Chiral Violations in Perturbative Domain Wall QCD

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We calculate in lattice perturbation theory, using the exact propagators corresponding to the setting of a finite number of points in the fifth dimensions, the residual mass and other matrix elements which measure the breaking of chiral symmetry in domain-wall fermions. We present results for several choices of the domain-wall parameters.

Introduction

Domain-wall simulations: finite number of points N_s in the 5th dimension
$\Rightarrow$ violations of chiral symmetry

Only in the theoretical limit in which N_s becomes infinite the chiral modes can fully decouple from each other, yielding an exact chiral symmetry

We investigate the chirality-violating effects using perturbative calculations

The focus is at small N_s, where the simulations are currently performed

We have computed three quantities:

- the residual mass
- the difference $\Delta = Z_V - Z_A$
- a chirally-forbidden mixing (nonzero at finite N_s) of an operator which describes the lowest moment of the g_2 structure function

For all this, we had to derive the required propagator functions

We have computed the same quantities with the plaquette action as well as with improved gauge actions (Lüscher-Weisz, Iwasaki, DBW2)

In numerical simulations it has been seen that these improved gauge actions (especially DBW2) reduce the chiral violations

Domain-wall fermions

Five-dimensional domain-wall action by Shamir (1993):

$$\sum_x \sum_{s=1}^{N_s} \left[\frac{1}{2} \sum_\mu \left(\overline{\psi}_s(x)(\gamma_\mu - r) U_\mu(x) \psi_s(x+\hat{\mu}) - \overline{\psi}_s(x)(\gamma_\mu + r) U_\mu^\dagger(x-\hat{\mu}) \psi_s(x-\hat{\mu}) \right) \right.$$

$$+ \left(\overline{\psi}_s(x) P_+ \psi_{s+1}(x) + \overline{\psi}_s(x) P_- \psi_{s-1}(x) \right) + (M - 1 + 4r) \overline{\psi}_s(x) \psi_s(x)$$

$$+ m \sum_x \left(\overline{\psi}_{N_s}(x) P_+ \psi_1(x) + \overline{\psi}_1(x) P_- \psi_{N_s}(x) \right)$$

with:

- $0 < M < 2$
  $\Rightarrow$ correct structure of chiral modes (with no doublers) for $N_s \to \infty$
- $r = -1$

Chiral projectors: $P_\pm = (1 \pm \gamma_5)/2$

Mass matrix

In (four-dimensional) momentum space the action becomes

$$\delta_{s,t} \sum_{\mu} i\gamma_\mu \sin p_\mu + (W^+_{st}(p) + mM^+_{st})P_+ + (W^-_{st}(p) + mM^-_{st})P_-$$

where

$$W^+(p) = \begin{pmatrix} -W(p) & 1 & & \\ & -W(p) & \ddots & \\ & & \ddots & 1 \\ & & & -W(p) \end{pmatrix} \qquad M^+ = \begin{pmatrix} & & \\ 1 & & \\ & & \end{pmatrix}$$

$$W^-(p) = \begin{pmatrix} -W(p) & & & \\ 1 & -W(p) & & \\ & \ddots & \ddots & \\ & & 1 & -W(p) \end{pmatrix} \qquad M^- = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}$$

and one puts

$$W(p) = 1 - M + 2\sum_{\lambda} \sin^2 \frac{p_\lambda}{2}$$

Chiral modes

This is like having several flavors of lattice Dirac fermions, mixed via the mass matrix in a very special way, so that a large mass hierarchy is generated

To determine the chiral modes we need to diagonalize (in the fifth dimension) the mass matrix, which is not hermitian

We consider then its square , that is the second-order operators $DD^\dagger$ and $D^\dagger D$: they are hermitian and nonnegative and give a well-behaved spectrum

We then rotate (in the fifth dimension) the original quark fields $\psi_s(x)$ to the basis which diagonalizes the mass matrix, and obtain the chiral mode

$$\chi_0(x) = \sqrt{1-w_0^2} \sum_s (P_+ w_0^{s-1} \psi_s(x) + P_- w_0^{N_s - s} \psi_s(x))$$

where we put

$$w_0 = W(0) = 1 - M$$

We see from the damping factors w_0^{s-1} and $w_0^{N_s-s}$ that the chiral mode is exponentially localized near the boundaries of the fifth dimension

Chiral modes

However: the domain-wall height M, and thus w_0, become additively renormalized

The standard "physical" quark fields used in Monte Carlo simulations are instead constructed only from quark fields exactly located at the boundaries:

$$\begin{aligned} q(x) &= P_+\psi_1(x) + P_-\psi_{N_s}(x) \\ \overline{q}(x) &= \overline{\psi}_{N_s}(x)P_+ + \overline{\psi}_1(x)P_- \end{aligned}$$

These physical quark fields $q(x)$ are more convenient to use than $\chi_0(x)$, because of the renormalization of w_0

Moreover, at finite N_s there is anyway an additional issue: $\chi_0(x)$ is not exactly the chiral mode

$\chi_0(x)$ is in fact an eigenvector of the mass matrix only up to terms of order

$$N_s\, e^{-N_s\alpha(0)},$$

where $\alpha(0)$ is defined by

$$\cosh(\alpha(0)) = \frac{1+w_0^2}{2|w_0|}$$

Chiral violations

How small can N_s be while still avoiding large values of the residual mass?

To investigate this problem, we have computed various matrix elements for several choices of N_s and of the domain-wall height M

A thorough exploration of large regions in the two-dimensional space spanned by N_s and M would be quite expensive for Monte Carlo simulations

Perturbation theory remains then often the more practical and cheaper way for gathering hints of what is happening when the parameters move in this space

We want to study of the dependence of m_{res} (and other quantities that can act as indicators of chiral violations) on M and N_s

We have then carried out some selected one-loop calculations using the Feynman rules which exactly correspond to the theory at finite N_s

This is then not ths situation of past calculations where, in place of the exact quark propagators, their asymptotic expressions for large N_s were used

The purpose of this work is to calculate with the exact Feynman rules the deviations from the $N_s = \infty$ results in the case where N_s is limited to small values , of $O(10)$

Propagators

The construction of the tree-level quark propagator has been started by Narayanan, Neuberger, Shamir and then furthered by Aoki, Hirose, Vranas, Kikukawa, Neuberger, . . .

It has been completed and then used in the first full-fledged calculations of the renormalization of the quark self-energy and bilinears by Aoki, Izubuchi, Kuramashi, & Taniguchi; Blum, Wingate & Soni

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We now work at finite N_s, and the expressions that we have to use for the various quark propagators are then different

Some of them were derived by Aoki et al. , and in this work we have also computed the propagators that were still missing

Residual mass at tree level

The function that describes the propagation of the chiral fields is

$$\langle q(-p)\overline{q}(p)\rangle = \frac{1}{E(p)}\left(i\gamma_\mu \sin p_\mu \left(1-e^{-2N_s\alpha(p)}\right)+e^{-N_s\alpha(p)}\cdot 2W(p)\sinh(\alpha(p))\right)$$

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In the limit of small momentum this expression becomes

$$\langle q(-p)\overline{q}(p)\rangle_c = -(1-w_0^2)\,\frac{i\not{p} + w_0^{N_s}(1-w_0^2)}{p^2 + w_0^{2N_s}(1-w_0^2)^2}$$

Since $w_0 = e^{-\alpha(0)}$, it is easy to see that the terms which are proportional to $w_0^{N_s} = e^{-N_s\alpha(0)}$ rapidly approach zero when N_s becomes large

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Thus, domain-wall fermions present at finite N_s some new peculiar features

Although in the Lagrangian all quark fields are massless , the truncation at finite N_s generates already at tree level a nonvanishing residual mass of the physical fields (Vranas, . . .)

$$a\, m_{res}^{(0)} = -w_0^{N_s}(1-w_0^2) = -(1-M)^{N_s}\, M(2-M)$$

Residual mass at tree level

As expected, this tree-level residual mass vanishes when $N_s = \infty$

The sign can be inferred from the general expression of a fermion propagator of mass μ for small momentum in Euclidean space:

$$\frac{-i\not{p} + \mu}{p^2 + \mu^2} = \frac{1}{i\not{p} + \mu}$$

Since we work with even N_s (where the fermion determinant can be proven to be positive) , $m_{res}^{(0)}$ is always a negative quantity

With our calculations we have thus reproduced, up to a sign, the result for $m_{res}^{(0)}$ found by Shamir, Vranas, Kikukawa, Neuberger, Blum, Wingate, Soni, ...

That result was derived by considering the quadratic operator $D^\dagger D$, which could perhaps explain the sign discrepancy

Radiative corrections also give additional contributions to m_{res}

We will see that when the one-loop corrections are taken into account, the residual mass changes sign and becomes positive

Residual mass at tree level

Residual mass at tree level in lattice units.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-12.91556	-8.47390	-5.55973	-3.64774	-2.39328	-1.57023	-1.03023	-0.19090	0
0.2	-9.53767	-3.90663	-1.60015	-0.65542	-0.26846	-0.10996	-0.04504	-0.00127	0
0.3	-4.64274	-1.11472	-0.26764	-0.06426	-0.01543	-0.00370	-0.00089	0.00000	0
0.4	-1.69750	-0.22000	-0.02851	-0.00370	-0.00048	-0.00006	-0.00001	0.00000	0
0.5	-0.46264	-0.02891	-0.00181	-0.00011	-0.00001	0.00000	0.00000	0.00000	0
0.6	-0.08693	-0.00223	-0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00943	-0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00039	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00039	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.00943	-0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
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One-loop structure of the physical propagator

$$\langle q(-p)\overline{q}(p)\rangle_{1\ loop} = \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)} + \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)}\,\Sigma_q(p)\,\frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)}$$

$$= \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2) - (1-w_0^2)\,\Sigma_q(p)},$$

where the contribution to the g_0^2 order is

$$\Sigma_q(p) = \sum_{s,t} \frac{1}{1-w_0^{2N_s}} \Big[\Big(w_0^{N_s-s} - w_0^{2N_s} w_0^{-(N_s-s)}\Big) P_+ + \Big(w_0^{s-1} - w_0^{2N_s} w_0^{-(s-1)}\Big) P_-$$

$$-w_0 \frac{i\not{p} - w_0^{N_s}(1-w_0^2)}{1-w_0^2} \Big(\Big(w_0^{s-1} - w_0^{2(N_s-1)} w_0^{-(s-1)}\Big) P_+ + \Big(w_0^{N_s-s} - w_0^{2(N_s-1)} w_0^{-(N_s-s)}\Big) P_- \Big) \Big]$$

$$\cdot \Sigma_{st}(p)$$

$$\cdot \frac{1}{1-w_0^{2N_s}} \Big[\Big(w_0^{N_s-t} - w_0^{2N_s} w_0^{-(N_s-t)}\Big) P_- + \Big(w_0^{t-1} - w_0^{2N_s} w_0^{-(t-1)}\Big) P_+$$

$$-w_0 \Big(\Big(w_0^{t-1} - w_0^{2(N_s-1)} w_0^{-(t-1)}\Big) P_- + \Big(w_0^{N_s-t} - w_0^{2(N_s-1)} w_0^{-(N_s-t)}\Big) P_+ \Big) \frac{i\not{p} - w_0^{N_s}(1-w_0^2)}{1-w_0^2} \Big],$$

and the functions G_{st}^R and G_{st}^L enter only in $\Sigma_{st}(p)$

One-loop structure of the physical propagator

For the half-circle diagram of the self-energy we have

$$\Sigma_{st}(p) = \int \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} V_\mu^{(1)}(p+k)\,\langle\psi_s(-k)\overline{\psi}_t(k)\rangle\, V_\nu^{(1)}(p+k)\, G_{\mu\nu}(p-k)$$

and for the tadpole diagram $\Sigma_{st}(p) = \int \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} V_{\mu\nu}^{(2)}(2p)\, G_{\mu\nu}(k) \cdot \delta_{st}$

