A transverse angular momentum sum rule

Phys Rev D **70**, *114001 (2004)*

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Several years ago Elliot Leader and I set about seeing if we could understand how to write down a sum rule for transverse polarization (transversity? More on this later) analogous to the very important helicity-based sum rule that had such a dominating importance for the past couple decades.

We had in mind to follow Feynman's well-marked path for parton model sum-rules; sum-rules for charge, isospin, momentum, helicity:

- Take some operator whose nucleon matrix elements are known, at least in principle. For example, momentum, charge, helicity.
- Calculate the operator's parton —quark and gluon— matrix elements.
- Expand the nucleon state into a sum of Fock space states of quarks and gluons.
- Equate the nucleon matrix element to the sum of the Fock state matrix elements.
- Take the limit that the nucleon momentum goes to infinity.
- Voila! If all goes well you have the desired sum rule.

Many people worked on related problems and obtained some interesting results, but not exactly what we were looking for. Some ran into difficulty carrying through this scheme which works so well for longitudinal spin for the transverse spin case.

The sort of problem encountered, and the one we focused on, can be seen in the formula that was widely used for the matrix element of the angular momentum operator between fermion —proton or quark—states.

$$\langle J_i \rangle_{std} = \frac{1}{4Mp^0}\{(3p_0^2 - M^2)s_i - (\frac{3p_0 + M}{p_0 + M})(\boldsymbol{p}\cdot\boldsymbol{s})p_i\} \\ + i\epsilon_{ijk}p_j \frac{\partial}{\partial p_k}\delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p}\ ).$$

Jaffe & Manohar, NP B337, 509 (1990)

If $\mathbf{s}$ is parallel to $\mathbf{p}$ this becomes simply $\sigma_{\mathbf{i}}/\mathbf{2}$, which is good, exactly what is needed for the much studied helicity sum rule, but if $\mathbf{s}\cdot\mathbf{p}$=0, i.e. it is transversely polarized, this gives $3p_3/Ms_i$ as $p_3 \rightarrow \infty$ and so it cannot be used in the manner planned to obtain a parton model sum rule.

This may be an intrinsic problem with describing transverse spin, which there are other indications for, but before reaching this strong negative conclusion, it is prudent to check the derivation of this formula. There are complications which make this formula difficult to derive; in particular, the angular momentum operator is given by an integral of the energy-momentum tensor:

$$J_i = \int d^3x(x_j T^{0k} - x_k T^{0j}).$$

In order to get well-defined integrals here it is necessary to utilize wave-packets for non-forward scattering, always a messy and tedious business. It is especially so because of the factors x_i in the integrand which will make the convergence more problematic and in the end lead to derivatives of δ-functions.

Nevertheless with due care and diligence this can be carried through and we obtain for "boost" states quantized with spin m along the z-axis,

$$\langle p', m'|J_i|p, m\rangle \;=\; 2p_0(2\pi)^3 \left[\frac{1}{2}\sigma_i + i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\right]_{m'm} \delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p}\ ).$$

instead of the above, the very simple and intuitively compelling result showing spin plus orbital angular momentum. For massless particles, especially the gluon we need to use instead the helicity basis for which the same procedure yields

$$\langle p', \lambda'|J_i|p, \lambda\rangle = (2\pi)^3 2p_0 \left[\lambda\eta_i(\boldsymbol{p}) + i(\boldsymbol{p}\times\boldsymbol{\nabla}_{\boldsymbol{p}})_i\right]\delta^3(\boldsymbol{p}' - \boldsymbol{p})\delta_{\lambda\lambda'}.$$

where (θ, ϕ) are the polar angles of $\boldsymbol{p}$ and

$$\eta_x = \cos(\phi)\tan(\theta/2), \qquad \eta_y = \sin(\phi)\tan(\theta/2), \qquad \eta_z = 1.$$

We have in mind a QCD energy-momentum tensor of the form

$$T^{0k} = \tfrac{1}{2}\bar{\Psi}\gamma^0 i \overleftrightarrow{D}^k \Psi - \tfrac{1}{2}\bar{\Psi}\gamma^k i \overleftrightarrow{D}^0 \Psi + \frac{1}{4}F^{0a}F^k_a.$$

We will not be doing any dynamical calculations here with this; only the form is important but in later applications it may be necessary to have this equation.

