Introduction to Accelerator Science: Overview of Colliders with Focus on RHIC

Waldo MacKay

- longitudinal acceleration: radio frequencey cavities
- transverse motion
  - bending and focusing
  - Hamiltonian, variables
  - Equations of motion
  - Courant-Snyder functions, invariant, emittances
  - Tunes
  - Liouville's Theorem
  - dispersion function
  - transverse beam size
  - nonlinearities, chromaticity
- longitudinal motion, acceleration, ...
  - phase stability
  - momentum compaction
  - transition energy
- Luminosity
  - low-β interaction region
  - beam-beam interaction, tune shift

Why collide?

- To get to higher energy in the center of mass, of course.

$$A + B \to X$$
$$M_{\text{cm}}^2 = s = (P_A + P_B)^\mu (P_A + P_B)_\mu$$

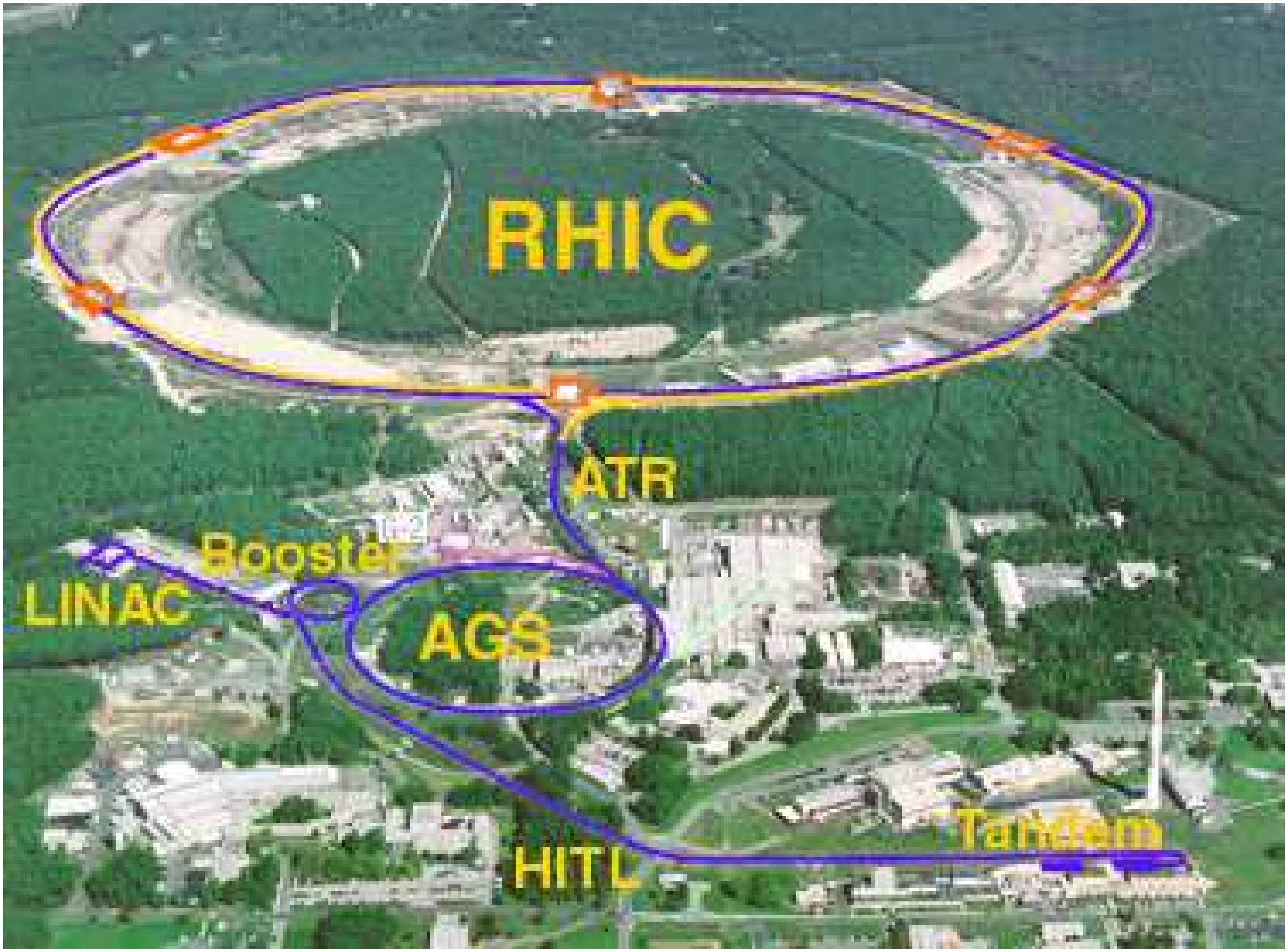
- fixed target (B): $s_{\text{ft}} = (E_A + M_B)^2 - P_B^2 \sim 2E_A M_B$
  assuming $E_A \gg M_A$.
- head-on collisions: $s_c = (E_A + E_B)^2 - (\vec{P}_A + \vec{P}_B)^2 \sim 4E_A E_B$
  assuming also that $E_B \gg M_B$ as well.

RHIC: 250 GeV×250 GeV protons $\sqrt{s} = 500$ GeV.
Equivalent fixed target would require a beam of 133 TeV protons.

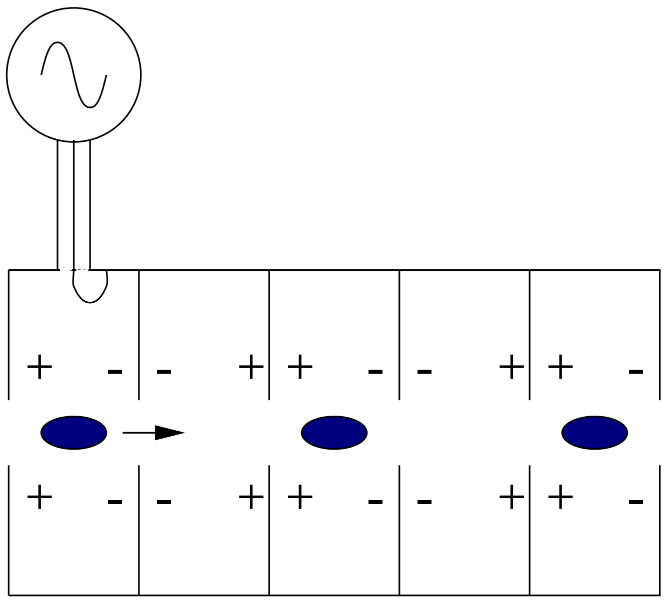
Some types of colliders

- Linear colliders $e^+ + e^-$ (SLC)
- Particle+anti-particle with a single ring
  - $e^+ + e^-$: many examples (SPEAR, VEPP4, LEP, CESR, ...)
  - $p + \bar{p}$ (Tevetron, SP$\bar{\text{P}}$S)
- Dual rings: (ISR, B-factories, HERA, RHIC, LHC)
  - Not restricted to same species for each beam, or the same energy.
- Future possibilities include:
  - Another electron+hadron collider
  - Linear collider for $e^+ + e^-$ in the TeV range
  - Muon collider

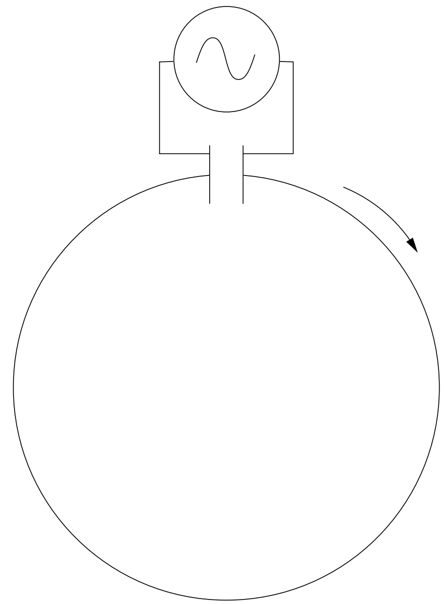
Disclaimer: The list of examples is not meant to be exhaustive.


RHIC
ATR
Booster
LINAC
AGS
Tandem
HITL

Acceleration with RF cavities



Linac: $\vec{F} = q\vec{E}(t)$.

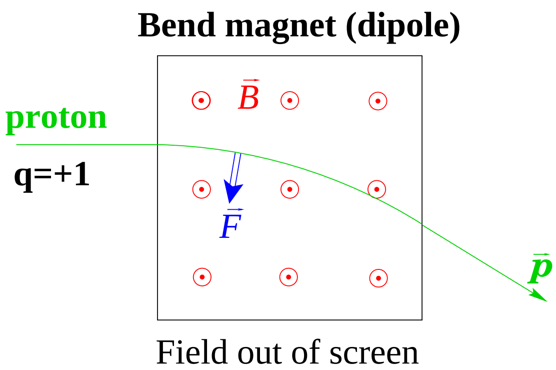


Ring with rf cavity

- Must maintain synchronism of bunch with rf phase.
- Particles oscillate in energy about the stable synchronous phase.

Particle Trajectories in Magnetic Fields

Dipole magnets bend the beam around the ring.

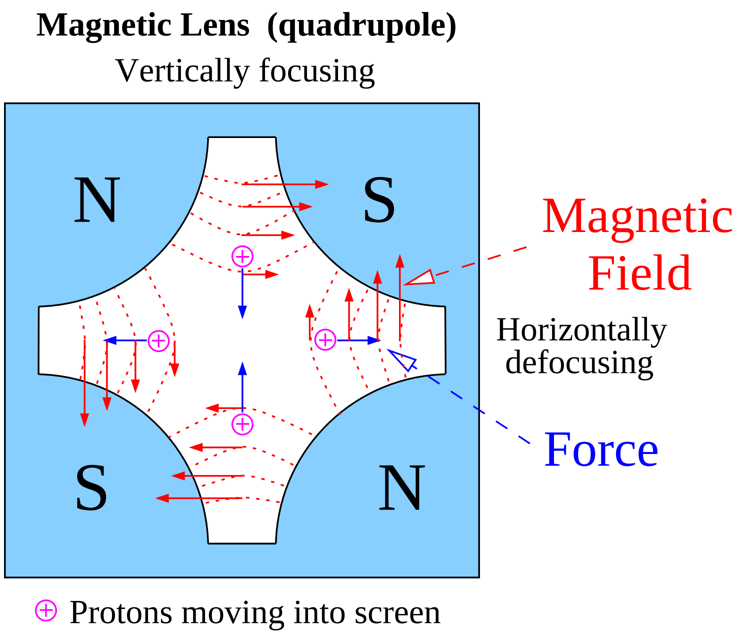


$$\rho = \frac{p}{qB_\perp}$$

Charged particles are deflected by magnetic fields. Lorentz Force:

$$\vec{F} = \frac{q}{\gamma m}\vec{p} \times \vec{B}$$

Quadrupole magnets focus the beam for stability.