The general form of $\Sigma_q(p)$ is

$$\Sigma_q(p) = \frac{\bar{g}^2}{1-w_0^2}\left[\frac{\Sigma_0}{a} + i\not{p}\left(c_{\Sigma_1}^{(N_s,M)}\log a^2p^2 + \Sigma_1\right) - \left(i\not{p} - w_0^{N_s}(1-w_0^2)\right)\frac{2w_0}{1-w_0^2}\,\Sigma_3\right],$$

where we call for brevity $\bar{g}^2 = (g_0^2/16\pi^2)\,C_F$

This Σ_q differs in a few aspects from its expression for infinite N_s:

- the coefficient of the logarithmic term now depends on N_s and M
- the coefficient of Σ_3 is slightly different
- there is a totally new contribution , Σ_0, which is proportional to $1/a$

One-loop structure of the physical propagator

Σ_0 originates from the breaking of chiral symmetry and comes from the terms of Σ_{st} which are of order zero in p

Σ_0 acts as a mass correction term: because

$$\langle q(-p)\overline{q}(p)\rangle_{1\ loop} = \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2) - (1-w_0^2)\,\Sigma_q(p)} = \frac{1-w_0^2}{i\not{p}\,Z_2^{-1} + m_{res}^{(1)}}\,Z_w\,,$$

the one-loop radiatively induced mass is indeed given by

$$a\,m_{res}^{(1)} = -w_0^{N_s}(1-w_0^2) - \bar{g}^2\,\Sigma_0$$

Thus, at finite N_s the Σ_0 contribution generates a finite additive renormalization to the quark mass, which is a measure of chirality-breaking effects

Of course higher loops and nonperturbative effects give further contributions to the shift of the critical mass

One-loop renormalization of operators

The renormalization of a composite operator $\overline{q}(x)\, O\, q(x)$ which is multiplicatively renormalizable can also be expressed in a simple way

The one-loop matrix element of such an operator between physical quark states can be written as

$$\langle\, (\,\overline{q} O q\,)\, q\overline{q}\,\rangle_{1\ loop} = \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)} \cdot A_O(p) \cdot \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)},$$

where $A_O(p)$ contains the contribution of the damping factors

For a logarithmically divergent operator $A_O(p)$ takes the form

$$A_O(p) = \bar{g}^2 \Big(-\gamma_O^{(N_s,M)} \log a^2 p^2 + B_O \Big)$$

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The anomalous dimension $\gamma_O^{(N_s,M)}$ turns out in general to be a function of N_s and M, even at lowest order

Some pathologies

The coefficients of the logarithmic divergences turn out to be different from their continuum values, and they depend on N_s and M

It is only when $N_s \to \infty$ that the anomalous dimensions approach the ones calculated in the continuum theory

In particular, the vector and axial-vector currents have a nonzero anomalous dimension at any finite N_s

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In particular, the vector and axial-vector currents have a nonzero anomalous dimension at any finite N_s

The coefficient of the logarithm of the one-loop quark self-energy is

$$c_{\Sigma_1}^{(N_s,M)} = c_{\Sigma_1}^{\infty} \cdot \left(1 - N_s\, w_0^{2N_s}\, \frac{1-w_0^2}{1-w_0^{2N_s}}\right) \cdot \left(1 - N_s\, w_0^{2N_s}\, \frac{1-w_0^2}{1-w_0^{2N_s}} + 2\, \frac{w_0^{2(N_s+1)}}{1-w_0^{2N_s}} \left(1 - N_s\, \frac{1-w_0^2}{w_0^2\,(1-w_0^{2N_s})}\right)\right),$$

where $c_{\Sigma_1}^{\infty}$ $(= \lambda$ in covariant gauge $)$ is the value of the coefficient in the case of exact chiral symmetry

Some pathologies

Coefficient of the logarithmic term for Σ_1, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	0.11387	0.41183	0.64719	0.80302	0.89547	0.94650	0.97333	0.99859	1
0.2	0.71677	0.92503	0.98280	0.99634	0.99926	0.99985	0.99997	1.00000	1
0.3	0.94927	0.99550	0.99965	0.99997	1.00000	1.00000	1.00000	1.00000	1
0.4	0.99443	0.99986	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
0.5	0.99964	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
0.6	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
0.7	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
0.8	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
0.9	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
1.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
1.1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
1.2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
1.3	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
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1.8	0.71677	0.92503	0.98280	0.99634	0.99926	0.99985	0.99997	1.00000	1
1.9	0.11387	0.41183	0.64719	0.80302	0.89547	0.94650	0.97333	0.99859	1

Some pathologies

Coefficient of the logarithmic term of the vertex diagram of the vector current, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.84792	-1.02208	-1.05864	-1.04852	-1.03113	-1.01779	-1.00952	-1.00058	-1
0.2	-1.04275	-1.02123	-1.00592	-1.00138	-1.00030	-1.00006	-1.00001	-1.00000	-1
0.3	-1.01354	-1.00158	-1.00014	-1.00001	-1.00000	-1.00000	-1.00000	-1.00000	-1
0.4	-1.00176	-1.00005	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
0.5	-1.00012	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
0.6	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
0.7	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
0.8	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
0.9	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.0	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.1	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.2	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.3	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.4	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.5	-1.00012	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.6	-1.00176	-1.00005	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.7	-1.01354	-1.00158	-1.00014	-1.00001	-1.00000	-1.00000	-1.00000	-1.00000	-1
1.8	-1.04275	-1.02123	-1.00592	-1.00138	-1.00030	-1.00006	-1.00001	-1.00000	-1
1.9	-0.84792	-1.02208	-1.05864	-1.04852	-1.03113	-1.01779	-1.00952	-1.00058	-1

Some pathologies

In Feynman gauge one gets for the anomalous dimension of the vector current

$$\gamma_V^{(N_s,M)} = 2\,w_0^{2N_s}\cdot\left(1 - N_s\,w_0^{2N_s}\,\frac{1-w_0^2}{1-w_0^{2N_s}}\right)$$
$$\cdot\left(N_s\,\frac{1-w_0^2}{1-w_0^{2N_s}}\left(2+\frac{1}{1-w_0^{2N_s}}\right) - 2 - \frac{w_0^2}{1-w_0^{2N_s}}\right)$$

Although numerically the deviations from gauge invariance remain in most practical cases rather small, we encounter here another of the pathological features of the domain-wall theory truncated at finite N_s

Some pathologies

Anomalous dimension of the vector current, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	**0.73405**	**0.61025**	**0.41144**	**0.24550**	**0.13567**	**0.07128**	**0.03619**	**0.00199**	**0**
0.2	**0.32598**	**0.09620**	**0.02312**	**0.00504**	**0.00104**	**0.00021**	**0.00004**	**0.00000**	**0**
0.3	**0.06427**	**0.00608**	**0.00049**	**0.00004**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.4	**0.00733**	**0.00020**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.5	**0.00048**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.6	**0.00002**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.7	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.8	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.9	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.0	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.1	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.2	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.3	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.4	**0.00002**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.5	**0.00048**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.6	**0.00733**	**0.00020**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.7	**0.06427**	**0.00608**	**0.00049**	**0.00004**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.8	**0.32598**	**0.09620**	**0.02312**	**0.00504**	**0.00104**	**0.00021**	**0.00004**	**0.00000**	**0**
1.9	**0.73405**	**0.61025**	**0.41144**	**0.24550**	**0.13567**	**0.07128**	**0.03619**	**0.00199**	**0**

Some pathologies

Thus, for domain-wall fermions, for any finite N_s already at the lowest order the anomalous dimensions of the continuum and lattice versions of an operator are not the same

This raises some issues about the correct procedure with which in this case one has to carry out the matching of lattice operators to a continuum scheme

In fact, the matching formula

$$\frac{\langle q|O^{\overline{\rm MS}}|q\rangle}{\langle q|O^{lat}|q\rangle} = 1 - \bar{g}^2\Big(- \gamma_O \log a^2\mu^2 + R^{lat} - R^{\overline{\rm MS}}\Big)$$

is only valid provided the coefficients of the logarithmic terms in

$$\langle q|O^{lat}|q\rangle = \Big(1 + \bar{g}^2\Big(- \gamma_O \log a^2 p^2 + R^{lat}\Big)\Big) \cdot \langle q|O^{tree}|q\rangle$$

and

$$\langle q|O^{\overline{\rm MS}}|q\rangle = \Big(1 + \bar{g}^2_{\overline{\rm MS}}\Big(- \gamma_O \log \frac{p^2}{\mu^2} + R^{\overline{\rm MS}}\Big)\Big) \cdot \langle q|O^{tree}|q\rangle$$

are the same

Some pathologies

But this does not happen for domain-wall fermions at finite N_s: in this case the γ_O in the equation of $\langle q|O^{lat}|q\rangle$ must be replaced with a $\gamma_O^{N_s,M}$ which depends on N_s:

$$\langle q|O^{lat}|q\rangle = \Big(1 + \bar{g}^2\Big(-\gamma_O^{N_s,M}\log a^2p^2 + R^{lat}\Big)\Big)\cdot\langle q|O^{tree}|q\rangle$$

$$\langle q|O^{\overline{\rm MS}}|q\rangle = \Big(1 + \bar{g}^2_{\overline{\rm MS}}\Big(-\gamma_O\log\frac{p^2}{\mu^2} + R^{\overline{\rm MS}}\Big)\Big)\cdot\langle q|O^{tree}|q\rangle$$

Thus, this logarithmic term <u>cannot be combined</u> with the one of $\langle q|O^{lat}|q\rangle$, whose coefficient is fixed by the continuum theory

In particular, the p dependence cannot be eliminated from the matching formula

(. . . some food for thought . . .)