We can use this for both the nucleon and the quarks within (if we quantize in the "instant" form which is more suitable for discussing rotations than the currently more popular "front" form.) Notice that the limit as $p_z \to \infty$ of the quark-spin matrix elements are totally trivial; in the same limit, the transverse spin matrix elements for gluons vanish, as they should.

Elementary but general derivation

which is based on the purely kinematic role of rotations in the *instant* form. Consider a rotation about axis-i through an angle β. The unitary operator which effects this is given in terms of the angular momentum operator J_i:

$$R_i(\beta) = \exp(-i\beta J_i)$$

and for a particle of spin-$\frac{1}{2}$

$$R_i(\beta))|p,m\rangle = |R_i(\beta)p,n\rangle \mathcal{D}^{\frac{1}{2}}_{nm}(R(\beta)).$$

So

$$\begin{aligned}
\langle p',m'|R_i(\beta)|p,m\rangle &= \langle p',m'|R_i(\beta)p,n\rangle \mathcal{D}^{\frac{1}{2}}_{nm}(R_i(\beta)) \\
&= 2p_0(2\pi)^3\delta^{(3)}(\boldsymbol{p}' - R_i(\beta)\boldsymbol{p})\mathcal{D}^{\frac{1}{2}}_{m'm}(R_i(\beta)),
\end{aligned}$$

and

$$\begin{aligned}\langle p', m'|J_i|p, m\rangle &= i\,\frac{\partial}{\partial\beta}\langle p', m'|(R_i(\beta)|p, m\rangle|_{\beta=0} \\ &= 2p_0(2\pi)^3\left(i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\delta_{m'm} + \right. \\ &\qquad \left. i\frac{\partial}{\partial\beta}\mathcal{D}^{\,s}_{m'm}(R_i(\beta))\Big|_{\beta=0}\right)\delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p}\ ).\end{aligned}$$

Now

$$i\frac{\partial}{\partial\beta}\mathcal{D}^{\,\frac{1}{2}}_{m'm}(R_i(\beta))\Big|_{\beta=0} = \tfrac{1}{2}(\sigma_i)_{m'm}$$

Thus, our final result for the matrix elements of the angular momentum becomes

$$\langle p', m'|J_i|p, m\rangle = 2p_0(2\pi)^3\left[\tfrac{1}{2}\sigma_i + i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\right]_{m'm}\delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p}\ ). \qquad (1)$$

We next use the Fock-space momentum wave functions to calculate the density matrices for quarks, anti-quarks and gluons: for the quarks and anti-quarks we use the canonical z-axis quantization for spin, for gluons the helicity basis and then equate the proton matrix element of J_i for any i to the corresponding parton contribution:

$$\begin{aligned}\tfrac{1}{2}(\boldsymbol{\sigma}_i)_{m'\,m} &= \int d^3\boldsymbol{k}\ \left[\tfrac{1}{2}(\boldsymbol{\sigma}_i)_{\sigma'\,\sigma}\ \rho^{m'\,m}_{\sigma'\,\sigma}(\boldsymbol{k},\boldsymbol{k})^{q+\bar{q}} + \lambda\ \eta_i(\boldsymbol{k})\ \rho^{m'\,m}_{\lambda\,\lambda}(\boldsymbol{k},\boldsymbol{k})^{G}\right] \\ &\quad + \langle L_i\rangle^{q+\bar{q}}_{m'\,m} + \langle L_i\rangle^{G}_{m'\,m}\end{aligned}$$

In order to calculate the orbital pieces we need to know the momentum dependence of ρ; we did not address this important question.

Note that the density matrix elements are not invariant under Lorentz transformation so although the equation has the same form in any frame, the values depend on the frame. This is standard parton model lore and to make contact with other calculations we must go to the infinite-momentum frame by way of an infinite boost along the z-axis. For J_z we just reproduce the classic sum-rule. The quark a spin contribution to the RHS (an identical expression holds for the antiquarks) is

$$\tfrac{1}{2}\int d^3\boldsymbol{k}\tfrac{1}{2}\left[\rho^{+\,+}_{+\,+} - \rho^{-\,-}_{+\,+} - \rho^{+\,+}_{-\,-} + \rho^{-\,-}_{-\,-}\right]^a$$

just the difference between the density matrices for quark spin parallel or anti-parallel to the proton spin. Because $\eta_z = 1$ the gluon enters in exactly the same way and they just add, along with the anti-quarks, to complete the sum-rule.