$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

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Momentum compaction and transition

- Momentum compaction: $\alpha_p = \frac{dL}{L} \Big/ \frac{dp}{p}$, L is ring's circumference.

$$\omega_{\text{rev}} = \frac{2\pi}{\tau} = \frac{2\pi\beta c}{L}$$

$$\frac{d\omega_{\text{rev}}}{\omega_{\text{rev}}} = -\frac{d\tau_{\text{rev}}}{\tau_{\text{rev}}} = \frac{d\beta}{\beta} - \frac{dL}{L} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p}$$

- Phase slip factor: $\eta_{\text{ph}} = \gamma^{-2} - \alpha_p$, $\gamma_t = 1/\sqrt{\alpha_p}$ (RHIC: $\gamma_t \sim 23$)

- Transition energy: $U_{\text{tr}} = \frac{mc^2}{\sqrt{\alpha_p}}$, when $\eta_{\text{ph}} = 0$.
  - Below transition: velocity increase dominates ($d\beta/\beta$).
  - Above transition: particle is relativistic (β close to c), momentum compaction dominates (dL/L).

- Homework: Show that the momentum compaction of Earth is -2.

Longitudinal equation of motion

$$\ddot{\phi} + \frac{\omega_{\rm rf}\eta_{\rm ph}qV}{2\pi\beta^2\gamma mc^2}(\sin\phi - \sin\phi_s) = 0$$

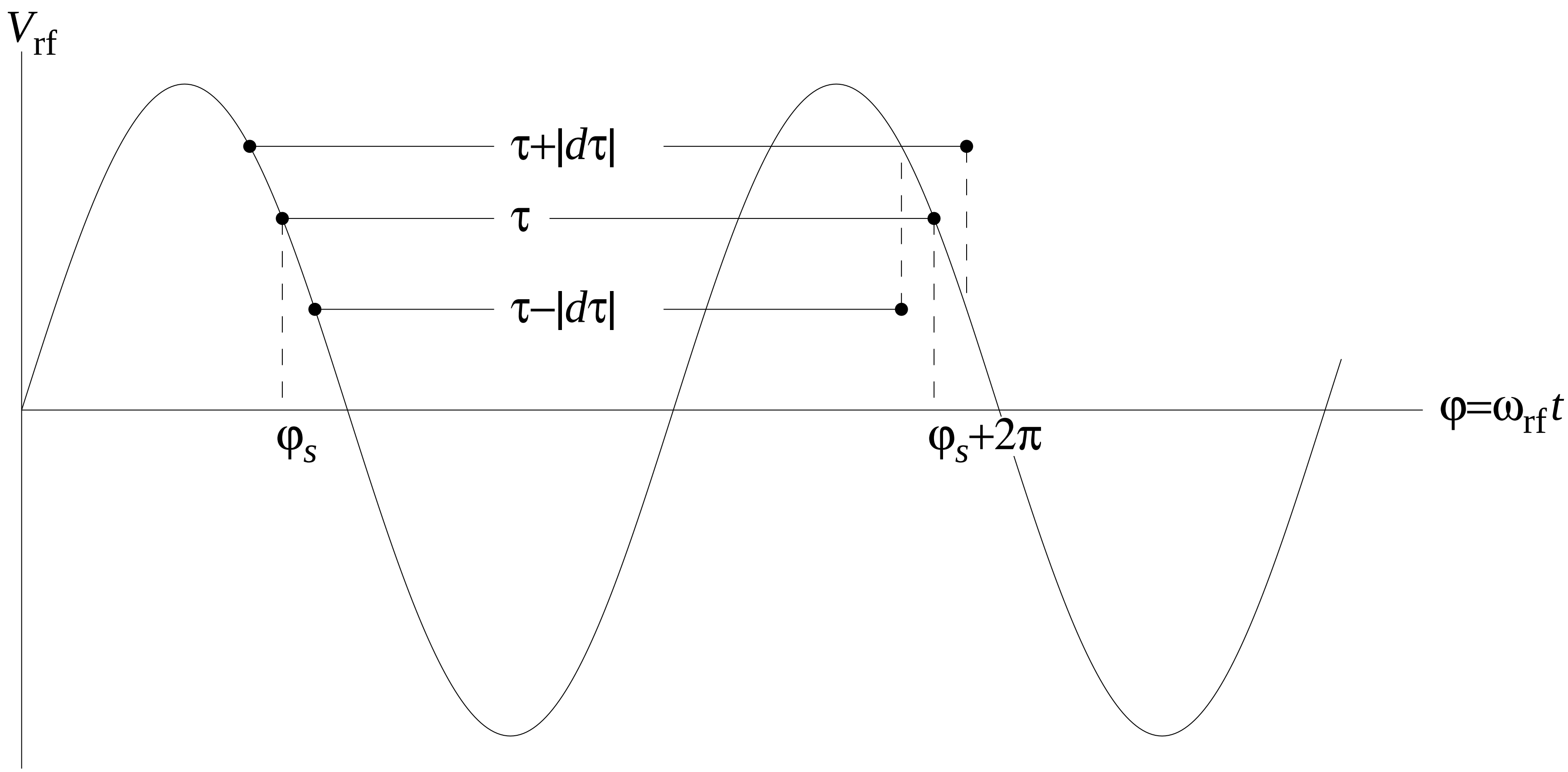
$$\omega_{\rm rf} = h\omega_{\rm rev}$$

- ϕ is the rf phase of the beam particle when it crosses the cavity.
- h is harmonic number.
  - RHIC: h =360 for 28 Mhz, or $7 \times 360 = 2520$ for 197 MHz.
- q, m are charge and mass of beam particle.
- V is the peak voltage of the rf cavity.

$$E(s,t) = V\sin(\omega_{\rm rf}t + \phi_s)\sum_{n=-\infty}^{\infty}\delta(s - nL)$$

- for fixed energy the synchronous phase $\phi_s = 0$.

Longinutinal motion above transition



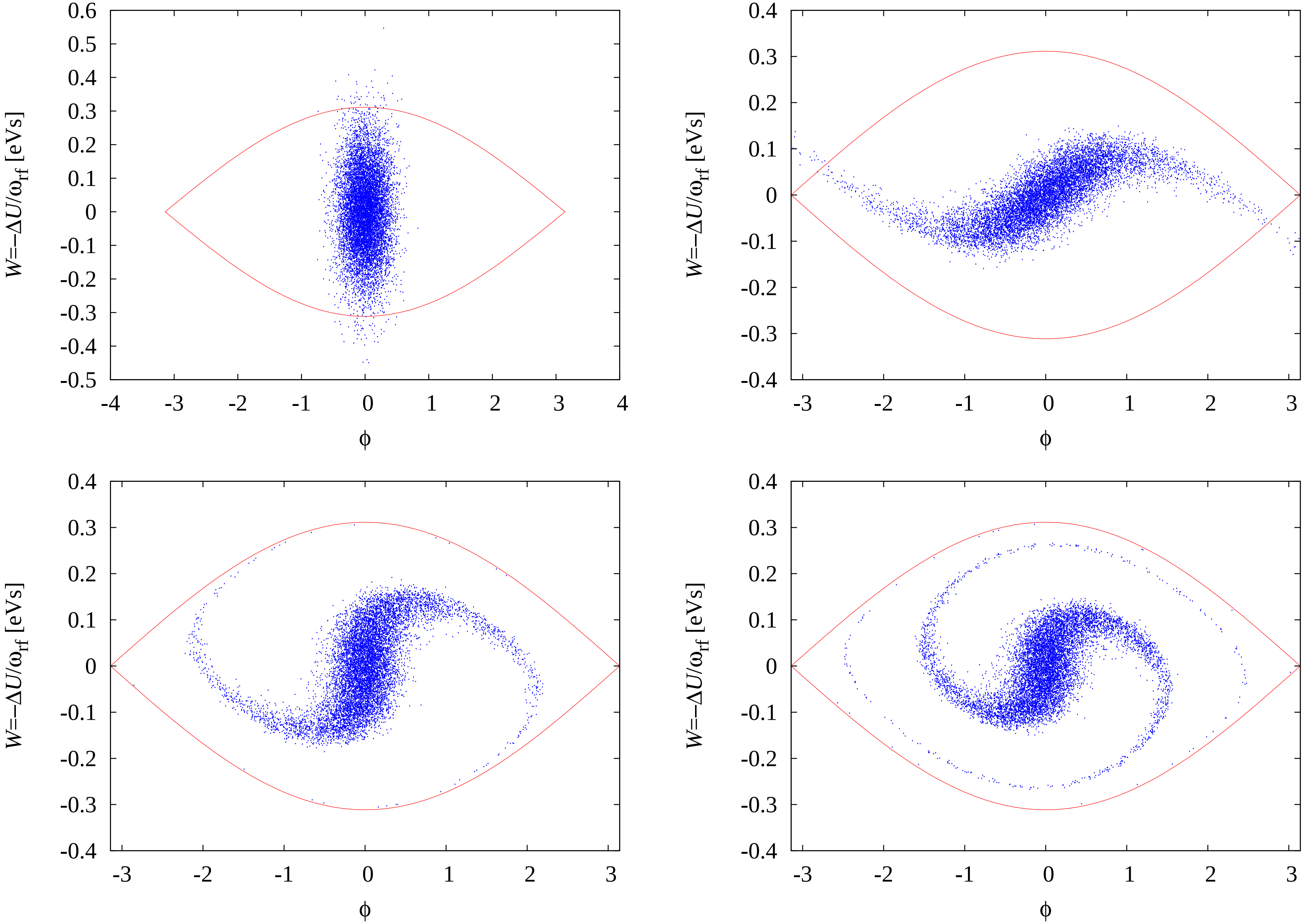
- Illustrated for harmonic number $h = 1$.