Some pathologies

Furthermore, the quantities that we have studied turn out to lose gauge invariance when N_s is not infinite

We can express Σ_0, the residual mass and the other quantities in the form $A + (1 - \lambda)\, B$, where A and $A + B$ provide the answers in Feynman and Landau gauge

B remains the same when using fermion formulations rather diverse like domain-wall with an infinite extent of the fifth dimension, Wilson or overlap

B remains the same also when improved gluon actions are used

Instead, for domain-wall with finite N_s we have found that in general the part proportional to $1 - \lambda$ presents a nontrivial dependence on N_s and M

Thus the results for the residual mass and in general for the renormalization factors and mixing coefficients turn out not to be gauge invariant

It could be that this constitutes a limitation of 1-loop (and higher-loops) perturbation theory, although it is also possible that a small gauge dependence is present in numerical simulations as well

Tadpoles

The behavior of the tadpole diagrams as N_s and M change is particularly interesting

Since no pure 5-dimensional quark propagators appear in the tadpoles, for these diagrams the $\Sigma_{st}(p)$ is diagonal in the fifth dimension and proportional to $(i\not{k} - 4r/a)\, G_{\mu\nu}(k)$

This is the same integrand of the tadpoles for Wilson fermions, and for that action in the case of the tadpole diagram contributing to Σ_1 it gave the result (in a general covariant gauge)

$$T_l = 8\pi^2 Z_0 \left(1 - 1/4\,(1-\lambda)\right),$$

where $Z_0 = 0.154933390231\ldots$ is a well-known integral

It is then clear that for domain-wall fermions the behavior of the tadpole diagrams as a function of N_s (and M) is completely determined by the damping factors in the fifth dimension

Their general effect can be already seen by looking at their leading contributions for large N_s

Tadpoles

For large N_s the damping factors enter the game in the combinations

$$\sum_{s=1}^{N_s} w_0^{s-1} w_0^{N_s-s} = N_s w_0^{N_s-1}$$

$$\sum_{s=1}^{N_s} (w_0^2)^{s-1} = \sum_{s=1}^{N_s} (w_0^2)^{N_s-s} = \frac{1-w_0^{2N_s}}{1-w_0^2}$$

These are indeed the leading expressions, in units of

$$T_d = \frac{(1-w_0^2)\, T_l}{(1-w_0^{2N_s})^2},$$

for the tadpole contributions to Σ_0 and Σ_1 respectively, in the limit of large N_s

Already from these asymptotic expressions (before going to the exact results) one can immediately see that the tadpole of Σ_0 vanishes when $N_s = \infty$, while the tadpole of Σ_1 gives in this limit the well-known Wilson number, T_l

Tadpoles

The damping factors thus play a primary rôle in determining the results of the domain-wall tadpoles

We have calculated their exact expressions which included all subleading terms in N_s, and the tadpole contribution to Σ_0 turns out to be equal to

$$4\,T_d\left[N_s\,(1+w_0^{2(N_s+1)})\,w_0^{N_s-1}-2\,w_0^{N_s+1}\,\frac{1-w_0^{2N_s}}{1-w_0^2}\right],$$

while the tadpole contribution to Σ_1 turns out to be equal to

$$T_d\left[(1+w_0^{2(N_s+1)})\,\frac{1-w_0^{2N_s}}{1-w_0^2}-2\,N_s\,w_0^{2N_s}\right]$$

Tadpoles

Tadpole contribution to Σ_0, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	15.10361	16.98204	16.85601	15.31833	13.06260	10.63233	8.36032	2.59484	0
0.2	18.33890	12.98046	7.73201	4.17637	2.12604	1.04115	0.49623	0.02182	0
0.3	12.61592	4.97585	1.66803	0.51427	0.15080	0.04277	0.01184	0.00006	0
0.4	6.03146	1.23559	0.21903	0.03602	0.00566	0.00086	0.00013	0.00000	0
0.5	2.10263	0.20309	0.01717	0.00135	0.00010	0.00001	0.00000	0.00000	0
0.6	0.51309	0.02003	0.00069	0.00002	0.00000	0.00000	0.00000	0.00000	0
0.7	0.07598	0.00093	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.00476	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00004	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	0
1.2	-0.00476	-0.00001	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	0
1.3	-0.07598	-0.00093	-0.00001	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	0
1.4	-0.51309	-0.02003	-0.00069	-0.00002	-0.00000	-0.00000	-0.00000	-0.00000	0
1.5	-2.10263	-0.20309	-0.01717	-0.00135	-0.00010	-0.00001	-0.00000	-0.00000	0
1.6	-6.03146	-1.23559	-0.21903	-0.03602	-0.00566	-0.00086	-0.00013	-0.00000	0
1.7	-12.61592	-4.97585	-1.66803	-0.51427	-0.15080	-0.04277	-0.01184	-0.00006	0
1.8	-18.33890	-12.98046	-7.73201	-4.17637	-2.12604	-1.04115	-0.49623	-0.02182	0
1.9	-15.10361	-16.98204	-16.85601	-15.31833	-13.06260	-10.63233	-8.36032	-2.59484	0

Tadpoles

Tadpole contribution to Σ_1, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	6.88684	8.89791	10.28165	11.14950	11.65586	11.93540	12.08339	12.22491	12.23305
0.2	10.71422	11.82437	12.13713	12.21230	12.22878	12.23220	12.23289	12.23305	12.23305
0.3	11.95987	12.20785	12.23105	12.23290	12.23304	12.23305	12.23305	12.23305	12.23305
0.4	12.20239	12.23224	12.23303	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
0.5	12.23104	12.23304	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
0.6	12.23299	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
0.7	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
0.8	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
0.9	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.0	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.1	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.2	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.3	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.4	12.23299	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.5	12.23104	12.23304	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.6	12.20239	12.23224	12.23303	12.23305	12.23305	12.23305	12.23305	12.23305	12.23305
1.7	11.95987	12.20785	12.23105	12.23290	12.23304	12.23305	12.23305	12.23305	12.23305
1.8	10.71422	11.82437	12.13713	12.21230	12.22878	12.23220	12.23289	12.23305	12.23305
1.9	6.88684	8.89791	10.28165	11.14950	11.65586	11.93540	12.08339	12.22491	12.23305

Tadpoles

The tadpole contributions present wide variations with N_s and M, so that sometimes they turn out to be small while sometimes they are large

This suggests that some care should be used when talking about tadpole dominance in relation to domain-wall fermions

Both tadpoles (of Σ_0 and Σ_1) even decrease toward zero for $M \to 0$ or $M \to 2$

Tadpoles

The tadpole contributions present wide variations with N_s and M, so that sometimes they turn out to be small while sometimes they are large

This suggests that some care should be used when talking about tadpole dominance in relation to domain-wall fermions

Both tadpoles (of Σ_0 and Σ_1) even decrease toward zero for $M \to 0$ or $M \to 2$

A central point also is that there are two kinds of tadpoles in the game here:

- the tadpole of order zero in p, which tends to zero for $N_s \to \infty$, which contributes to Σ_0 and the residual mass
- the tadpole of order ap, which tends to its Wilson value for $N_s \to \infty$, which contributes to Σ_1 and the renormalization factors

They really behave differently , in the large N_s case as well as in the exact case for any finite N_s

Essentially, the $i\not{p}$ in the first order flips the chirality of the damping factors, and they then combine in a different way

Tadpoles

This is unlike the Wilson case, where the tadpole of Σ_0 is just proportional to the tadpole of Σ_1:

$$T_{(\Sigma_0)} = -4\, T_{(\Sigma_1)}$$

Tadpole improvement seems then not to be appropriate for the residual mass: the tadpole which contributes to m_{res} goes to zero for large N_s or for $M \to 1$, while for small N_s it assumes a certain spectrum of values

Our interest is small N_s, and it is unclear what tadpole improvement could mean in this case

Moreover, there is no tadpole at all in the calculations of Δ and c_{mix}

The tadpole of Σ_1 for large N_s is instead rather close to its Wilson value, and that is why people have used tadpole improvement in the calculations of the renormalization factors (when $N_s = \infty$)

Automation of the calculations

At one loop two diagrams contribute to Σ_0 (at order zero in p) and enter in the calculation of the residual mass: the half-circle (or sunset) and the tadpole diagrams

We have automated the calculations of the half-circle (and vertex) diagrams by developing suitable **FORM codes**, integrating afterwards the corresponding expressions by means of Fortran codes

With these codes one is able to compute matrix elements for general values of N_s and M

The numbers that we have obtained are valid both in the quenched and unquenched cases, because at one loop internal quark loops can never appear in the diagrams

In addition to running the standard numerical integration in 6 dimensions , we have also redone the computation of the half-circle diagram by hand , including the calculation of the gamma algebra and the explicit exact evaluation of the sums over the fifth-dimensional indices

This provides a rather strong check of our calculations, and also saves 2 dimensions in the numerical integration

Residual mass

The deviations from the case of exact chiral symmetry are much more pronounced when M is close to 0.1 or 1.9

The values of $am_{res}^{(1)}$ are positive only for $M \geq 1.2$ (at least for even N_s and if the coupling is not very small)

Looking at the values for $N_s = 8$, our results would indicate that the minimal amount of chiral violations is attained for $M \sim 1.2$ (in any gauge)

The optimal choice of M from the point of view of 1-loop calculations would then be around this value

This is the renormalization of M, which is not protected by chiral symmetry, and is then moved by radiative corrections away from its free value $M = 1$

One can conjecture that higher-loop corrections and nonperturbative effects would shift this optimal value further on, until the minimal point is near $M = 1.8$ (which provides the smallest residual mass in Monte Carlo simulations)

It could also be that the results of our one-loop calculations for $M \sim 1.8$ are not substantially changed even once higher-loop corrections are included, while for $M < 1.8$ they would be increased by renormalization, so that the minimum finally ends up located near $M = 1.8$

Residual mass

Residual mass in lattice units at $\beta = 6$ in Landau gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.22770	-0.21582	-0.19469	-0.16708	-0.13735	-0.10905	-0.08425	-0.02520	0
0.2	-0.23978	-0.15064	-0.08471	-0.04430	-0.02209	-0.01067	-0.00503	-0.00022	0
0.3	-0.15560	-0.05674	-0.01833	-0.00553	-0.00160	-0.00045	-0.00012	0.00000	0
0.4	-0.07331	-0.01430	-0.00248	-0.00040	-0.00006	-0.00001	0.00000	0.00000	0
0.5	-0.02610	-0.00247	-0.00021	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00690	-0.00028	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00132	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00048	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00328	0.00014	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01317	0.00137	0.00012	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.03632	0.00830	0.00154	0.00026	0.00004	0.00001	0.00000	0.00000	0
1.7	0.07054	0.03276	0.01172	0.00374	0.00112	0.00032	0.00009	0.00000	0
1.8	0.08620	0.08078	0.05326	0.03026	0.01587	0.00793	0.00383	0.00017	0
1.9	0.03905	0.08531	0.10461	0.10507	0.09478	0.07991	0.06434	0.02093	0

Residual mass

Residual mass in lattice units at $\beta = 5.2$ in Landau gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.25014	-0.24077	-0.21923	-0.18923	-0.15614	-0.12430	-0.09621	-0.02889	0
0.2	-0.26738	-0.17001	-0.09619	-0.05047	-0.02523	-0.01220	-0.00576	-0.00025	0
0.3	-0.17502	-0.06438	-0.02089	-0.00632	-0.00183	-0.00051	-0.00014	0.00000	0
0.4	-0.08294	-0.01629	-0.00283	-0.00046	-0.00007	-0.00001	0.00000	0.00000	0
0.5	-0.02966	-0.00283	-0.00024	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00787	-0.00032	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00151	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00056	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00387	0.00016	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01565	0.00161	0.00014	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.04357	0.00979	0.00181	0.00030	0.00005	0.00001	0.00000	0.00000	0
1.7	0.08592	0.03889	0.01378	0.00438	0.00131	0.00038	0.00011	0.00000	0
1.8	0.10875	0.09702	0.06302	0.03555	0.01857	0.00925	0.00446	0.00020	0
1.9	0.05763	0.10669	0.12612	0.12479	0.11169	0.09373	0.07524	0.02433	0

