rightarrow Z + X. \]

Factored form of cross section:

\[
\frac{d\sigma}{dy} \approx \sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \ f_{a/A}(\xi_A, \mu) \ f_{b/B}(\xi_B, \mu) \ \frac{d\tilde{\sigma}_{ab}(\mu)}{dy}.
\]

This has corrections of order \( m/M \).
Some history. For $A + B \rightarrow \mu^+ + \mu^- + X$ one has the formula

$$\frac{d\sigma}{dQ^2 dy} \approx \sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \ f_{a/A}(\xi_A, \mu) \ f_{b/B}(\xi_B, \mu) \ \frac{d\hat{\sigma}_{ab}(\mu)}{dQ^2 dy}$$

where $Q^2$ is the squared mass of the muon pair.

- Before QCD, one had partons and QED, which did a good job of explaining deeply inelastic scattering.
- But there were other ways to explain DIS.

- Drell and Yan proposed to explain the Lederman et al. experiment using the lowest order version of this formula.

- It worked!
Discussion of factorization

For $A + B \to \mu^+ + \mu^- + X$ one has the formula

$$\frac{d\sigma}{dQ^2dy} \approx \sum_{a, b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \ f_{a/A}(\xi_A, \mu) \ f_{b/B}(\xi_B, \mu) \ \frac{d\hat{\sigma}_{ab}(\mu)}{dQ^2dy} + \mathcal{O}(m^2/Q^2).$$

This result is not so obvious, and in fact does not hold graph by graph.

- Need unitarity.
- Need causality.
- Need gauge invariance.
Jet production

One can also measure cross sections to make jets,

\[ A + B \rightarrow \text{jet} + X. \]

The idea is that the partons in the final state turn into collimated sprays of physical particles ("jets"). The cross section has the factored form

\[
\frac{d\sigma}{dE_Td\eta} \approx \sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \ f_{a/A}(\xi_A, \mu) \ f_{b/B}(\xi_B, \mu) \ \frac{d\hat{\sigma}_{ab}(\mu)}{dE_Td\eta}.
\]
What does one mean by a jet?

Consider

\[ \frac{d \sigma}{d E_T \ d \eta} \]

\( E_T = \) transverse energy

\( \eta = \) rapidity

- Substantial \( E_T \) at large angles \( \Rightarrow \) care with the definition.

- In particular, the definition should be infrared safe.

- Cone definition often used.

- "\( k_T \)" definition adapted from \( e^+e^- \) is nicer.
**$k_T$ algorithm**

An iterative successive combination algorithm, as in $e^+e^-$. The main idea is to use $E_T$, $\eta$ and $\phi$ as variables, and to take the many low $E_T$ particles into account.

- Choose a merging parameter $R$.
- Start with a list of “protojets” with momenta $p_1^\mu, \ldots, p_N^\mu$.
- We also start with an empty list of finished jets.
- Result is a list of momenta $p_k$ of jets, ordered in $E_T$.

- Many will have small $E_T$ and are really minijets, or just part of low $E_T$ debris.
1. For each pair of protojets define

\[ d_{ij} = \min(E^2_{T,i}, E^2_{T,j}) \left[ (\eta_i - \eta_j)^2 + (\phi_i - \phi_j)^2 \right]/R^2 \]

For each protojet define

\[ d_i = E^2_{T,i} \]

2. Find the smallest of all the \( d_{ij} \) and the \( d_i \). Call it \( d_{\min} \)

3. If \( d_{\min} \) is a \( d_{ij} \), merge protojets \( i \) and \( j \) into a new protojet \( k \) with

\[
\begin{align*}
E_{T,k} &= E_{T,i} + E_{T,j} \\
\eta_k &= \frac{[E_{T,i}\eta_i + E_{T,j}\eta_j]}{E_{T,k}} \\
\phi_k &= \frac{[E_{T,i}\phi_i + E_{T,j}\phi_j]}{E_{T,k}}
\end{align*}
\]

4. If \( d_{\min} \) is a \( d_i \), then protojet \( i \) is “not mergable.” Remove it from the list of protojets and add it to the list of jets.

5. If protojets remain, go to 1.
Summary of second half

• $\alpha_s$ and parton distributions depend on resolution scales.

• Usually want $\mu$ and $\mu_F$ approximately $Q$.

• Dependence on choice decreases with order of calculation.

• Parton distributions have operator definitions.

• Parton distributions determined from experiments.

• High $P_T$ cross sections in hadron collisions factor: $f \otimes f \otimes \hat{\sigma}$.

• Predict Drell-Yan, heavy quark, jet ... cross sections.