Small oscillations and synchrotron frequency

Linearize Eq. of motion for small amplitudes:

$$\phi = \phi_s + \varphi$$
$$0 \simeq \ddot{\varphi} + \Omega_s^2 \varphi$$

- Angular synchrotron frequency: $\Omega_s = \omega_{\mathrm{rev}} \sqrt{\frac{h\eta_{\mathrm{ph}} \cos\phi_s}{2\pi\beta^2\gamma} \, \frac{qV}{mc^2}}$
- In general $|\Omega_s| \ll 1$.
- Synchotron tune: $Q_z = \frac{\Omega_s}{\omega_{\mathrm{rev}}}$



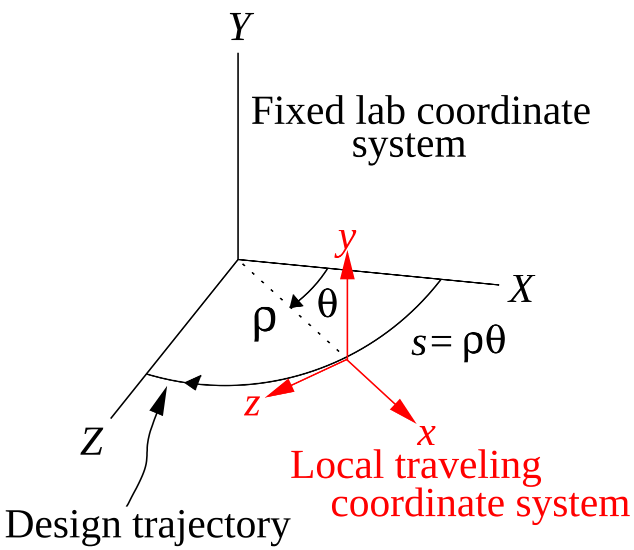
W=$-\Delta U/\omega_{rf}$ [eVs]
ϕ
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-4
-3
-2
-1
0
1
2
3
4

Hamiltonian

$$H(X, P_X, Y, P_Y, Z, P_Z; t) = \sqrt{(\vec{P} - q\vec{A})^2 + m^2c^4} + q\phi$$

After a bunch of canonical transformations and $\phi = 0$, $\vec{A} = (0, 0, A_s)$:

$$\mathcal{H}(x, x', y, y', z, \delta p/p_0; s) \simeq -\frac{q}{p_0}A_s - \left(1 + \frac{x}{\rho}\right)\left(1 + \frac{\delta p}{p_0} - \frac{1}{2}(x'^2 + y'^2) + \cdots\right)$$



$$\rho = \frac{p}{qB_\perp}$$

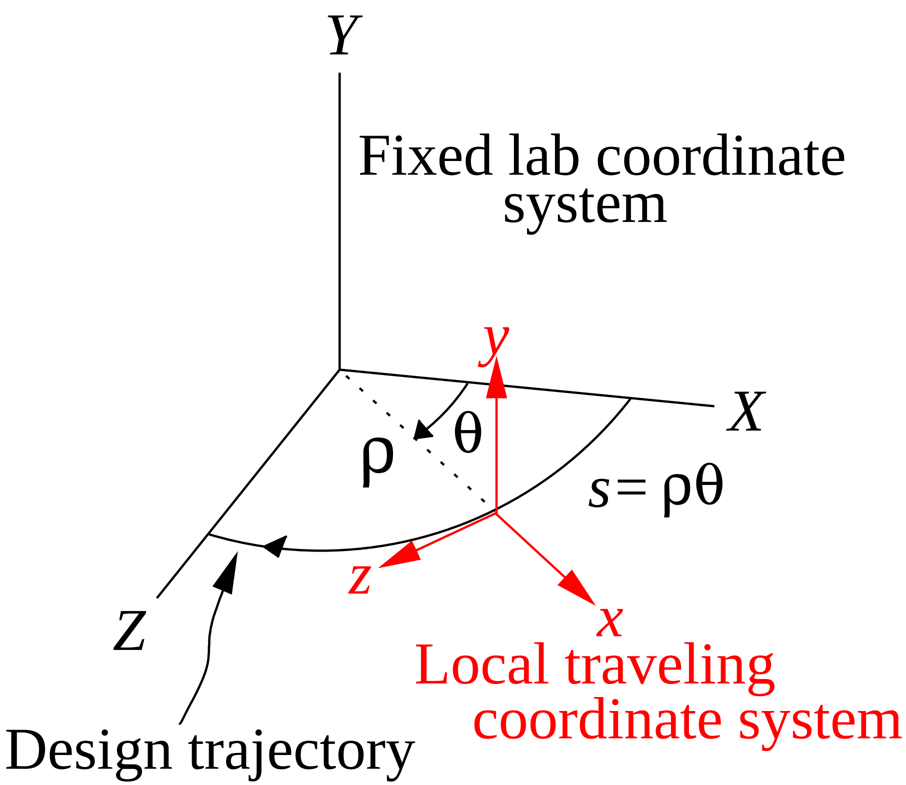
$$x' = \frac{dx}{ds}$$

$$y' = \frac{dy}{ds}$$

Paraxial approx.: $|x'|, |y'| \ll 1$

Paraxial coordinates

Expand about the design trajectory.



$$x' = \frac{dx}{ds} \simeq \frac{dp_x}{p_0}$$

$$y' = \frac{dy}{ds} \simeq \frac{dp_y}{p_0}$$

$$z = s - v_0 t$$

$$\delta = \frac{\Delta p}{p_0}$$

$$\vec{\mathbf{X}} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta = \frac{\Delta p}{p_0} \end{pmatrix}$$

Hill's Equations

$$x'' + k_x(s)x = \frac{\delta}{\rho(s)},$$

$$y'' + k_y(s)y = 0,$$

$$\text{with} \quad \delta = \frac{\Delta p}{p_0}.$$

For quadrupoles:

$$k_x = \frac{q}{p}\frac{\partial B_y}{\partial x}$$

$$k_y = -\frac{q}{p}\frac{\partial B_y}{\partial x}$$

Harmonic oscillator with periodic spring constant.

Periodic conditions: $k_j(s+L) = k_j(s), \qquad \rho(s+L) = \rho(s)$

where L is length of periodic cell.

- Horizontal motion has inhomogeneous dispersion term.
  - Ignore it for now.

Solutions to Hill's Equation

Use Floquet's (Block's) Theorem $\Rightarrow$
Quasi-periodic solutions of form:

$$x(s) = \sqrt{\mathcal{W}\beta(s)}\,\cos(\psi(s)), \quad \text{with}$$
$$\psi'(s) = \frac{1}{\beta(s)}.$$

Periodicity of β-function: $\qquad \beta(s+L) = \beta(s).$

Note: In general $\psi(s+L) \neq \psi(s) + n2\pi$. $\qquad$ Resonances are bad!

$$x'(s) = -\sqrt{\frac{\mathcal{W}}{\beta}}\left[\alpha(s)\cos\psi(s) + \sin\psi(s)\right],$$

$$\text{with} \quad \alpha(s) = -\tfrac{1}{2}\beta', \quad \text{and so} \quad \alpha(s+L) = \alpha(s).$$

Courant–Snyder Invariant

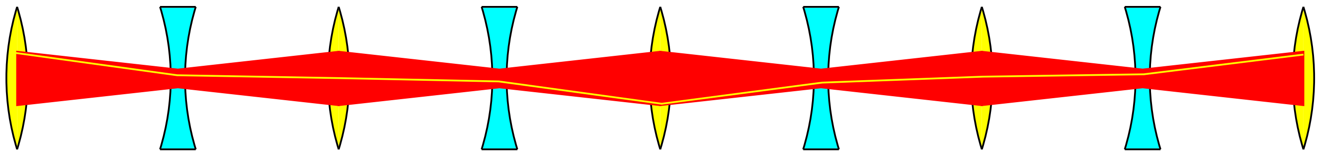
For a particular trajectory with initial conditions:

- Solve for $\sin\psi$ and $\cos\psi$ from equations for x and x'.
- Use $\sin^2\psi + \cos^2\psi = 1$ to get an invariant:

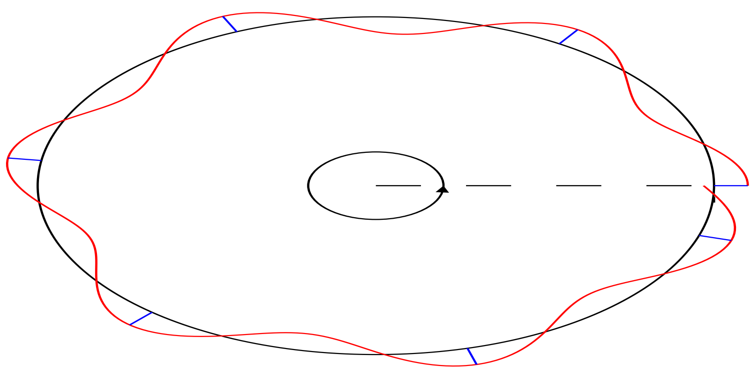
$$\mathcal{W} = \frac{1}{\beta}\left[y^2 + (\alpha y + \beta y')^2\right] \qquad \text{(Action variable: } J = \tfrac{1}{2}\mathcal{W}\text{)}$$

- Functions of s: $y(s)$, $y'(s)$, $\beta(s)$, $\alpha(s)$. (β and α are periodic.)
- Eq. (1) is the equation for an ellipse.
  - Area of ellipse $= \pi\mathcal{W}$.
- Beam envelope: $\pm\sqrt{\beta(s)\epsilon_{\rm rms}}$
  - $\pi\epsilon_{\rm rms}$ is the rms emittance

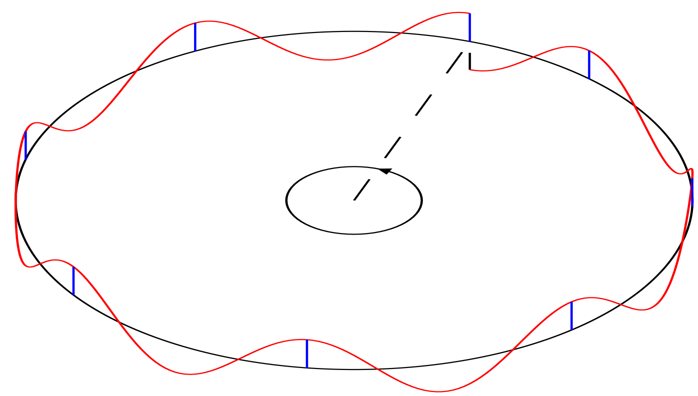
Transport and Betatron Oscillations



FODO: Alternate focusing and defocusing lenses for stability.

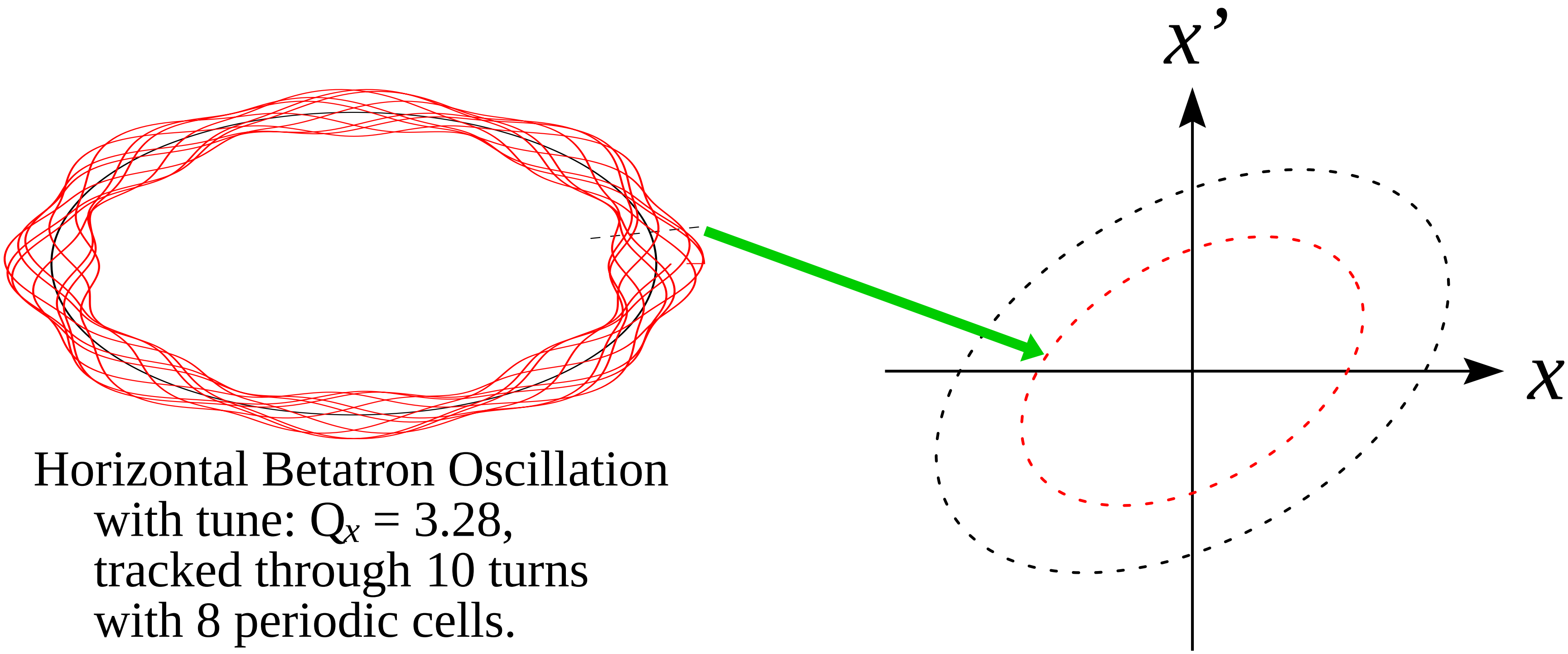


Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation
with tune: $Q_v = 7.5$,
i.e., 7.5 oscillations per turn.