Residual mass

Although not completely consistent in the framework of the one-loop theory, it is interesting to look at the perturbative numbers for $M = 1.8$

Focusing on this value of M, we observe that at one loop the critical mass seems to be still large up to $N_s = 16$ or $N_s = 20$, becoming then somewhat smaller for $N_s = 24$ and higher

If we take $M = 1.8$ and $N_s = 16$, and quenched $\beta = 6.0$ (2 GeV), our one-loop calculations give $am_{res} = 0.05207 \sim 104\ MeV$ in Feynman gauge and $am_{res} = 0.05326 \sim 107\ MeV$ in Landau gauge

Numerically, the dependence on the gauge seems to be very small

Notice also that if $m_{res}^{(0)}$ were positive, the values of $m_{res}^{(1)}$ would be larger

In the unquenched case, the same lattice spacing of 2 GeV corresponds to a different bare coupling ($\beta = 5.2$), and we obtain $am_{res} = 0.06164 \sim 123\ MeV$ in Feynman gauge and $am_{res} = 0.06302 \sim 126\ MeV$ in Landau gauge

Results from dynamical domain-wall simulations show that the residual mass in full QCD assumes much higher values than in the quenched case

Residual mass

One-loop perturbative results point in the same direction, although here the difference between quenched and full QCD is not as pronounced , suggesting that higher-order corrections and especially nonperturbative effects presumably play a significant rôle

It is also interesting to compare the one-loop domain-wall results for Σ_0 to the numbers that one obtains using Wilson fermions, which are -51.43471 for the unimproved and -31.98644 for the improved (with $c_{sw} = 1$) case

We can then see that the domain-wall results are much smaller than the Wilson numbers , even for N_s as small as 8

The tree level and one-loop contributions are of a different sign, and if the two-loop contribution enters again with a negative sign it would damp the chiral violations that we have obtained

But if higher-order corrections do not at the end strongly compensate the results of the residual mass at one loop, all this would suggest that values like $N_s = 24$ could be better choices for the simulations

Notice also that for $M = 1.9$ the residual mass at $N_s = 12$ is larger than at $N_s = 8$, and at $N_s = 16$ is even larger

Improved gauge actions

In the plaquette case the residual mass, negative at tree level, changes sign and becomes positive when the one-loop corrections are taken into account

The same thing happens with the Lüscher-Weisz and Iwasaki actions, and (except for very small N_s, or M very close to 2) with the DBW2 action

Exactly as in the case of the plaquette action, also for improved gauge actions it happens that the residual mass $am_{res}^{(1)}$ is positive only for $M \geq 1.2$

This shows that improved gauge actions do not behave too differently also with respect to the additive renormalization undergone by w_0

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This shows that improved gauge actions do not behave too differently also with respect to the additive renormalization undergone by w_0

The use of improved gauge actions produces a suppression of m_{res}, when comparisons are made at the same coupling

The Iwasaki action gives a stronger suppression than the Lüscher-Weisz action

The DBW2 action is the most effective in generating large suppressions

There is a monotonic decrease of the residual mass as c_1 grows, but for the DBW2 action this suppression can go too far and sometimes we obtain (small) negative values for the residual mass

Improved gauge actions

If comparisons between the various actions are made at the same energy scale , the picture that comes out is somewhat different

For quenched QCD at $2\ GeV$, one has to take $\beta = 5.7$ for the Lüscher-Weisz action, $\beta = 2.6$ for the Iwasaki action, and $\beta = 1.04$ for the DBW2 action

We keep $\beta = 6/g_0^2$ also for improved actions, instead of $\beta' = 6(1 - 8c_1)/g_0^2$

The numbers for the Iwasaki action are rather close to those for the DBW2 action, and surprisingly they lie in general slightly above the plaquette values

However, for the quenched DBW2 action at 2 GeV we have $g_0^2 = 5.77$

$\Rightarrow$ this is not a reasonable perturbative expansion parameter

Moreover, these values of the coupling are fixed from numerical simulations, and they then contain informations of a nonperturbative nature , so that a mismatch can arise when one only takes into account the results of the one-loop diagrams calculated at these values of the coupling

Notice that the Lüscher-Weisz action gives in the above case the best results in terms of suppression, and indeed g_0^2 is still close to 1 even at 2 GeV

Improved gauge actions

Residual mass at $\beta = 5.2$, in Landau gauge for the Lüscher-Weisz action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.21605	-0.20260	-0.18145	-0.15496	-0.12696	-0.10057	-0.07757	-0.04763	0
0.2	-0.22551	-0.14044	-0.07859	-0.04098	-0.02040	-0.00984	-0.00464	-0.00149	0
0.3	-0.14568	-0.05280	-0.01700	-0.00512	-0.00148	-0.00042	-0.00011	-0.00002	0
0.4	-0.06850	-0.01331	-0.00230	-0.00037	-0.00006	-0.00001	0.00000	0.00000	0
0.5	-0.02438	-0.00231	-0.00019	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00646	-0.00026	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00124	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00021	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00042	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00285	0.00012	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01138	0.00120	0.00011	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.03114	0.00724	0.00135	0.00023	0.00004	0.00001	0.00000	0.00000	0
1.7	0.05963	0.02849	0.01029	0.00330	0.00099	0.00029	0.00008	0.00002	0
1.8	0.07020	0.06959	0.04664	0.02670	0.01406	0.00704	0.00341	0.00123	0
1.9	0.02569	0.07052	0.09008	0.09195	0.08365	0.07088	0.05726	0.03713	0

Improved gauge actions

Residual mass at $\beta = 5.2$, in Landau gauge for the Iwasaki action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.17110	-0.15236	-0.13179	-0.10995	-0.08865	-0.06942	-0.05309	-0.03202	0
0.2	-0.17025	-0.10145	-0.05542	-0.02847	-0.01404	-0.00672	-0.00315	-0.00098	0
0.3	-0.10687	-0.03748	-0.01187	-0.00354	-0.00102	-0.00028	-0.00008	-0.00001	0
0.4	-0.04933	-0.00936	-0.00160	-0.00026	-0.00004	-0.00001	0.00000	0.00000	0
0.5	-0.01731	-0.00161	-0.00013	-0.00001	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00453	-0.00018	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00086	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00014	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00023	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00153	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.00586	0.00067	0.00006	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.01508	0.00394	0.00077	0.00013	0.00002	0.00000	0.00000	0.00000	0
1.7	0.02561	0.01502	0.00577	0.00190	0.00058	0.00017	0.00005	0.00001	0
1.8	0.02027	0.03401	0.02538	0.01519	0.00820	0.00417	0.00204	0.00073	0
1.9	-0.01566	0.02356	0.04321	0.04923	0.04715	0.04113	0.03384	0.02181	0

Improved gauge actions

Residual mass at $\beta = 5.2$, in Landau gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.12472	-0.10075	-0.08091	-0.06392	-0.04952	-0.03764	-0.02814	-0.01655	0
0.2	-0.11324	-0.06137	-0.03163	-0.01566	-0.00752	-0.00354	-0.00163	-0.00049	0
0.3	-0.06676	-0.02167	-0.00657	-0.00191	-0.00054	-0.00015	-0.00004	-0.00001	0
0.4	-0.02942	-0.00525	-0.00087	-0.00014	-0.00002	0.00000	0.00000	0.00000	0
0.5	-0.00991	-0.00088	-0.00007	-0.00001	0.00000	0.00000	0.00000	0.00000	0
0. 6	-0.00248	-0.00010	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00045	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00023	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.00040	0.00014	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.6	-0.00087	0.00066	0.00019	0.00004	0.00001	0.00000	0.00000	0.00000	0
1.7	-0.00818	0.00159	0.00125	0.00051	0.00017	0.00005	0.00002	0.00000	0
1.8	-0.02927	-0.00149	0.00411	0.00367	0.00232	0.00128	0.00066	0.00024	0
1.9	-0.05656	-0.02323	-0.00366	0.00640	0.01050	0.01123	0.01029	0.00673	0

Improved gauge actions

Residual mass at $\beta = 5.7$, in Landau gauge for the Lüscher-Weisz action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.20427	-0.18953	-0.16862	-0.14340	-0.11716	-0.09262	-0.07133	-0.04383	0
0.2	-0.21102	-0.13029	-0.07259	-0.03775	-0.01876	-0.00904	-0.00425	-0.00137	0
0.3	-0.13548	-0.04879	-0.01566	-0.00471	-0.00136	-0.00038	-0.00010	-0.00002	0
0.4	-0.06344	-0.01227	-0.00212	-0.00034	-0.00005	-0.00001	0.00000	0.00000	0
0.5	-0.02250	-0.00212	-0.00018	-0.00001	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00594	-0.00024	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00114	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00019	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00038	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00255	0.00011	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01012	0.00108	0.00010	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.02746	0.00649	0.00122	0.00021	0.00003	0.00001	0.00000	0.00000	0
1.7	0.05182	0.02537	0.00924	0.00297	0.00090	0.00026	0.00007	0.00002	0
1.8	0.05874	0.06132	0.04166	0.02399	0.01268	0.00636	0.00309	0.00111	0
1.9	0.01626	0.05963	0.07909	0.08186	0.07499	0.06379	0.05167	0.03349	0

Improved gauge actions

Residual mass at $\beta = 2.6$, in Landau gauge for the Iwasaki action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.26041	-0.25106	-0.22837	-0.19680	-0.16214	-0.12890	-0.09967	-0.05976	0
0.2	-0.28010	-0.17816	-0.10070	-0.05280	-0.02638	-0.01275	-0.00602	-0.00185	0
0.3	-0.18434	-0.06790	-0.02204	-0.00667	-0.00193	-0.00054	-0.00015	-0.00002	0
0.4	-0.08791	-0.01732	-0.00302	-0.00049	-0.00008	-0.00001	0.00000	0.00000	0
0.5	-0.03170	-0.00304	-0.00026	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00852	-0.00035	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00167	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00029	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00053	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00362	0.00015	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01465	0.00152	0.00013	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.04090	0.00928	0.00172	0.00029	0.00005	0.00001	0.00000	0.00000	0
1.7	0.08062	0.03709	0.01323	0.00421	0.00126	0.00036	0.00010	0.00002	0
1.8	0.10093	0.09276	0.06089	0.03453	0.01809	0.00903	0.00436	0.00157	0
1.9	0.05046	0.10077	0.12163	0.12156	0.10945	0.09220	0.07421	0.04790	0

Improved gauge actions

Residual mass at $\beta = 1.04$, in Landau gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.29645	-0.28910	-0.26373	-0.22721	-0.18696	-0.14842	-0.11460	-0.06565	0
0.2	-0.32459	-0.20788	-0.11763	-0.06168	-0.03081	-0.01489	-0.00703	-0.00198	0
0.3	-0.21621	-0.08011	-0.02607	-0.00790	-0.00229	-0.00065	-0.00018	-0.00003	0
0.4	-0.10412	-0.02066	-0.00361	-0.00059	-0.00009	-0.00001	0.00000	0.00000	0
0.5	-0.03784	-0.00366	-0.00031	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.01022	-0.00043	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00200	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00033	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00009	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00049	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00336	0.00014	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01371	0.00143	0.00013	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.03866	0.00889	0.00165	0.00028	0.00005	0.00001	0.00000	0.00000	0
1.7	0.07672	0.03617	0.01301	0.00416	0.00125	0.00036	0.00010	0.00002	0
1.8	0.09526	0.09152	0.06108	0.03493	0.01839	0.00921	0.00446	0.00166	0
1.9	0.04437	0.09848	0.12253	0.12440	0.11311	0.09591	0.07755	0.05076	0