For transverse spin we follow the same path. For fermions we write, say for spin in the x-direction

$$
\begin{aligned}
|\boldsymbol{p},\uparrow\rangle &= \frac{1}{\sqrt{2}}\{|\boldsymbol{p},m=1/2\rangle + |\boldsymbol{p},m=-1/2\rangle\},\\
|\boldsymbol{p},\downarrow\rangle &= \frac{1}{\sqrt{2}}\{|\boldsymbol{p},m=1/2\rangle - |\boldsymbol{p},m=-1/2\rangle\}.
\end{aligned}
$$

For the proton, we assume the spin is in the x-direction and we take its matrix element of J_x so again get $1/2$ on left-hand side.

The quark spin matrix element are all off-diagonal in m and take the value 1 so we get for transverse spin from the quarks

$$\tfrac{1}{2}\int d^3\boldsymbol{k}\tfrac{1}{2}\left[\rho_{+\,-}^{+\,-}+\rho_{-\,+}^{+\,-}+\rho_{-\,+}^{-\,+}+\rho_{+\,-}^{-\,+}\right]^a$$

This can be straightforwardly rearranged to give for transversity states

$$\tfrac{1}{2}\int d^3\boldsymbol{k}\tfrac{1}{2}\left[\rho_{\uparrow\,\uparrow}^{\uparrow\,\uparrow}-\rho_{\uparrow\,\uparrow}^{\downarrow\,\downarrow}-\rho_{\downarrow\,\downarrow}^{\uparrow\,\uparrow}+\rho_{\downarrow\,\downarrow}^{\downarrow\,\downarrow}\right]^a$$

so once again it is parallel minus anti-parallel. It is interesting that this form is formally independent of the frame. Not so for the gluon: rather we get parallel minus antiparallel multiplied by $\eta_x = \cos(\phi)\tan(\theta/2)$. For finite k_T we have $\theta \to 0$ as $p_z \to \infty$ and so the gluon spin contribution vanishes in the parton model limit.

This can be written in a form familiar from the helicity sum rule by writing

$$\begin{aligned}\tfrac{1}{2}\int d^3\boldsymbol{k}\tfrac{1}{2}\left[\rho^{\uparrow\,\uparrow}_{\uparrow\,\uparrow}-\rho^{\downarrow\,\downarrow}_{\uparrow\,\uparrow}-\rho^{\uparrow\,\uparrow}_{\downarrow\,\downarrow}+\rho^{\downarrow\,\downarrow}_{\downarrow\,\downarrow}\right]^a &= \tfrac{1}{2}\int dx(q^{\uparrow\uparrow}_{\uparrow\uparrow}(x)-q^{\uparrow\uparrow}_{\downarrow\downarrow}(x))^a \\ &\equiv \tfrac{1}{2}\Delta_T q^a\end{aligned}$$

so our sum rule is

$$\tfrac{1}{2} \quad = \quad \tfrac{1}{2}\Sigma_a(\Delta_T q^a+\Delta_T q^{\bar{a}})+\langle L_x\rangle^{a+\bar{a}}+\langle L_x\rangle^G$$

This is different from the sum-rule of Jaffe and Ji (PRL 67, 552, (1991)) which they call the transversity sum-rule

$$\delta q=\Sigma_a(\Delta_T q^a-\Delta_T q^{\bar{a}})+\langle L_x\rangle^{a-\bar{a}}$$