2d Phase Space Plots



Horizontal Betatron Oscillation
with tune: $Q_x = 3.28$,
tracked through 10 turns
with 8 periodic cells.

$$\text{Ellipse area} = \pi\mathcal{W} = \oint x'\,dx$$
$$\simeq \frac{1}{p_0}\oint p_x\,dx$$

Poincaré plot of particle on successive turns for one location in the ring.

Liouville's Theorem

- Most beams have a low enough density, so that we ignore hard collisions between particles.
  - Thus we can use a 6d phase space rather than a 6N-d phase space.
- In the phase space of coordinates and their corresponding canonical momenta, the phase flow of the particle trajectories evolves so that the volumes of differential volume elements are preserved.
  - In other words, the Jacobian determinant is 1.
- Emittance is the area of the projection of the beam's phase-space volume onto a particular (x_i, P_i) plane.

Expansion of Trajectories

Expand trajectory $\mathbf{Y} = \mathbf{T}(\mathbf{X})$ in Taylor series about design orbit:

$$\vec{\mathbf{Y}} = \vec{\mathbf{T}}_0 + \sum_{i=1}^{6} \frac{\partial \vec{\mathbf{T}}}{\partial \Delta X_i}(0) X_i + \sum_{i,j=6}^{6} \frac{\partial^2 \vec{\mathbf{T}}}{\partial \Delta X_i \Delta X_j}(0) X_j X_k + \ldots$$

Linear first order derivative is Jacobian matrix $\mathbf{M}$ of transformation with

$$M_{ij} = \frac{\partial Y_i}{\partial X_j} = \frac{\partial T_i}{\partial X_j}$$

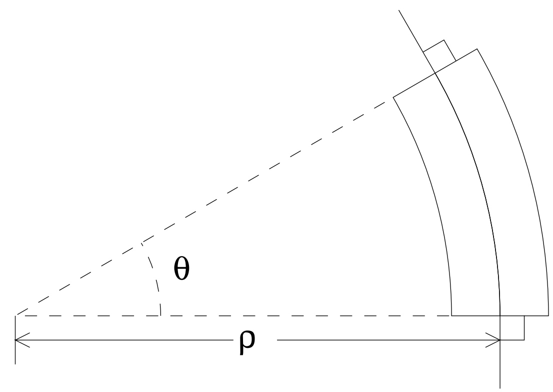
- Liouville's theorem requires that $|\mathbf{M}| = 1$.
- A more restrictive requirement is that $\mathbf{M}$ be symplectic: $\mathbf{M} \in \mathrm{Sp}(2n, \mathrm{R})$
  (e.g., see Goldstein's *Classical Mechanics*, 2nd Ed.)

Matrices for various elements

- In linear modeling we just calculate the Jacobian matrix for each individual element (drift, dipole, quad).
- Then to propagate through a group of elements, we just need to multiply matrices.
- Matrix for a drift of length ℓ

$$\mathbf{M}_{\mathrm{d}} = \begin{pmatrix} 1 & \ell & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- For a horizontal bend, i.e., sector dipole of design radius ρ and bend angle θ):



$$\mathbf{M}_{\rm b} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\theta & -\rho(1-\cos\theta) & 0 & 0 & 0 & -\rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- For a quadrupole of length ℓ and strength $k = \frac{q}{p}\frac{\partial B_y}{\partial x}$ (k of Hill's Eqn.):
  - (This is a horizontally focusing quad if $k > 0$.)

$$\mathbf{M}_{\rm qf} = \begin{pmatrix} \cos(\sqrt{k}\ell) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\ell) & 0 & 0 & 0 & 0 \\ -\sqrt{k}\,\sin(\sqrt{k}\ell) & \cos(\sqrt{k}\ell) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k}\ell) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\ell) & 0 & 0 \\ 0 & 0 & \sqrt{k}\,\sinh(\sqrt{k}\ell) & \cosh(\sqrt{k}\ell) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- If $k < 0$, quad is horiz defocusing; replace upper-left 4×4 block with:

$$\begin{pmatrix} \cosh(\sqrt{|k|}\ell) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}\ell) & 0 & 0 \\ \sqrt{|k|}\,\sinh(\sqrt{|k|}\ell) & \cosh(\sqrt{|k|}\ell) & 0 & 0 \\ 0 & 0 & \cos(\sqrt{|k|}\ell) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}\ell) \\ 0 & 0 & -\sqrt{|k|}\,\sin(\sqrt{|k|}\ell) & \cos(\sqrt{|k|}\ell) \end{pmatrix}$$

- For a solenoid of length ℓ and field B
  - Define $k = \frac{qB_0}{p}$

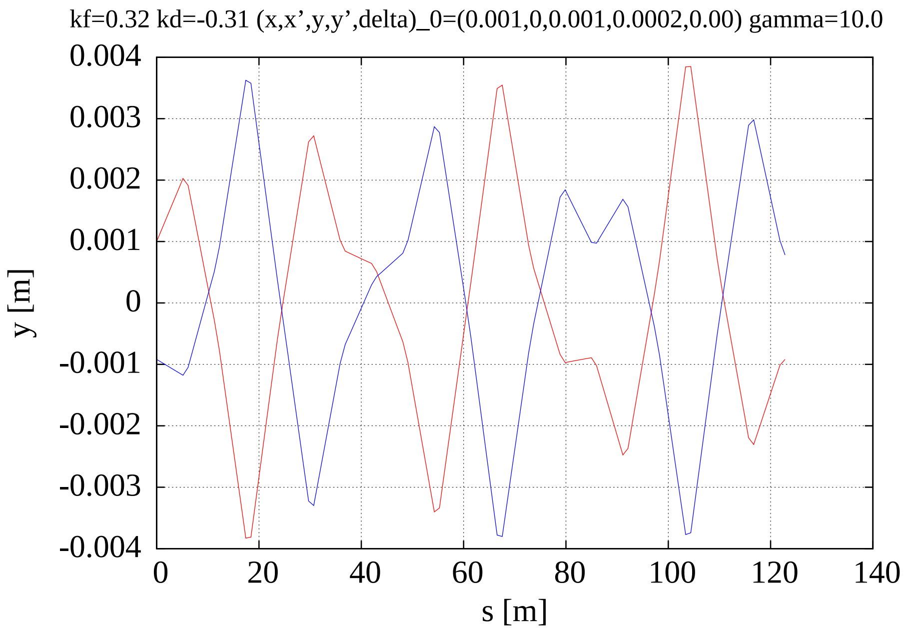
$$\mathbf{M}_{\rm sol} = \begin{pmatrix} \frac{1+\cos kl}{2} & \frac{\sin kl}{k} & \frac{\sin kl}{2} & \frac{1-\cos kl}{k} & 0 & 0 \\ -\frac{k\ \sin kl}{4} & \frac{1+\cos kl}{2} & -k\frac{1-\cos kl}{4} & \frac{\sin kl}{2} & 0 & 0 \\ -\frac{\sin kl}{2} & -\frac{1-\cos kl}{k} & \frac{1+\cos kl}{2} & \frac{\sin kl}{k} & 0 & 0 \\ k\frac{1-\cos kl}{4} & -\frac{\sin kl}{2} & -\frac{k \sin kl}{4} & \frac{1+\cos kl}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

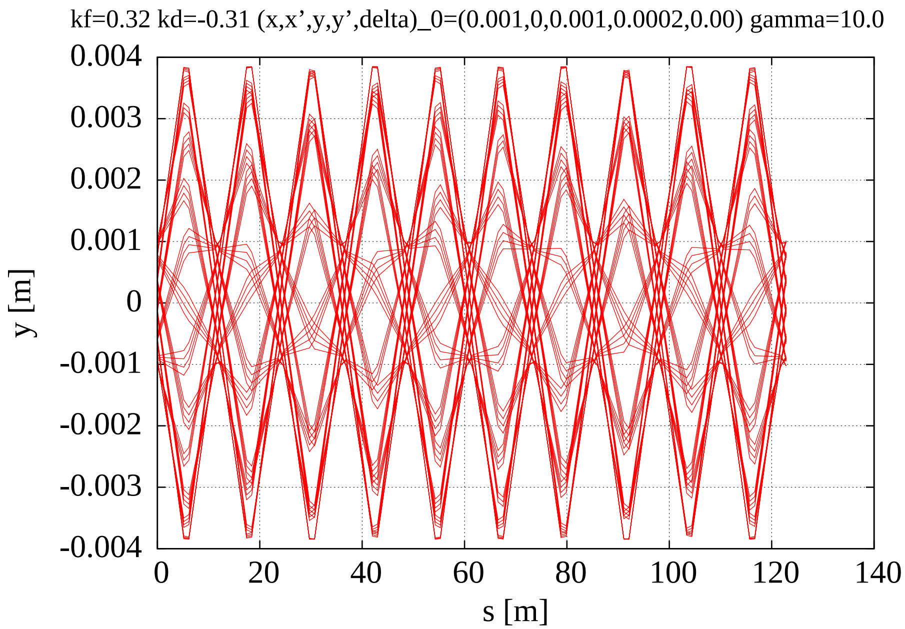
- Notice that this matrix couples the horiz. and vert. motion.
- For experiment solenoids at high energy, the coupling terms are generally small but nonnegligble.