Bilinear differences

Finite differences of chirally-related bilinear operators

The vector and axial-vector currents renormalize differently when chiral symmetry is broken

An estimate of chirality-breaking effects can then be given by how much for a given finite N_s the perturbative results for these currents differ from each other

The quantity $\Delta = Z_V - Z_A = -(Z_S - Z_P)/2$ provides then such an estimate

Δ tends to zero in the limit of infinite N_s (when chiral symmetry is restored)

We can successfully obtain all results for Δ by computing only finite lattice diagrams, because:

- the anomalous dimensions of the operators which are chirally related are the same also when one considers the subleading orders
- the continuum values of the finite parts are also equal for these pairs
- the mismatch between lattice (at finite N_s) and continuum anomalous dimensions is also the same for the chirally-related operators, and thus it cancels in their differences

Bilinear differences

The amount of chirality breaking connected to Δ follows a pattern similar to the one of the residual mass:

- Δ is rather large for small N_s and large $|1-M|$
- Δ decreases when N_s grows or when $|1-M|$ tends towards zero

The part proportional to $(1-\lambda)$ of Δ shows a violation of gauge invariance (otherwise it would vanish, like for Wilson or overlap fermions)

If we look at the results for $M=1.8$ and $N_s=16$ for $a^{-1}\sim 2\,GeV$, our numbers imply in QCD a chiral violation of about $2\,MeV$, with the quenched case giving the slightly lower value

These numbers are much smaller , at a given M and N_s, than the ones for the residual mass

For quantities such as four-quark operators it has been suggested (Aoki et al., Christ) that their chiral violations are of $O(m_{res}^2)$

Given the smallness of the numbers that we have obtained for Δ, it is possible that something similar is occurring here

Bilinear differences

It is also interesting to compare the domain-wall numbers with the results that one obtains for this quantity when Wilson fermions are used

In this case one gets $\Delta = 4.82152$ for the unimproved case and $\Delta = 1.53633$ for the improved case, which correspond to chiral violations which are at least of order $30\ MeV$ for $\beta \sim 6.0$

The use of improved gauge actions produces small changes in the results for Δ compared to the plaquette action, and thus does not further suppress this quantity (if comparisons are made at the same g_0)

Thus, somewhat surprisingly, in the case of Δ improved gauge actions do not seem to improve things compared with the plaquette action

If for this quantity a reduction due to the use of these actions will come out from numerical simulations, it should be probably be traced to nonperturbative effects

Bilinear differences

It is also interesting to compare the domain-wall numbers with the results that one obtains for this quantity when Wilson fermions are used

In this case one gets $\Delta = 4.82152$ for the unimproved case and $\Delta = 1.53633$ for the improved case, which correspond to chiral violations which are at least of order $30\ MeV$ for $\beta \sim 6.0$

The use of improved gauge actions produces small changes in the results for Δ compared to the plaquette action, and thus does not further suppress this quantity (if comparisons are made at the same g_0)

Thus, somewhat surprisingly, in the case of Δ improved gauge actions do not seem to improve things compared with the plaquette action

If for this quantity a reduction due to the use of these actions will come out from numerical simulations, it should be probably be traced to nonperturbative effects

If one compares the numbers at the same energy scale, then Δ is even enhanced by improved gauge actions

Bilinear differences

Coefficient of $\bar{g}^2$ for $\Delta = Z_V - Z_A = -(Z_S - Z_P)/2$, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	6.9399	4.4434	2.6941	1.5749	0.8916	0.4899	0.2622	0.0179	0
0.2	2.3272	0.7211	0.1945	0.0477	0.0110	0.0025	0.0005	0.0000	0
0.3	0.5894	0.0668	0.0064	0.0006	0.0000	0.0000	0.0000	0.0000	0
0.4	0.0994	0.0034	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.5	0.0106	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.6	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0083	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.0701	0.0029	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.7	0.3140	0.0501	0.0056	0.0005	0.0000	0.0000	0.0000	0.0000	0
1.8	0.8354	0.3672	0.1288	0.0374	0.0096	0.0022	0.0005	0.0000	0
1.9	2.2075	1.5323	1.0348	0.6904	0.4490	0.2811	0.1686	0.0153	0

Bilinear differences

Coefficient of $\bar{g}^2$ for the part proportional to $1-\lambda$ of Δ.

M	$N_s=8$	$N_s=12$	$N_s=16$	$N_s=20$	$N_s=24$	$N_s=28$	$N_s=32$	$N_s=48$	$N_s=\infty$
0.1	-1.1115	-0.8058	-0.5284	-0.3257	-0.1914	-0.1081	-0.0590	-0.0042	0
0.2	-0.4075	-0.1438	-0.0416	-0.0106	-0.0025	-0.0006	-0.0001	0.0000	0
0.3	-0.1092	-0.0140	-0.0014	-0.0001	0.0000	0.0000	0.0000	0.0000	0
0.4	-0.0191	-0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.5	-0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.6	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	-0.0022	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	-0.0186	-0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.7	-0.0879	-0.0134	-0.0015	-0.0001	0.0000	0.0000	0.0000	0.0000	0
1.8	-0.2750	-0.1064	-0.0353	-0.0100	-0.0025	-0.0006	-0.0001	0.0000	0
1.9	-0.9106	-0.5678	-0.3473	-0.2140	-0.1314	-0.0790	-0.0461	-0.0040	0

Bilinear differences

Coefficient of $\bar{g}^2$ for Δ, in Feynman gauge for the Lüscher-Weisz action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	**6.6748**	**4.3283**	**2.6444**	**1.5535**	**0.8823**	**0.4859**	**0.2605**	**0.0178**	**0**
0.2	**2.2664**	**0.7107**	**0.1928**	**0.0474**	**0.0110**	**0.0024**	**0.0005**	**0.0000**	**0**
0.3	**0.5778**	**0.0662**	**0.0064**	**0.0006**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
0.4	**0.0976**	**0.0034**	**0.0001**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
0.5	**0.0104**	**0.0001**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
0.6	**0.0006**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
0.7	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
0.8	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
0.9	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.0	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.1	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.2	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.3	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.4	**0.0005**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.5	**0.0083**	**0.0001**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.6	**0.0698**	**0.0029**	**0.0001**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.7	**0.3112**	**0.0500**	**0.0056**	**0.0005**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0**
1.8	**0.8168**	**0.3640**	**0.1283**	**0.0373**	**0.0095**	**0.0022**	**0.0005**	**0.0000**	**0**
1.9	**2.1037**	**1.4869**	**1.0150**	**0.6819**	**0.4453**	**0.2795**	**0.1679**	**0.0152**	**0**

Bilinear differences

Coefficient of $\bar{g}^2$ for Δ, in Feynman gauge for the Iwasaki action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	6.2243	4.1314	2.5590	1.5166	0.8664	0.4791	0.2575	0.0177	0
0.2	2.1620	0.6927	0.1897	0.0469	0.0109	0.0024	0.0005	0.0000	0
0.3	0.5578	0.0650	0.0063	0.0006	0.0000	0.0000	0.0000	0.0000	0
0.4	0.0945	0.0033	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.5	0.0100	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.6	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0082	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.0694	0.0029	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.7	0.3073	0.0497	0.0056	0.0005	0.0000	0.0000	0.0000	0.0000	0
1.8	0.7891	0.3592	0.1275	0.0372	0.0095	0.0022	0.0005	0.0000	0
1.9	1.9492	1.4179	0.9846	0.6686	0.4396	0.2770	0.1668	0.0152	0

Bilinear differences

Coefficient of $\bar{g}^2$ for Δ, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	5.4947	3.8063	2.4168	1.4548	0.8396	0.4675	0.2525	0.0175	0
0.2	1.9894	0.6623	0.1845	0.0460	0.0108	0.0024	0.0005	0.0000	0
0.3	0.5250	0.0629	0.0062	0.0005	0.0000	0.0000	0.0000	0.0000	0
0.4	0.0896	0.0032	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.5	0.0094	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.6	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0082	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.0688	0.0029	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.7	0.3021	0.0494	0.0055	0.0005	0.0000	0.0000	0.0000	0.0000	0
1.8	0.7516	0.3521	0.1262	0.0369	0.0095	0.0022	0.0005	0.0000	0
1.9	1.7396	1.3168	0.9384	0.6480	0.4305	0.2730	0.1651	0.0151	0

A power-divergent mixing

Many of the mixings which one encounters in the study of physical processes arise as a consequence of the breaking of Lorentz symmetry (or, in other cases, of other symmetries apart from chirality)

We are interested here in an operator mixing which gets completely suppressed only when chiral symmetry is fully operative

The nonzero amount of residual mixing present for any given finite N_s provides then another quantitative measure of chiral violations

One of the simplest examples of this kind of mixings is probably furnished by

$$O_{d_1} = \bar{q}\gamma_{[4}\gamma_5 D_{1]}q,$$

which taken together with $O_{a_2,d} = \bar{q}\gamma_{\{1}\gamma_5 D_{4\}}q$ determines the first moment of the g_2 structure function, which measures the distribution of the (chiral even) transverse spin of quarks inside hadrons, and also receives contributions from twist-3 operators

Matrix elements of the operator O_{d_1} have been simulated using quenched domain-wall fermions with the DBW2 gauge action by Orginos et al.

A power-divergent mixing

The power-divergent mixing on the lattice of O_{d_1} with an operator of lower dimension , which can be written as

$$c_{mix} \cdot \frac{i}{a} \bar{q}\sigma_{41}\gamma_5 q,$$

is only caused by the breaking of chirality, and hence it provides a quantitative measure of how much chiral symmetry has been broken

For infinite N_s c_{mix} vanishes, and O_{d_1} becomes in this case multiplicatively renormalized (like it also is when overlap fermions are used)

As in the case of Δ, these chiral violations seems to be of higher order in m_{res}, and in general rather small

The results show that large chiral violations are present only for very small N_s, or when M is rather close to 0 or 2

Again, let us look at the numbers that come out for $M = 1.8$ and $N_s = 16$

For $a^{-1} \sim 2\ GeV$ the chiral violations from c_{mix} come out of about $3\ MeV$, with the quenched case giving the slightly lower value

A power-divergent mixing

Again, gauge invariance is lost in the theory at finite N_s, and recovered only when $N_s = \infty$

Wilson fermions also suffer from this power-divergent mixing which is caused by the breaking of chirality, but the mixing coefficient is in this case gauge invariant

However, for Wilson fermions one has $c_{mix} = 16.243762$ for the unimproved and $c_{mix} = 8.798732$ for the improved case with $c_{sw} = 1$

Thus, compared to domain-wall fermions at the standard nonperturbative choices of N_s and M, the Wilson violations are about two orders of magnitude higher

At variance with Δ, we can see that improved gauge actions do suppress the amount of mixing c_{mix} (at the same coupling) , with the DBW2 action producing a suppression of about one order of magnitude

However we can also notice that when c_1 increases from the Iwasaki to the DBW2 action the results change sign in the region around $M = 1.8 - 1.9$, perhaps again indicating that the suppression goes too far and two-loop corrections are to be taken into account here

A power-divergent mixing

Coefficient of $\bar{g}^2$ for c_{mix}, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-14.1244	-8.4448	-5.3682	-3.4884	-2.2846	-1.4996	-0.9848	-0.1829	0
0.2	-5.8166	-2.4014	-0.9910	-0.4075	-0.1674	-0.0688	-0.0283	-0.0008	0
0.3	-2.4733	-0.6070	-0.1475	-0.0359	-0.0087	-0.0021	-0.0005	0.0000	0
0.4	-0.9361	-0.1265	-0.0169	-0.0022	-0.0003	0.0000	0.0000	0.0000	0
0.5	-0.3094	-0.0213	-0.0014	-0.0001	0.0000	0.0000	0.0000	0.0000	0
0.6	-0.0893	-0.0029	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	-0.0231	-0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	-0.0059	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	-0.0015	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0031	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0127	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0546	0.0029	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.1969	0.0207	0.0024	0.0003	0.0000	0.0000	0.0000	0.0000	0
1.7	0.5774	0.1129	0.0229	0.0049	0.0011	0.0002	0.0001	0.0000	0
1.8	1.3844	0.4817	0.1715	0.0615	0.0224	0.0083	0.0031	0.0001	0
1.9	2.9825	1.5917	0.9340	0.5665	0.3480	0.2148	0.1328	0.0198	0

A power-divergent mixing

Coefficient of $\bar{g}^2$ for the part proportional to $1-\lambda$ of c_{mix}.