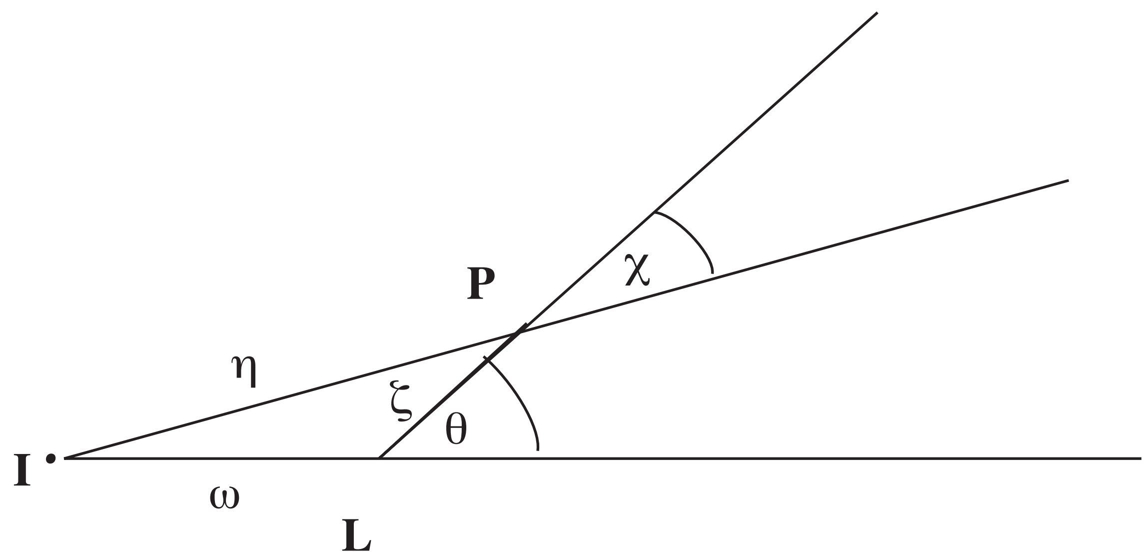
They call δq the *tensor charge*; some people call it g_T which confuses it with another quantity denoted the same way. It evidently has no sea quark or gluon contribution.

To be more precise, Jaffe & Ji's result has the quark spin be given as the quark light-cone helicity. This is obtained by a rotation through the famous Melosh angle from our fixed z quantization direction: For large energy and limited transverse momentum, this correction which has

$$d^2 k_T \rho(k) \rightarrow d^2 k_T \cos(\Theta_M)$$

is very small. For a similar effect, see Ma and Schmidt, Phys Rev D 58, 096008.

Velocity space diagram relating lab frame to
infinite momentum frame and
illustrating Melosh angle Θ_M



$$\Theta_M = \theta - \chi$$

$\sin\chi = \sin\theta / \exp\zeta$
by hyperbolic trig for ω very large
(infinite momentum frame)

Unlike the wonderful helicity sum-rule, neither of these two lead directly to methods for measuring $\Delta_T q^a$. You can imagine ways in Drell-Yan and semi-inclusive DIS, but I have not attempted to do that. Likewise, the orbital angular momentum remains a challenge for both of them. Perhaps they can be used together to make progress?

Transversity:

Just a few comments to help clarify the term: The name has been around since at least the early 1960's when it was introduced as a parallel quantity to helicity: spin quantized normal to momentum. (A. Kotanski, Acta Phys. Pol., 30: 629-45(Oct. 1966), Cohen-Tannoudji et al, Nuovo Cim. (10), 50A: 1025-8(Aug. 21, 1967).) This is especially useful for $2 \to 2$ reactions where all particles can share the normal as the quantization axis. Transversity amplitudes have been used intermittently since that time. More recently, in a paper cited by Jaffe and Ji, Goldstein and Moravcsik (PR D 32,303 (1985))used them for describing the polarization in high energy pp scattering.

J&J were searching for a term to avoid confusion of notation that was plaguing the transverse spin business, so they took the term "transversity" to mean specifically the structure function which enters their sum-rule. Unfortunately Leader and I are so old that we continue to use the original, broader definition of transversity so confusion continues:

email from Ji to Leader:

Dear Elliot, I was asked occasionally about the role of the trasverstity distribution in the angular momentum sum rule. My answer has been "none" because transversity operator is not part of the angular momentum operator and therefore a spin sum rule cannot naturally involve transversity. This is a general observation .... [In fact, angular momentum operator is chirally-even, and transversity operator is chirally odd.] I would be happpy to hear your argument against this if you don;t agree.

Best regards, Xiangdong

So whatever you want to call it, we have derived a simple, intuitive sum rule for transverse polarized protons. Its derivation is essentially kinematic, involving no dynamics, rather like the charge sum rule, but a little more complicated.

It is distinct from the *tensor charge* sum rule which does involve the Jaffe-Ji transversity distribution $h1(x)$, but they contain related quantities and so might be complementary.

Its usefulness depends on how well the various quantities that enter into it can be determined (cf. e.g.: Barone et al Phys Rep 359,1 (2002), Boer and Mulders, Phys Rev D 57, 5780 (1998)); it is especially important to determine the Fock space wave functions and orbital angular momentum part.(e.g. Hoodbhoy et al Phys Rev D 59, 014013 (1998).