$$\lim_{p\to\infty} \mathbf{M}_{\rm sol} = \mathbf{M}_{\rm d}.$$

FODO: Trajectory and Envelope

Lattice with 10 FODO cells

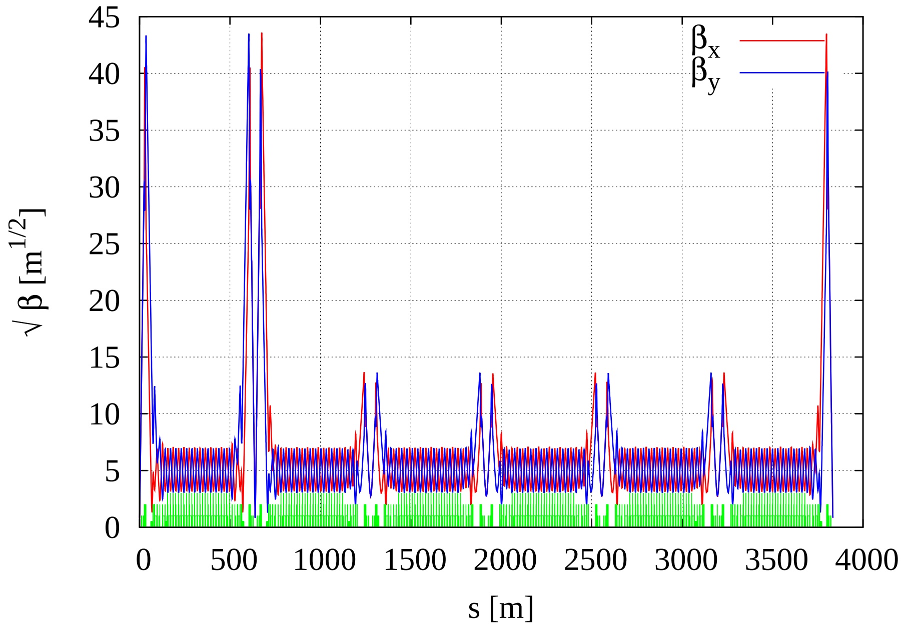


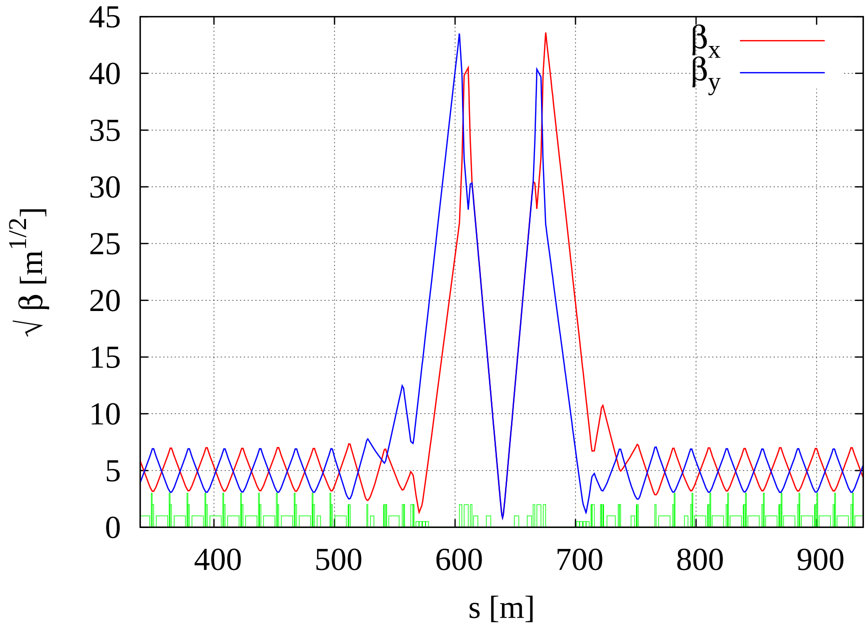


- Red curve is 1st turn.
- Blue curve is 2nd turn.

- 50 turns begin to fill up envelope.
- Envelope $\propto \sqrt{\beta(s)}$.

RHIC beta functions





- Polarized proton lattice at 100 GeV.
- Right plot is zoomed in around PHENIX.
- STAR and PHENIX: $\beta^* = 0.7$ m; 7.5 m at other IR's.

- One turn matrix: $\mathbf{M}_{1\text{turn}} = \prod_{j=1}^{N} \mathbf{M}_j$
- Can get tunes from eigenvalues: $\lambda_{\pm j} = e^{\pm i 2\pi Q_j}$
- Stable motion for real Q_j; unstable if Q_j complex (not real).
- Can permute product of matrices to propagate the 1-turn matrix.
    - Tunes don't depend on starting point, i.e. eigenvalues invariant under similarity transformations.

Dispersion Function

- Inhomogeneous solution Hill's Equation:

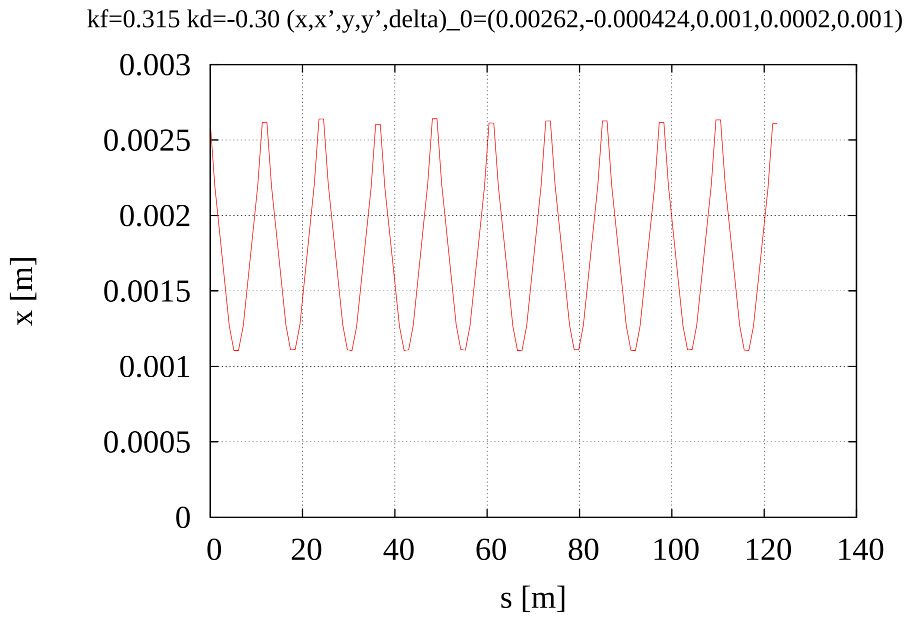
$$x'' + k_x(s)x = \frac{\delta}{\rho(s)}$$

$$\eta_x'' + k_x(s)\eta_x = \frac{\delta}{\rho(s)}$$

- Dispersion function is periodic: $\eta_x(s+L) = \eta_x(s)$.

$$\begin{aligned} x(s) &= x_\beta(s) + x_\delta(s) \\ &= x_\beta(s) + \eta_x(s)\delta \qquad \text{with } \delta = \tfrac{\Delta p}{p} \end{aligned}$$

Dispersion of FODO Lattice

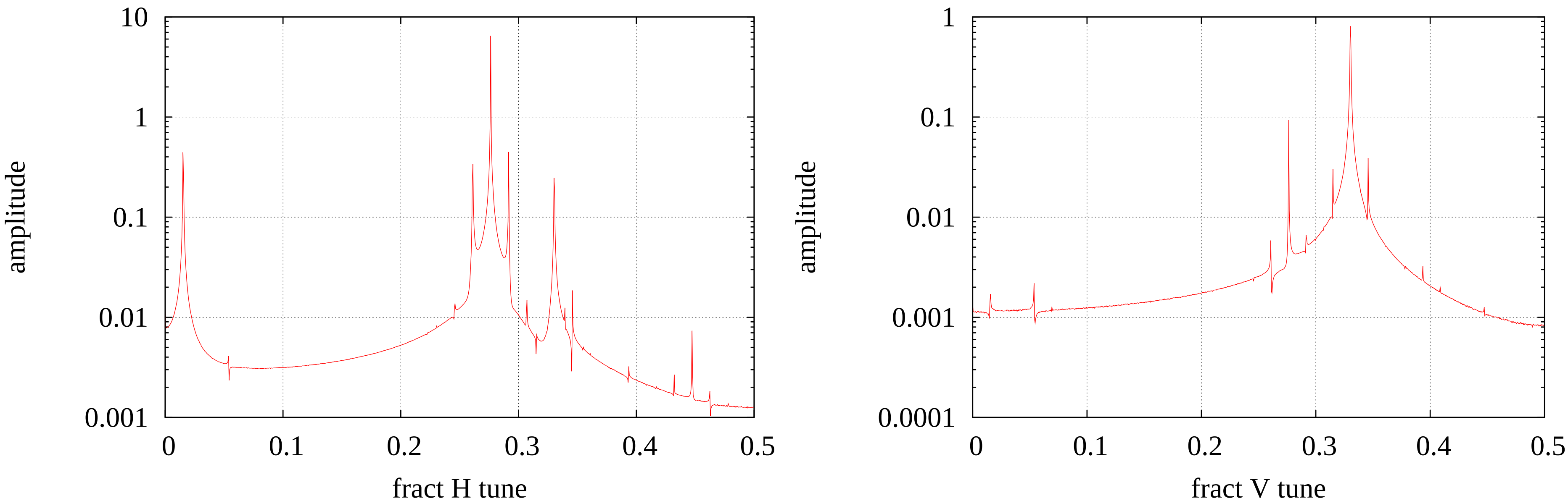


$$V_{\rm rf} = 0$$

$$\delta = \frac{\Delta p}{p} = 0.001$$

$$\eta_{\rm max} = 2.62 \text{ m}$$

Tunes for the simple FODO lattice



$$q_x = 0.2766, \quad q_y = 0.3301, \quad Q_s = 0.0153$$

- Single particle tracked for 2048 turns.
- With sextupoles to correct chromaticities (i.e. a nonlinearity).
- With coupling (rotate 1st quad by 0.007 radian).
- Strong synchrotron sidebands