M	$N_s=8$	$N_s=12$	$N_s=16$	$N_s=20$	$N_s=24$	$N_s=28$	$N_s=32$	$N_s=48$	$N_s=\infty$
0.1	0.3241	0.1056	0.0395	0.0162	0.0071	0.0033	0.0015	0.0001	0
0.2	0.0480	0.0093	0.0019	0.0004	0.0001	0.0000	0.0000	0.0000	0
0.3	0.0093	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.4	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.5	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	-0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.7	-0.0060	-0.0007	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.8	0.0035	-0.0043	-0.0015	-0.0004	-0.0001	0.0000	0.0000	0.0000	0
1.9	0.4015	0.0982	0.0218	0.0029	-0.0011	-0.0013	-0.0009	-0.0001	0

A power-divergent mixing

Coefficient of $\bar{g}^2$ for c_{mix}, in Feynman gauge for the Lüscher-Weisz action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-12.2811	-7.3668	-4.6923	-3.0533	-2.0014	-1.3145	-0.8635	-0.1604	0
0.2	-5.0766	-2.1043	-0.8697	-0.3578	-0.1471	-0.0605	-0.0249	-0.0007	0
0.3	-2.1594	-0.5314	-0.1293	-0.0315	-0.0077	-0.0019	-0.0005	0.0000	0
0.4	-0.8140	-0.1104	-0.0148	-0.0020	-0.0003	0.0000	0.0000	0.0000	0
0.5	-0.2675	-0.0186	-0.0012	-0.0001	0.0000	0.0000	0.0000	0.0000	0
0.6	-0.0768	-0.0025	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	-0.0197	-0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	-0.0050	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	-0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0024	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0097	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0409	0.0021	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.1467	0.0142	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000	0
1.7	0.4250	0.0767	0.0142	0.0028	0.0006	0.0001	0.0000	0.0000	0
1.8	0.9787	0.3200	0.1056	0.0345	0.0113	0.0038	0.0013	0.0000	0
1.9	1.8728	0.9467	0.5303	0.3068	0.1790	0.1043	0.0605	0.0064	0

A power-divergent mixing

Coefficient of $\bar{g}^2$ for c_{mix}, in Feynman gauge for the Iwasaki action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-9.4574	-5.7117	-3.6538	-2.3844	-1.5659	-1.0297	-0.6769	-0.1259	0
0.2	-3.9356	-1.6453	-0.6821	-0.2810	-0.1156	-0.0476	-0.0196	-0.0006	0
0.3	-1.6731	-0.4139	-0.1010	-0.0247	-0.0060	-0.0015	-0.0004	0.0000	0
0.4	-0.6240	-0.0853	-0.0115	-0.0015	-0.0002	0.0000	0.0000	0.0000	0
0.5	-0.2020	-0.0142	-0.0010	-0.0001	0.0000	0.0000	0.0000	0.0000	0
0.6	-0.0570	-0.0019	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	-0.0143	-0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	-0.0035	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	-0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0015	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0054	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0218	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.0773	0.0053	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.7	0.2172	0.0273	0.0024	-0.0001	-0.0001	0.0000	0.0000	0.0000	0
1.8	0.4328	0.1029	0.0172	-0.0016	-0.0035	-0.0023	-0.0012	0.0000	0
1.9	0.4010	0.0932	-0.0035	-0.0366	-0.0444	-0.0417	-0.0352	-0.0113	0

A power-divergent mixing

Coefficient of $\bar{g}^2$ for c_{mix}, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-5.8666	-3.5954	-2.3230	-1.5265	-1.0072	-0.6642	-0.4374	-0.0815	0
0.2	-2.4673	-1.0511	-0.4388	-0.1812	-0.0747	-0.0308	-0.0128	-0.0004	0
0.3	-1.0435	-0.2604	-0.0639	-0.0157	-0.0039	-0.0010	-0.0002	0.0000	0
0.4	-0.3771	-0.0521	-0.0072	-0.0010	-0.0001	0.0000	0.0000	0.0000	0
0.5	-0.1165	-0.0084	-0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.6	-0.0313	-0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.7	-0.0074	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.8	-0.0017	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.9	-0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.1	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.2	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.3	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.4	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.5	0.0030	-0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1.6	0.0115	-0.0031	-0.0007	-0.0001	0.0000	0.0000	0.0000	0.0000	0
1.7	0.0279	-0.0171	-0.0082	-0.0026	-0.0007	-0.0002	0.0000	0.0000	0
1.8	-0.0434	-0.0836	-0.0583	-0.0325	-0.0161	-0.0074	-0.0033	-0.0001	0
1.9	-0.8296	-0.6069	-0.4382	-0.3152	-0.2253	-0.1599	-0.1125	-0.0256	0

Conclusions

- Our perturbative calculations show that the numerical deviations from the case of infinite N_s depend, apart from N_s and to a smaller extent from the bare coupling g_0, very strongly on the choice of M
- These deviations can become rather pronounced when M is close to the borders of the region of allowed values
- The values of $am_{res}^{(1)}$ are positive only for $M \geq 1.2$
- Looking at the numbers for $N_s = 8$, our results would indicate that the minimal amount of chiral violations is attained for $M \sim 1.2$
- The pattern of the deviations from the case of exact chirality turns out to be approximately the same for all quantities studied
- For $M = 1.8$, a standard choice in Monte Carlo simulations, chiral violations are still not small for $N_s = 16$
- For example, m_{res} for a lattice spacing of $2\ GeV$ is equal to about $100\ MeV$ in the quenched case and about $120\ MeV$ in full QCD
- For the difference between the vector and axial-vector renormalization constants as well as for the power-divergent mixing, the chiral violations are instead of about $2 - 3\ MeV$, suggesting that they are of higher order in m_{res}

Conclusions

- We have also investigated the effectiveness of improved gauge actions in suppressing the chiral violations of domain-wall fermions
- When comparisons are made at the same value of the coupling, our calculations show substantial reductions of the residual mass, the largest suppression being obtained when the DBW2 action is used
- Similar effects can also be observed for a power-divergent mixing coefficient which is chirally suppressed
- No significant reduction can instead be seen in the case of the difference between the vector and axial-vector renormalization constants when improved gauge actions are used in place of the plaquette action
- A lesson from this could be that some (but by no means not all) quantities which measure chiral violations but are of higher order in m_{res} will get little or no reduction when improved gauge actions are used

- When comparisons are made at the same energy scale, perturbation theory gives results that are quite distant from naive expectations
- Its use appears to be problematic for these actions, also because of the large values that g_0 assumes at the scale of 2 GeV

SOME DISCARDED SLIDES

Residual mass

At finite N_s chiral symmetry is broken, and the resulting overlap between the modes exponentially confined on the two opposite walls generates a nonzero residual mass

Its leading effects have been analytically estimated (by looking at the eigenfunctions of the five-dimensional transfer matrix) as (Christ, Golterman, Shamir, . . .)

$$m_{res} \sim R_e^4 \, \rho_e(\lambda_c) \, \frac{\exp\left(-\lambda_c N_s\right)}{N_s} + R_l^4 \, \rho_l(0) \, \frac{1}{N_s}$$

Here ρ is the density per unit spacetime volume of the eigenvalues of the logarithm of the transfer matrix, while l stands for the localized and e for the extended modes

The energy threshold from localized to extended modes is given by the mobility edge λ_c, which is thus also responsible for the speed with which the chiral violations decay with N_s

One-loop structure of the physical propagator

The other quantities appearing in $\langle q(-p)\overline{q}(p)\rangle_{1\ loop}$ are

$$Z_2 = 1 + \bar{g}^2 \left(c_{\Sigma_1}^{(N_s,M)} \log a^2 p^2 + \Sigma_1 \right),$$

which is the quark wave function renormalization factor, and

$$Z_w = 1 - \frac{2w_0}{1-w_0^2}\, \bar{g}^2\, \Sigma_3 = 1 + \bar{g}^2\, z_w,$$

which represents an additive renormalization to w_0 (and hence to the domain-wall height M), as can be inferred from

$$(1-w_0^2)\, Z_w = 1 - \left(w_0 + \bar{g}^2\, \Sigma_3\right)^2 + O(\bar{g}^4)$$

There is indeed no chiral symmetry which can protect this mass, even at $N_s = \infty$

Some pathologies

Anomalous dimension of the scalar density, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	**3.27780**	**3.67649**	**3.58735**	**3.39105**	**3.22907**	**3.12464**	**3.06474**	**3.00372**	**3**
0.2	**3.45424**	**3.15990**	**3.04087**	**3.00918**	**3.00193**	**3.00039**	**3.00008**	**3.00000**	**3**
0.3	**3.10489**	**3.01080**	**3.00089**	**3.00007**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
0.4	**3.01260**	**3.00036**	**3.00001**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
0.5	**3.00085**	**3.00001**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
0.6	**3.00003**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
0.7	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
0.8	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
0.9	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.0	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.1	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.2	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.3	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.4	**3.00003**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.5	**3.00085**	**3.00001**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.6	**3.01260**	**3.00036**	**3.00001**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.7	**3.10489**	**3.01080**	**3.00089**	**3.00007**	**3.00000**	**3.00000**	**3.00000**	**3.00000**	**3**
1.8	**3.45424**	**3.15990**	**3.04087**	**3.00918**	**3.00193**	**3.00039**	**3.00008**	**3.00000**	**3**
1.9	**3.27780**	**3.67649**	**3.58735**	**3.39105**	**3.22907**	**3.12464**	**3.06474**	**3.00372**	**3**

Some pathologies

The domain-wall theory at finite N_s is then loaded with a few pathologies

In passing from $N_s = \infty$ to any finite value of N_s, the analytical Atiyah-Singer index (which protects the quarks from acquiring a nonzero mass) disappears

It is well known that in all formulations of chiral fermions that recover the correct quantum anomalies one needs (in one way or another) an infinite number of fermion fields

With a finite number of these fields what happens is that either the two chiralities cannot be completely separated or that the right anomalies cannot be reproduced

The lack of gauge invariance, and the "anomalous" anomalous dimensions presumably appears also at higher loops, since there is nothing here suggesting a peculiarity of the one-loop theory

However, they are probably negligible from a practical point of view, given that the optimal one-loop values of M lie near $M \sim 1.2$, although in principle they are present, and for any N_s and M and any loop order there are deviations (however small they are)

Tadpoles

Tadpoles of order zero for improved gauge actions:

$$T_d^{(0)} = T_l^{(0)} \, \frac{1 - w_0^2}{(1 - w_0^{2N_s})^2} \left(N_s \, (1 + w_0^{2(N_s+1)}) \, w_0^{N_s - 1} - 2 \, w_0^{N_s+1} \, \frac{1 - w_0^{2N_s}}{1 - w_0^2} \right),$$

where $T_l^{(0)}$ is (up to a sign) the result for Wilson fermions:

- $T_l^{(0)} = 48.932201 - 8\pi^2 Z_0 \, (1 - \lambda)$ for the plaquette action
- $T_l^{(0)} = 40.517749 - 8\pi^2 Z_0 \, (1 - \lambda)$ for the Lüscher-Weisz action
- $T_l^{(0)} = 29.927709 - 8\pi^2 Z_0 \, (1 - \lambda)$ for the Iwasaki action
- $T_l^{(0)} = 19.715871 - 8\pi^2 Z_0 \, (1 - \lambda)$ for the DBW2 action

Tadpoles

Coefficient of $\bar{g}^2$ for the tadpole of Σ_0, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	6.08558	6.84244	6.79166	6.17209	5.26321	4.28400	3.36856	1.04552	0
0.2	7.38915	5.23011	3.11540	1.68275	0.85663	0.41950	0.19994	0.00879	0
0.3	5.08323	2.00488	0.67208	0.20721	0.06076	0.01723	0.00477	0.00002	0
0.4	2.43021	0.49784	0.08825	0.01451	0.00228	0.00035	0.00005	0.00000	0
0.5	0.84720	0.08183	0.00692	0.00055	0.00004	0.00000	0.00000	0.00000	0
0.6	0.20674	0.00807	0.00028	0.00001	0.00000	0.00000	0.00000	0.00000	0
0.7	0.03061	0.00038	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.00192	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00192	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.03061	-0.00038	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.20674	-0.00807	-0.00028	-0.00001	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.84720	-0.08183	-0.00692	-0.00055	-0.00004	0.00000	0.00000	0.00000	0
1.6	-2.43021	-0.49784	-0.08825	-0.01451	-0.00228	-0.00035	-0.00005	0.00000	0
1.7	-5.08323	-2.00488	-0.67208	-0.20721	-0.06076	-0.01723	-0.00477	-0.00002	0
1.8	-7.38915	-5.23011	-3.11540	-1.68275	-0.85663	-0.41950	-0.19994	-0.00879	0
1.9	-6.08558	-6.84244	-6.79166	-6.17209	-5.26321	-4.28400	-3.36856	-1.04552	0

For the Iwasaki action all entries have to be multiplied by 1.51795, and for the Lüscher-Weisz action all entries have to be multiplied by 2.05508

Residual mass

Coefficient of $\bar{g}^2$ for the contribution of the half-circle diagram to Σ_0, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	2.75558	2.69676	2.39676	2.00277	1.60286	1.24317	0.94186	0.26825	0
0.2	3.48374	2.22615	1.24197	0.64382	0.31901	0.15331	0.07208	0.00306	0
0.3	2.67348	1.01173	0.33194	0.10111	0.02942	0.00830	0.00227	0.00001	0
0.4	1.52701	0.31852	0.05710	0.00946	0.00147	0.00022	0.00003	0.00000	0
0.5	0.68927	0.07235	0.00641	0.00051	0.00004	0.00000	0.00000	0.00000	0
0.6	0.24943	0.01188	0.00046	0.00002	0.00000	0.00000	0.00000	0.00000	0
0.7	0.07417	0.00157	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.02085	0.00029	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00770	0.00012	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00418	0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00286	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00304	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.01346	0.00021	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.06983	0.00270	0.00009	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.24400	0.02258	0.00186	0.00013	0.00001	0.00000	0.00000	0.00000	0
1.6	0.60517	0.11252	0.01902	0.00304	0.00045	0.00006	0.00001	0.00000	0
1.7	1.12385	0.36321	0.10935	0.03157	0.00886	0.00243	0.00064	0.00000	0
1.8	1.56704	0.77955	0.36471	0.16644	0.07499	0.03349	0.01486	0.00053	0
1.9	1.40741	1.01047	0.66666	0.41011	0.23700	0.12837	0.06403	-0.00472	0

Residual mass

Coefficient of $\bar{g}^2$ for the part proportional to $1-\lambda$ of the contribution of the half-circle diagram to Σ_0.

M	$N_s=8$	$N_s=12$	$N_s=16$	$N_s=20$	$N_s=24$	$N_s=28$	$N_s=32$	$N_s=48$	$N_s=\infty$
0.1	3.19742	3.77191	3.84954	3.56027	3.07177	2.52079	1.99387	0.62668	0
0.2	4.00781	2.95012	1.79210	0.97875	0.50169	0.24683	0.11804	0.00523	0
0.3	2.81125	1.14035	0.38726	0.12026	0.03543	0.01008	0.00280	0.00001	0
0.4	1.35920	0.28390	0.05078	0.00839	0.00133	0.00020	0.00003	0.00000	0
0.5	0.47745	0.04674	0.00398	0.00032	0.00002	0.00000	0.00000	0.00000	0
0.6	0.11737	0.00463	0.00016	0.00001	0.00000	0.00000	0.00000	0.00000	0
0.7	0.01755	0.00022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.00112	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00126	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.02044	-0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.13917	-0.00539	-0.00018	-0.00001	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.57387	-0.05480	-0.00461	-0.00036	-0.00003	0.00000	0.00000	0.00000	0
1.6	-1.65669	-0.33389	-0.05873	-0.00961	-0.00150	-0.00023	-0.00003	0.00000	0
1.7	-3.49885	-1.34763	-0.44675	-0.13687	-0.03997	-0.01130	-0.00312	-0.00002	0
1.8	-5.17527	-3.54181	-2.07409	-1.10946	-0.56133	-0.27375	-0.13008	-0.00568	0
1.9	-4.39070	-4.73341	-4.58402	-4.10098	-3.46030	-2.79565	-2.18639	-0.67074	0

Residual mass

Coefficient of $\bar{g}^2$ for the complete result of Σ_0, in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	17.85919	19.67880	19.25277	17.32109	14.66546	11.87550	9.30218	2.86309	0
0.2	21.82264	15.20660	8.97397	4.82019	2.44504	1.19446	0.56831	0.02488	0
0.3	15.28940	5.98758	1.99997	0.61538	0.18022	0.05106	0.01412	0.00007	0
0.4	7.55847	1.55410	0.27613	0.04548	0.00713	0.00108	0.00016	0.00000	0
0.5	2.79190	0.27543	0.02358	0.00186	0.00014	0.00001	0.00000	0.00000	0
0.6	0.76252	0.03191	0.00115	0.00004	0.00000	0.00000	0.00000	0.00000	0
0.7	0.15015	0.00250	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.02561	0.00031	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00774	0.00012	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00418	0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00283	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00172	0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.06252	-0.00073	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.44327	-0.01733	-0.00060	-0.00002	0.00000	0.00000	0.00000	0.00000	0
1.5	-1.85863	-0.18051	-0.01531	-0.00122	-0.00009	-0.00001	0.00000	0.00000	0
1.6	-5.42629	-1.12306	-0.20001	-0.03298	-0.00521	-0.00080	-0.00012	0.00000	0
1.7	-11.49207	-4.61264	-1.55867	-0.48269	-0.14194	-0.04034	-0.01120	-0.00006	0
1.8	-16.77186	-12.20090	-7.36730	-4.00994	-2.05105	-1.00765	-0.48138	-0.02129	0
1.9	-13.69620	-15.97157	-16.18935	-14.90821	-12.82561	-10.50396	-8.29629	-2.59956	0

Residual mass

Coefficient of $\bar{g}^2$ for the complete result of Σ_0, in Landau gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	17.28070	19.20520	18.88831	17.05178	14.47158	11.73820	9.20597	2.84106	0
0.2	21.24572	14.91161	8.83308	4.75484	2.41522	1.18100	0.56229	0.02466	0
0.3	14.94668	5.88397	1.97022	0.60708	0.17795	0.05045	0.01396	0.00007	0
0.4	7.40981	1.52910	0.27215	0.04487	0.00704	0.00107	0.00016	0.00000	0
0.5	2.74369	0.27141	0.02327	0.00184	0.00014	0.00001	0.00000	0.00000	0
0.6	0.75162	0.03153	0.00114	0.00004	0.00000	0.00000	0.00000	0.00000	0
0.7	0.14871	0.00249	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.02554	0.00031	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00774	0.00012	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00418	0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00283	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00179	0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.06397	-0.00074	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.45417	-0.01771	-0.00061	-0.00002	0.00000	0.00000	0.00000	0.00000	0
1.5	-1.90684	-0.18453	-0.01562	-0.00124	-0.00009	-0.00001	0.00000	0.00000	0
1.6	-5.57512	-1.14806	-0.20399	-0.03359	-0.00530	-0.00081	-0.00012	0.00000	0
1.7	-11.83694	-4.71631	-1.58842	-0.49100	-0.14421	-0.04095	-0.01136	-0.00006	0
1.8	-17.36240	-12.49760	-7.50839	-4.07530	-2.08087	-1.02111	-0.48740	-0.02151	0
1.9	-14.31100	-16.45947	-16.55937	-15.17962	-13.02025	-10.64153	-8.39260	-2.62158	0

Residual mass

Residual mass in lattice units at $\beta = 6$ in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.23258	-0.21982	-0.19777	-0.16935	-0.13898	-0.11021	-0.08507	-0.02538	0
0.2	-0.24466	-0.15313	-0.08590	-0.04485	-0.02234	-0.01078	-0.00508	-0.00022	0
0.3	-0.15850	-0.05761	-0.01858	-0.00560	-0.00162	-0.00045	-0.00012	0.00000	0
0.4	-0.07457	-0.01452	-0.00251	-0.00041	-0.00006	-0.00001	0.00000	0.00000	0
0.5	-0.02650	-0.00251	-0.00021	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00699	-0.00028	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00133	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00047	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00319	0.00013	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01276	0.00134	0.00012	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.03507	0.00809	0.00151	0.00026	0.00004	0.00001	0.00000	0.00000	0
1.7	0.06763	0.03189	0.01147	0.00367	0.00110	0.00032	0.00009	0.00000	0
1.8	0.08121	0.07828	0.05207	0.02971	0.01562	0.00781	0.00378	0.00017	0
1.9	0.03385	0.08119	0.10149	0.10278	0.09314	0.07875	0.06353	0.02074	0

Residual mass

Residual mass in lattice units at $\beta = 5.2$ in Feynman gauge.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.25578	-0.24538	-0.22278	-0.19185	-0.15803	-0.12564	-0.09715	-0.02910	0
0.2	-0.27300	-0.17289	-0.09756	-0.05111	-0.02552	-0.01233	-0.00582	-0.00025	0
0.3	-0.17836	-0.06539	-0.02118	-0.00640	-0.00185	-0.00052	-0.00014	0.00000	0
0.4	-0.08439	-0.01653	-0.00287	-0.00047	-0.00007	-0.00001	0.00000	0.00000	0
0.5	-0.03013	-0.00287	-0.00024	-0.00002	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00798	-0.00033	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00152	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00055	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00377	0.00015	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.01518	0.00158	0.00014	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.6	0.04212	0.00955	0.00177	0.00030	0.00005	0.00001	0.00000	0.00000	0
1.7	0.08256	0.03788	0.01349	0.00430	0.00129	0.00037	0.00010	0.00000	0
1.8	0.10300	0.09413	0.06164	0.03492	0.01828	0.00912	0.00440	0.00020	0
1.9	0.05165	0.10194	0.12252	0.12214	0.10980	0.09239	0.07430	0.02412	0

Odd N_s

As an aside, we have also investigated what happens when one chooses an odd value of N_s

When M is smaller than about 1, the residual mass has always a negative sign whether N_s is even or odd

In the more interesting case of M greater than 1, the one-loop residual mass turns out to be:

- positive when N_s is even
- negative when N_s is odd

This could be related to the suggestion made by Neuberger some years ago, according to which for odd N_s the domain-wall theory which is simulated could correspond to the $\theta = \pi$ regime of QCD

Odd N_s

Residual mass at $\beta = 6.0$ for some odd N_s, in Landau gauge.