Chromaticity

$$\xi_x = \frac{dQ_x}{d\delta}, \quad \xi_y = \frac{dQ_y}{d\delta}$$

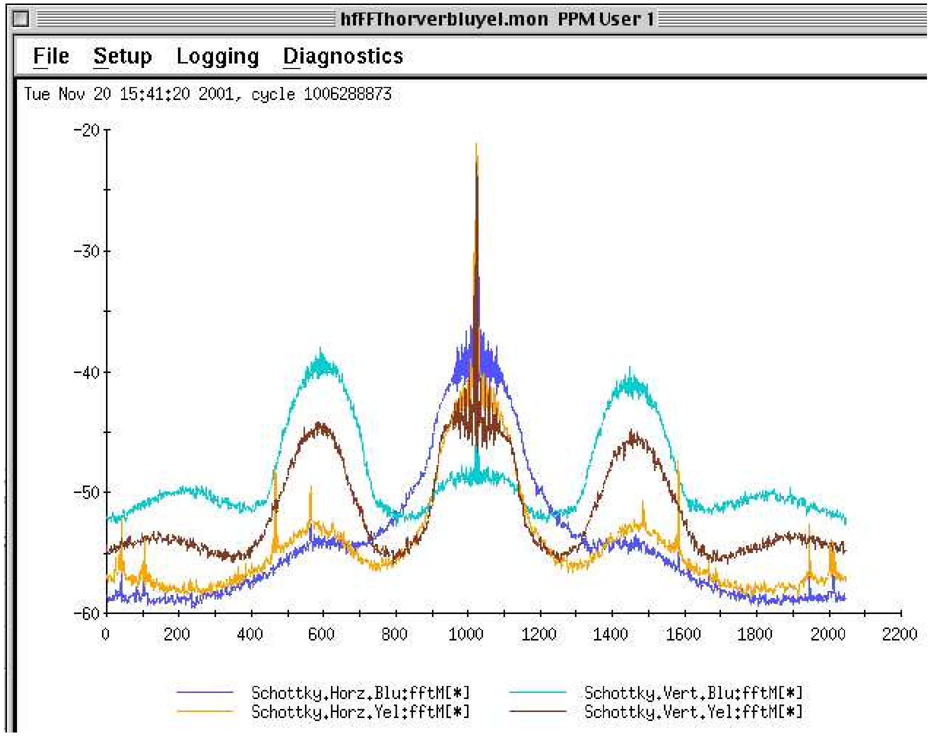
Recall Homogeneous Hill's Eqn.: $x'' + k(s)x = 0$

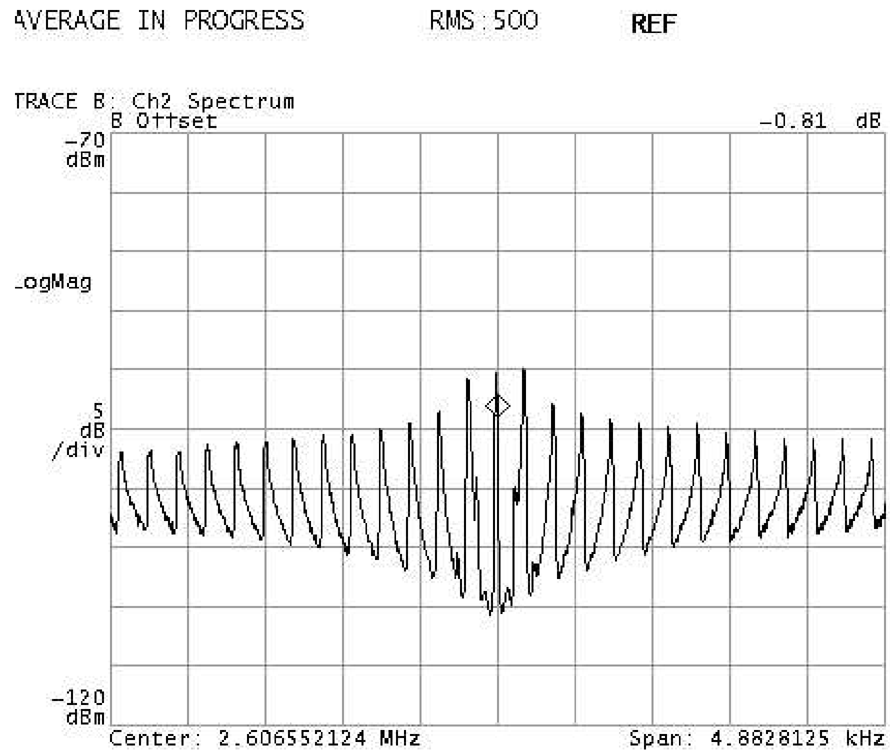
For a quadrupole:

$$k = \frac{q}{p}\frac{\partial B_y}{\partial x} = \frac{q}{p_0(1+\delta)}\frac{\partial B_y}{\partial x} \simeq k_0(1-\delta)$$

- Shifting the momentum changes the focusing of each quad.
  - Thus it shifts the tunes.
- Compensate with sextupole magnets: $B_y = b_2(x^2 - y^2), \quad B_x = 2b_2xy.$
  - $x = x_\beta + \eta_x\delta$
  - Downside: introduces amplitude dependent tune shifts.

RHIC tunes from Schottky cavity





- Can see coherent and incoherent tune information.
- Left shows high freq Schottky harmonics (∼2 GHz) with betatron sidebands.
- Right shows synchrotron sidebands around revolution harmonic.

Transverse beam size

$$\begin{aligned}\sigma_x^2 = \langle(x^2 - \langle x\rangle^2)\rangle &= \langle(x_\beta^2 + x_\delta^2 + 2x_\beta x_\delta - \langle x_\beta\rangle^2 - \langle x_\delta\rangle^2 - 2\langle x_\beta\rangle\langle x_\delta\rangle)\rangle \\ &= \langle x_\beta^2\rangle - \langle x_\beta\rangle^2 + \langle x_\delta^2\rangle - \langle x_\delta\rangle^2 \\ &= \sigma_{x_\beta}^2 + \sigma_{x_\delta}^2 \\ &= \beta_x \epsilon_{x,\mathrm{rms}} + \eta_x^2 \sigma_\delta^2,\end{aligned}$$

since the betatron and synchrotron oscillations are independent and should be added in quadrature.

Transverse emittances

The rms contour ellipse in the xx' phase space has area$= \pi\epsilon_{x,\mathrm{rms}}$ with

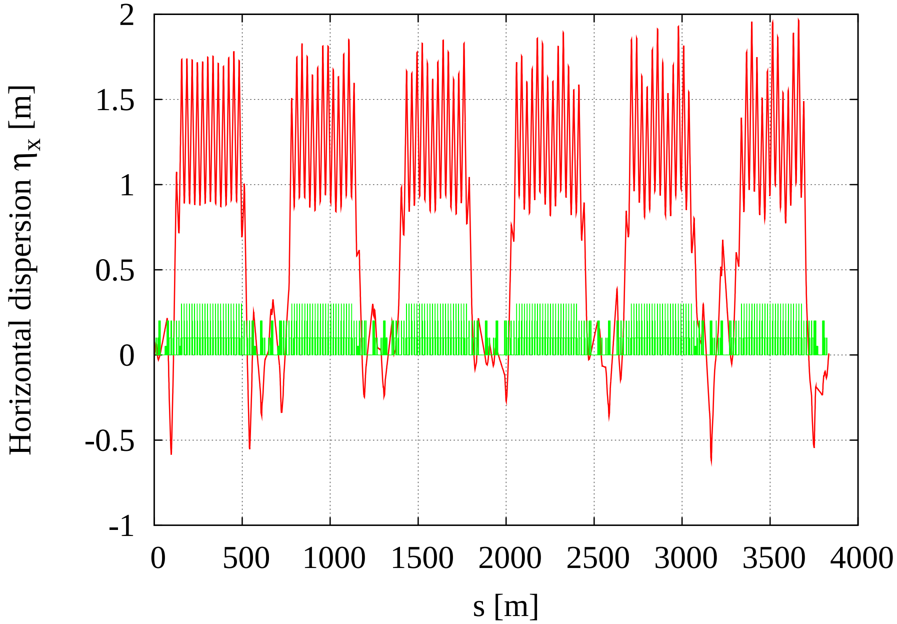
$$\epsilon_{x,\mathrm{rms}} = \frac{1}{\pi}\oint x'_{\mathrm{rms}}\,dx \simeq \frac{1}{\pi}\oint \frac{p_{x,\mathrm{rms}}}{p_0}\,dx = \frac{1}{\pi\beta\gamma mc} I_x$$

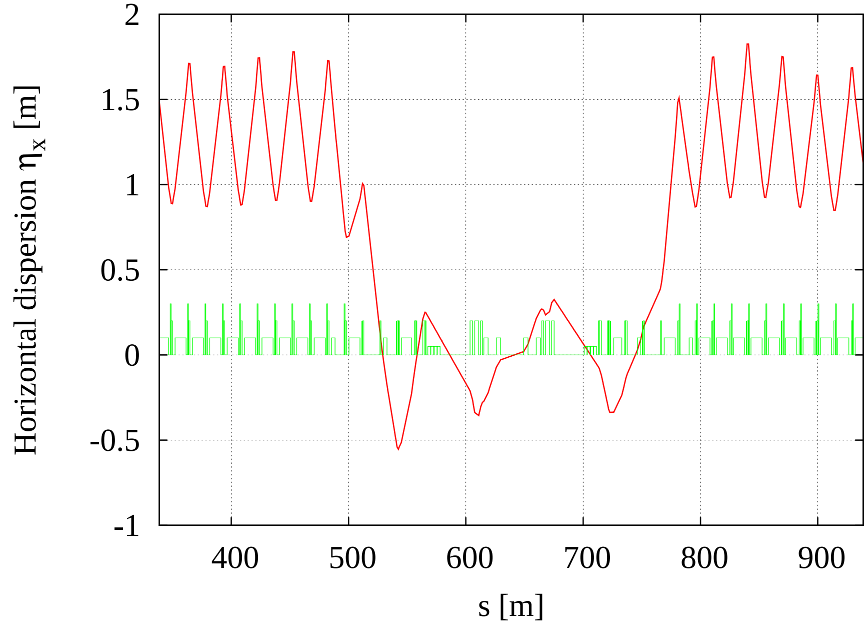
with the invariant $I_x = \oint p_{x,\mathrm{rms}}$

- Normalized emittance: $\pi\epsilon^{\mathrm{N}}_{\mathrm{rms}} = \pi\epsilon_{\mathrm{rms}} \times \beta\gamma$
  - The decrease of $\epsilon(\propto \frac{1}{\beta\gamma})$ with energy is referred to as *adiabatic damping.*
- For a 95% contour ellipse: $\epsilon^{95\%} \simeq 6\epsilon_{\mathrm{rms}}$, assuming a 2d Gaussian dist.

$$\sigma_x = \sqrt{\frac{\beta_x \epsilon_x^{\mathrm{N},95\%}}{6\beta\gamma} + \eta_x^2\left(\frac{\sigma_p}{p}\right)^2}.$$

RHIC dispersion function





- RHIC dispersion with STAR and PHENIX $\beta^* = 0.7$ m
- Right: zoomed in around PHENIX.

Resonances

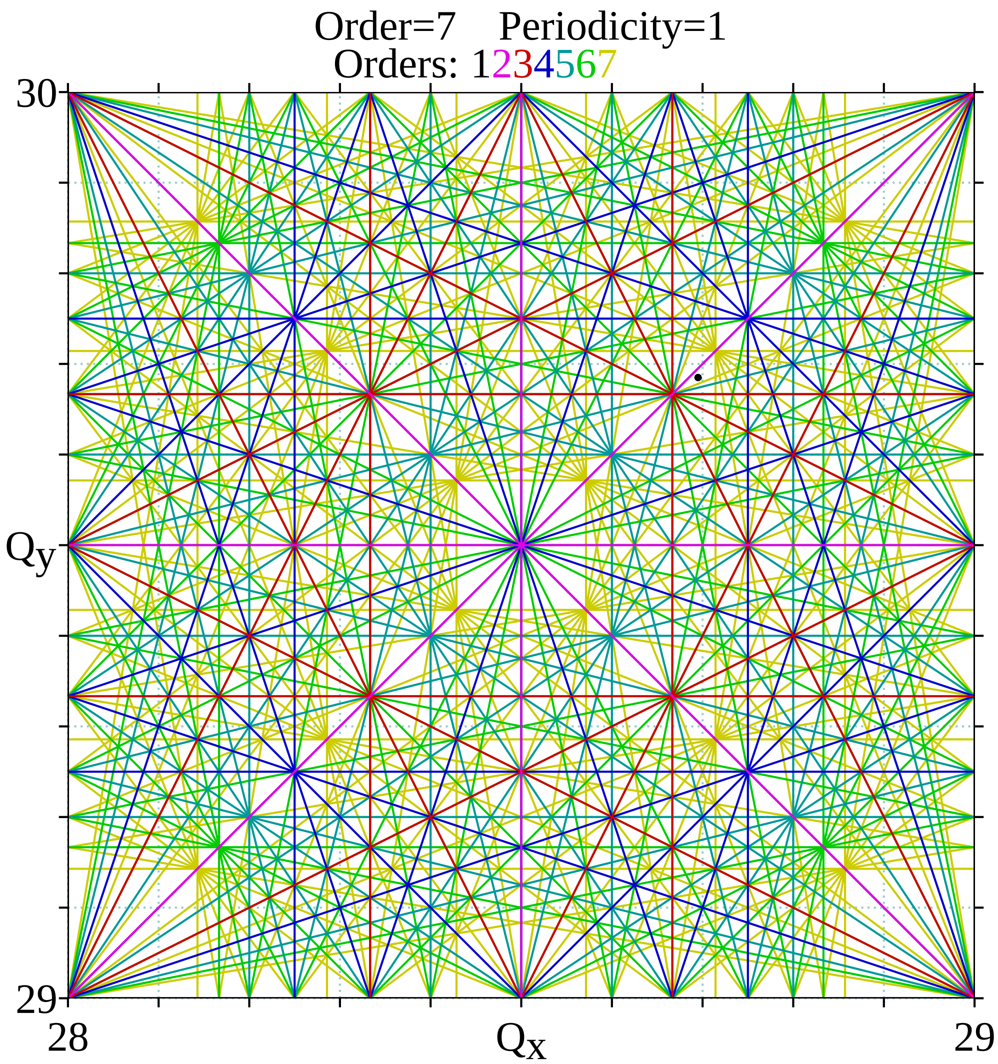
Resonance conditions can occur when

$$N_x Q_x + N_y Q_y + N_z Q_z = N, \quad \text{for} \quad N_x, N_y, N_z, N \in Z$$

The beam can blow up or be driven out of the ring by a resonance.

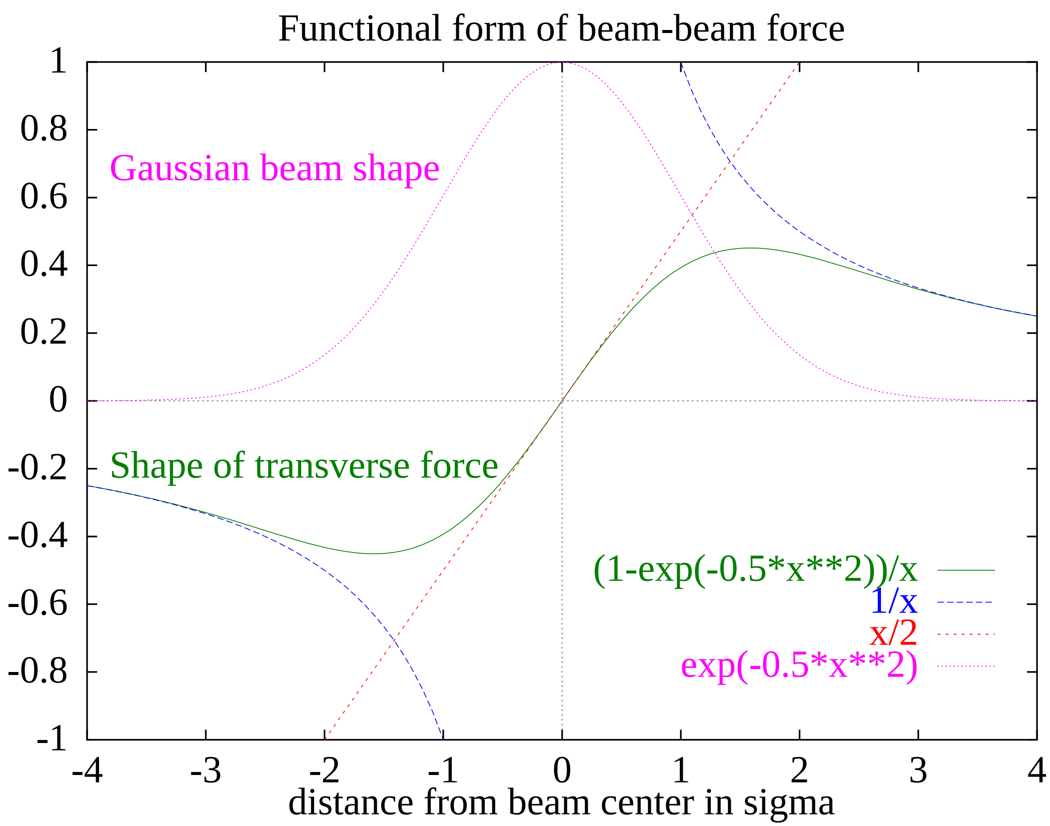
- Integer resonances are typically driven by imperfections in the lattice, especially a misalignment.
  - An integer resonance has a linear growth in amplitude.
- Quadrupoles can drive the half-integer resonances.
- Sextupoles typically drive $\frac{1}{3}$-integer resonances.

RHIC tune plane



- $N_x Q_x + N_y Q_y = N$
- Order: $N_x + N_y$
- slope>0 tend to be stable
- slope≤ 0 are unstable
- RHIC tunes for present run:
  - $Q_x = 28.695$, $Q_y = 29.685$
  - indicated by black dot.
- Nonlinearities smear the dot.

Beam-beam interaction



$$\rho = \frac{N}{\sqrt{2\pi}\sigma\ell} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

$$F_y = \frac{NZ^2e^2(1+\frac{v}{c})}{2\pi\epsilon_0\ell\sigma} f(\frac{y}{\sigma})$$

$$f(x) = \frac{1-\exp(-\frac{x^2}{2})}{x}$$

$$\frac{\Delta Q_{\mathrm{bb}}}{N_{\mathrm{IP}}} \sim \frac{r_0}{2}\,\frac{N}{\epsilon^{N,95\%}}$$

- Same sign: repulsive force, i.e., lowers tunes

- A test particle sees an almost linear gradient in the core of the beam.
- The beam-beam force produces odd harmonics.
- r_0 is classical radius; N is number of particles in bunch.

RHIC protons:

$$\begin{aligned} r_0 &= 1.5 \times 10^{-18} \text{ m} \\ N &= 1.3 \times 10^{11} \\ \epsilon^{N,95\%} &= 20 \times 10^{-6} \text{ m} \\ N_{\text{IP}} &= 2 \\ \Delta Q_{\text{bb}} &\simeq 0.01 \end{aligned}$$

Some $\mathrm{e}^+ + \mathrm{e}^-$ colliders have achieved $|\Delta Q_{\text{bb}}| \sim 0.04$.

Luminosity

Interaction rate:

$$\frac{dN}{dt} = \sigma\mathcal{L} = \int \sigma(\vec{\mathbf{X}}_2, \vec{\mathbf{X}}_1)\rho_2(\vec{\mathbf{X}}_2, t)\;|\vec{v}_1 - \vec{v}_2|\;\rho_1(\vec{\mathbf{X}}_1, t)\;d^6\mathbf{X}_2\,d^6\mathbf{X}_1,$$

$$= |\vec{v}_1 - \vec{v}_2|\;f_0\,N_b\,\sigma \int \rho_2(\vec{x} - \vec{v}_2 t)\;\rho_2(\vec{x} - \vec{v}_1 t)\;d^3x\,dt$$

$\vec{v}_j$ is the velocity of a particle in the j^{th} beam.
σ is the total cross section, and
ρ_1 and ρ_2 are the density distributions of the two beams.
N_b is the number of bunch crossings per turn.

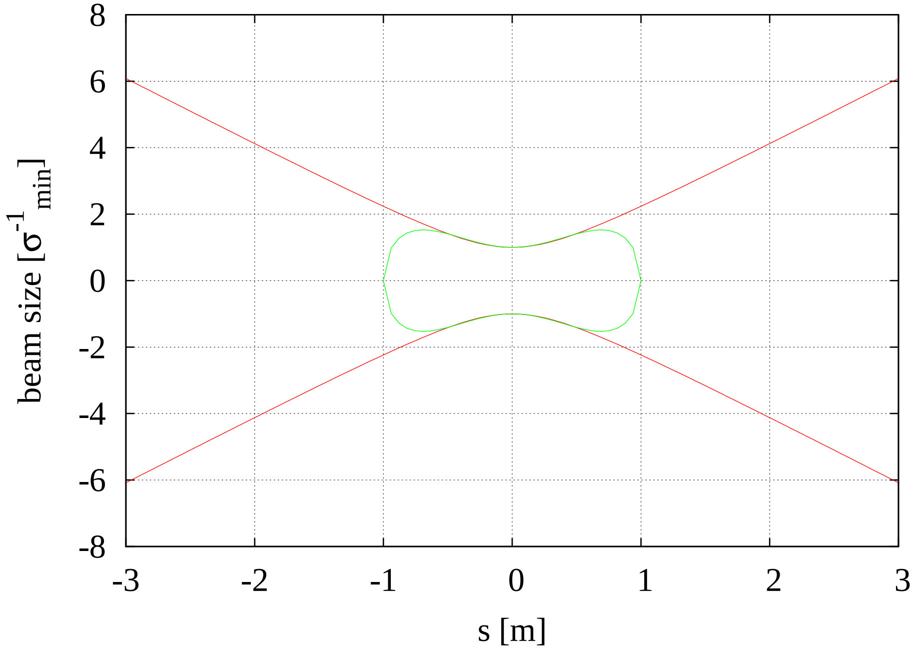
Instantaneous luminosity: $\mathcal{L}_0 \simeq N_b \frac{f_{\text{rev}} N_1 N_2}{4\pi\sigma_x\sigma_y}$

IR beam envelope shape: hour glass effect

In a field free region the beta functions are parabolic:

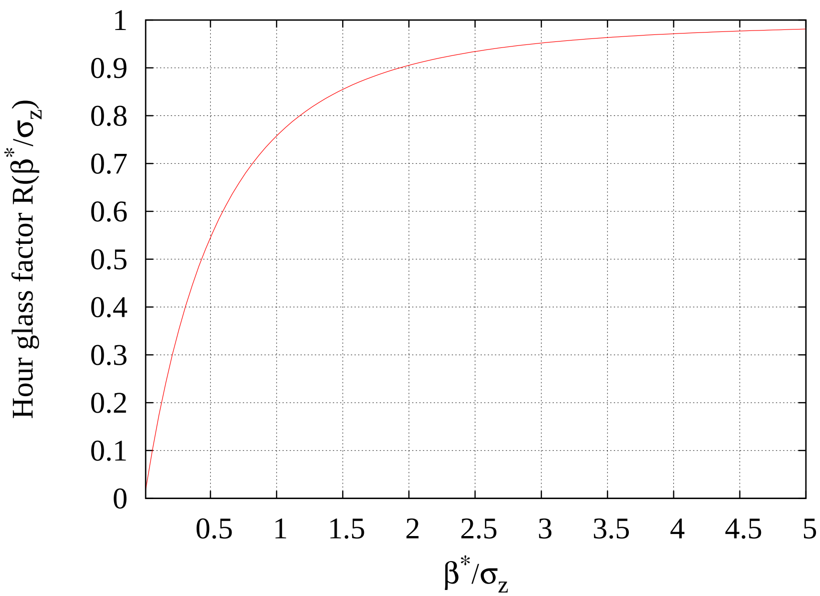
$$\beta_i(s) = \beta_i^* + \frac{s^2}{\beta_i^*}, \qquad \text{for } i \in \{x, y\}$$

$$\frac{\sigma}{\sigma_{\min}} = \sqrt{1 + \left(\frac{s}{\beta^*}\right)^2}$$



Plotted for $\beta^* = 0.5$ m.

Hour glass factor



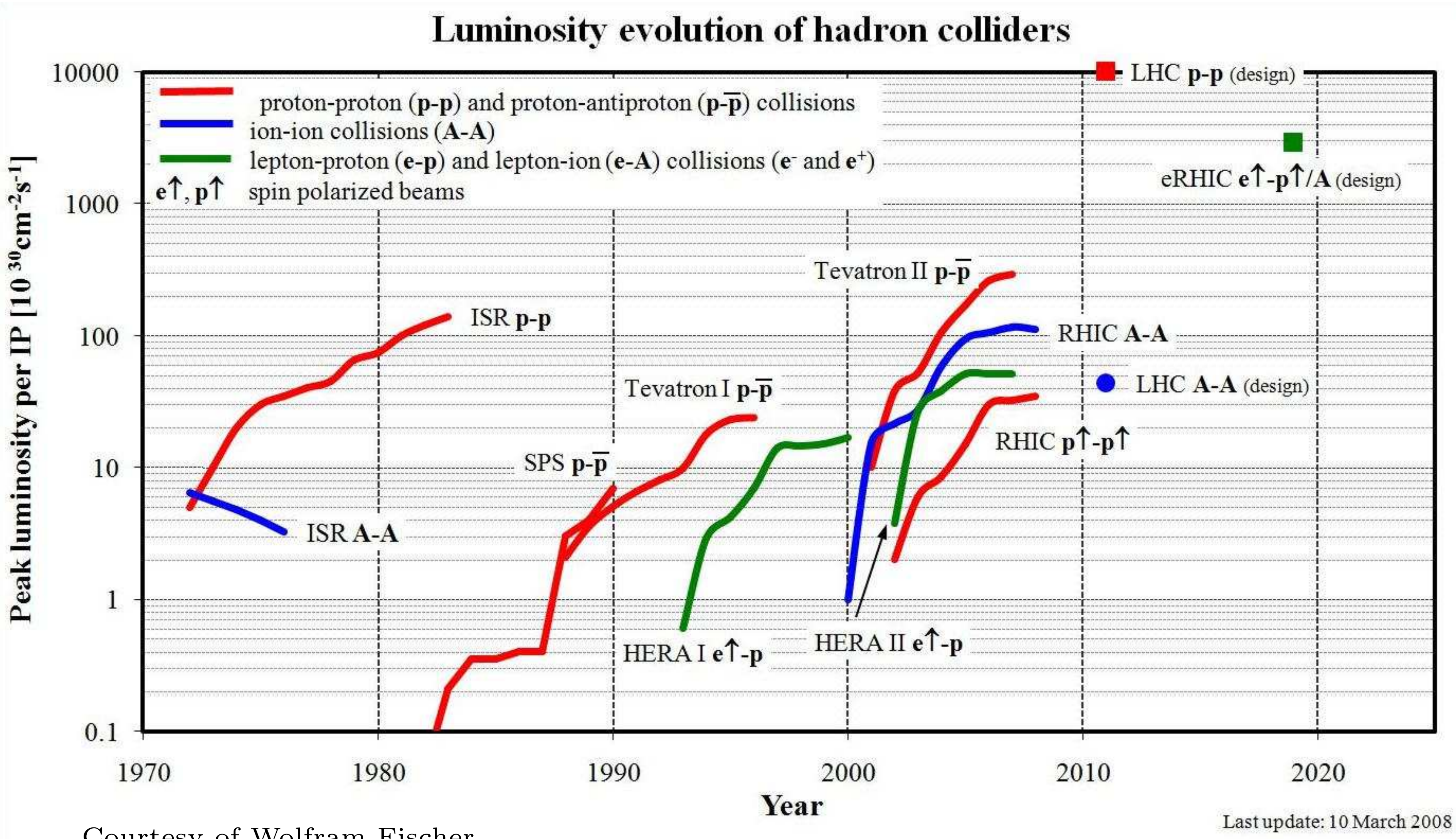
$$\mathcal{L} = \mathcal{L}_0 R(\beta^*/\sigma_z)$$

For round beams with

$$\beta^*_{1x} = \beta^*_{1y} = \beta^*_{1x} = \beta^*_{1y}$$

$$\epsilon_{1x} = \epsilon_{1y} = \epsilon_{1x} = \epsilon_{1y}$$

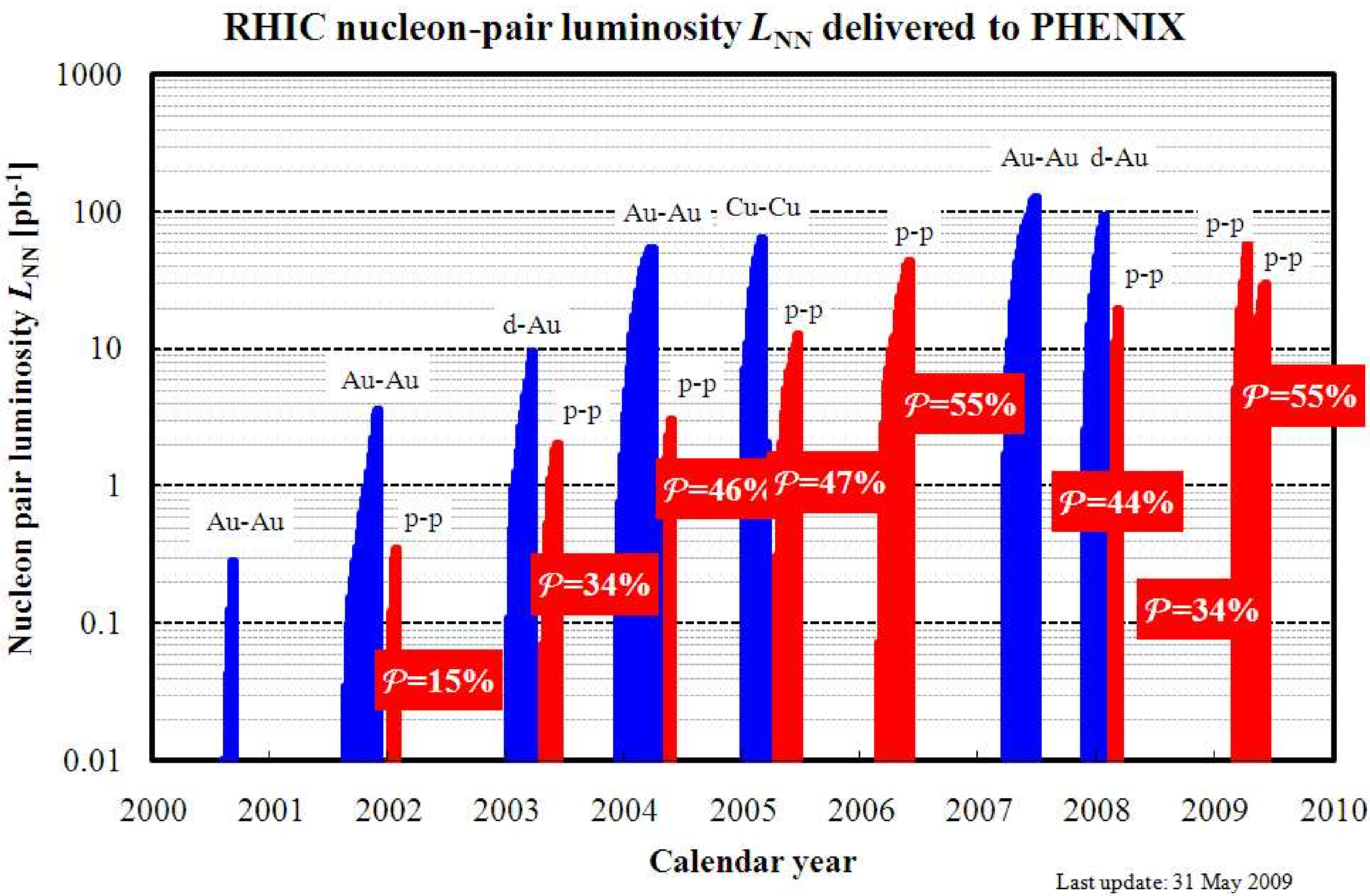
$$R(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{1 + \left(\frac{t}{\xi}\right)^2}\, dt = \sqrt{\pi}\, \xi\, e^{\xi^2} [1 - \mathrm{erf}(\xi)].$$



Courtesy of Wolfram Fischer

2009: RHIC Polarized proton collisions at 250 GeV

$\mathcal{L}_{\text{peak}} \simeq 0.85 \times 10^{32}\ \text{cm}^{-2}\text{s}^{-1}$ (Design Manual: $2 \times 10^{32}\ \text{cm}^{-2}\text{s}^{-1}$)



Courtesy of Wolfram Fischer

Suggested reading

- M. Conte and W. W. MacKay, *An Introduction to the Physics of Particle Accelerators*, 2nd Ed., World Sci. (2008).
- S. Y. Lee, *Accelerator Physics*, 2nd Ed., World Sci. (2004).
- D. A. Edwards and M. J. Syphers, *An Introduction to the Physics of High Energy Accelerators*, John Wiley (1992).
- E. J. N. Wilson, *An Introduction to Particle Accelerators*, Oxford (2001).
- Bryant and Johnsen, *The Principles of Circular Accelerators and Storage Rings*, Cambridge (1993).
- H. Weidemann, *Particle Accelerator Physics*, 2nd Ed., Springer (2007).