M	$N_s = 13$	$N_s = 14$	$N_s = 15$	$N_s = 16$	$N_s = 17$	$N_s = 18$
0.1	-0.211	-0.206	-0.201	-0.195	-0.188	-0.181
0.2	-0.132	-0.114	-0.099	-0.085	-0.072	-0.062
0.3	-0.043	-0.033	-0.025	-0.018	-0.014	-0.010
0.4	-0.009	-0.006	-0.004	-0.002	-0.002	-0.001
0.5	-0.001	-0.001	0.000	0.000	0.000	0.000
0.6	0.000	0.000	0.000	0.000	0.000	0.000
0.7	0.000	0.000	0.000	0.000	0.000	0.000
0.8	0.000	0.000	0.000	0.000	0.000	0.000
0.9	0.000	0.000	0.000	0.000	0.000	0.000
1.0	0.000	0.000	0.000	0.000	0.000	0.000
1.1	0.000	0.000	0.000	0.000	0.000	0.000
1.2	0.000	0.000	0.000	0.000	0.000	0.000
1.3	0.000	0.000	0.000	0.000	0.000	0.000
1.4	0.000	0.000	0.000	0.000	0.000	0.000
1.5	-0.001	0.000	0.000	0.000	0.000	0.000
1.6	-0.006	0.004	-0.002	0.002	-0.001	0.001
1.7	-0.026	0.020	-0.015	0.012	-0.009	0.007
1.8	-0.074	0.067	-0.060	0.053	-0.047	0.041
1.9	-0.092	0.098	-0.102	0.105	-0.106	0.107

Odd N_s

Residual mass at $\beta = 5.2$ for some odd N_s, in Landau gauge.

M	$N_s = 13$	$N_s = 14$	$N_s = 15$	$N_s = 16$	$N_s = 17$	$N_s = 18$
0.1	-0.236	-0.231	-0.226	-0.219	-0.212	-0.205
0.2	-0.149	-0.129	-0.112	-0.096	-0.082	-0.070
0.3	-0.049	-0.037	-0.028	-0.021	-0.016	-0.012
0.4	-0.011	-0.007	-0.004	-0.003	-0.002	-0.001
0.5	-0.002	-0.001	0.000	0.000	0.000	0.000
0.6	0.000	0.000	0.000	0.000	0.000	0.000
0.7	0.000	0.000	0.000	0.000	0.000	0.000
0.8	0.000	0.000	0.000	0.000	0.000	0.000
0.9	0.000	0.000	0.000	0.000	0.000	0.000
1.0	0.000	0.000	0.000	0.000	0.000	0.000
1.1	0.000	0.000	0.000	0.000	0.000	0.000
1.2	0.000	0.000	0.000	0.000	0.000	0.000
1.3	0.000	0.000	0.000	0.000	0.000	0.000
1.4	0.000	0.000	0.000	0.000	0.000	0.000
1.5	-0.001	0.000	0.000	0.000	0.000	0.000
1.6	-0.006	0.004	-0.003	0.002	-0.001	0.001
1.7	-0.030	0.024	-0.018	0.014	-0.010	0.008
1.8	-0.089	0.080	-0.071	0.063	-0.055	0.048
1.9	-0.114	0.120	-0.124	0.126	-0.127	0.127

Improved gauge actions

Coefficient of $\bar{g}^2$ of the tadpole and half-circle diagrams, for $M = 1.8$ and $N_s = 16$ in Feynman gauge, and corresponding one-loop residual mass at $\beta = 5.2$:

action	tadpole	half-circle	$am_{res}^{(0)}$	$am_{res}^{(1)}$
plaquette	-7.73201	0.36471	-0.01013	0.06164
Lüscher-Weisz	-6.40240	0.71569	-0.01013	0.04527
Iwasaki	-4.72902	1.22505	-0.01013	0.02400
DBW2	-3.11540	1.79449	-0.01013	0.00274

Note that the dramatic suppression of the residual mass in the case of the DBW2 action is caused not only by the decreasing of the absolute value of the numbers for the tadpole (which are negative for $M > 1$), but also by the increasing of the numbers for the half-circle diagram (which are positive)

(Remember that Σ_0 enters with a negative sign in the definition of m_{res})

Improved gauge actions

At the end, what happens is that the total contribution of the one-loop diagrams becomes smaller as c_1 grows

Then, since the (negative) tree-level residual mass always remains the same, for the DBW2 action some large compensations between the tree-level and one-loop diagrams occur

For this action the final numbers for $m_{res}^{(1)}$ are then decisively smaller, and sometimes they even overshoot and become negative

Moreover, for the DBW2 action the violations of gauge invariance are rather pronounced

It is clear that one-loop perturbation theory encounters here some limitations and cannot provide the whole story

At least for the DBW2 action, 2-loop corrections should be considered if one is interested in more reliable numbers

The two-loop contributions become indeed more and more important as c_1 grows, because of the ever larger compensations between one-loop and tree-level diagrams

Improved gauge actions

Coefficient of $\bar{g}^2$ for the half-circle diagram of Σ_0, in Feynman gauge for the Lüscher-Weisz action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	1.85305	1.69896	1.41789	1.12004	0.85404	0.63588	0.46559	0.12125	0
0.2	2.33907	1.42243	0.76566	0.38732	0.18868	0.08958	0.04173	0.00173	0
0.3	1.83151	0.67820	0.21995	0.06655	0.01928	0.00542	0.00147	0.00001	0
0.4	1.08252	0.22556	0.04045	0.00670	0.00104	0.00015	0.00002	0.00000	0
0.5	0.50879	0.05403	0.00482	0.00038	0.00003	0.00000	0.00000	0.00000	0
0.6	0.19223	0.00935	0.00036	0.00001	0.00000	0.00000	0.00000	0.00000	0
0.7	0.05966	0.00129	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.01726	0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00635	0.00010	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00339	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00229	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00268	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.01527	0.00021	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.08659	0.00335	0.00011	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.32093	0.03003	0.00249	0.00018	0.00001	0.00000	0.00000	0.00000	0
1.6	0.84348	0.16170	0.02776	0.00448	0.00067	0.00010	0.00001	0.00000	0
1.7	1.65276	0.57478	0.18073	0.05366	0.01535	0.00427	0.00115	0.00000	0
1.8	2.37106	1.36227	0.71569	0.35718	0.17245	0.08134	0.03770	0.00154	0
1.9	2.08894	1.80335	1.46797	1.14614	0.86894	0.64513	0.47169	0.12268	0

Improved gauge actions

Coefficient of $\bar{g}^2$ for the half-circle diagram of Σ_0, in Feynman gauge for the Iwasaki action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	0.50821	0.21779	-0.03162	-0.18502	-0.25188	-0.26034	-0.23689	-0.09532	0
0.2	0.63587	0.22998	0.06006	0.00769	-0.00410	-0.00467	-0.00314	-0.00024	0
0.3	0.57859	0.18254	0.05361	0.01524	0.00423	0.00115	0.00029	0.00000	0
0.4	0.41985	0.08688	0.01562	0.00259	0.00039	0.00005	0.00001	0.00000	0
0.5	0.23861	0.02652	0.00242	0.00019	0.00001	0.00000	0.00000	0.00000	0
0.6	0.10593	0.00549	0.00022	0.00001	0.00000	0.00000	0.00000	0.00000	0
0.7	0.03747	0.00086	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.01169	0.00017	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00428	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00222	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00146	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00218	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.01789	0.00023	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.11072	0.00428	0.00014	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	0.43174	0.04077	0.00340	0.00025	0.00002	0.00000	0.00000	0.00000	0
1.6	1.18690	0.23278	0.04042	0.00656	0.00100	0.00015	0.00002	0.00000	0
1.7	2.41466	0.88093	0.28421	0.08571	0.02477	0.00695	0.00189	0.00001	0
1.8	3.52723	2.20554	1.22505	0.63440	0.31424	0.15100	0.07098	0.00301	0
1.9	3.06494	2.94845	2.63010	2.21616	1.78901	1.39824	1.06623	0.30874	0

Improved gauge actions

Coefficient of $\bar{g}^2$ for the half-circle diagram of Σ_0, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-1.10030	-1.53568	-1.73593	-1.71261	-1.54242	-1.30396	-1.05364	-0.34630	0
0.2	-1.38867	-1.17537	-0.76769	-0.43649	-0.22928	-0.11464	-0.05545	-0.00253	0
0.3	-0.90559	-0.40170	-0.14203	-0.04505	-0.01344	-0.00386	-0.00110	-0.00001	0
0.4	-0.36485	-0.07724	-0.01379	-0.00227	-0.00038	-0.00006	-0.00001	0.00000	0
0.5	-0.08232	-0.00635	-0.00045	-0.00004	0.00000	0.00000	0.00000	0.00000	0
0.6	0.00264	0.00080	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	0.01059	0.00032	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.00494	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00188	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00092	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00059	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00169	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.02083	0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.13739	0.00532	0.00017	0.00001	0.00000	0.00000	0.00000	0.00000	0
1.5	0.55390	0.05271	0.00442	0.00033	0.00002	0.00000	0.00000	0.00000	0
1.6	1.56471	0.31178	0.05455	0.00890	0.00137	0.00020	0.00003	0.00000	0
1.7	3.24961	1.22111	0.39990	0.12166	0.03537	0.00997	0.00273	0.00001	0
1.8	4.78426	3.14023	1.79449	0.94576	0.47393	0.22960	0.10857	0.00468	0
1.9	4.11043	4.20697	3.92344	3.41545	2.82481	2.24857	1.73889	0.52014	0

Improved gauge actions

Coefficient of $\bar{g}^2$ for Σ_0, in Feynman gauge for the Lüscher-Weisz action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	14.35942	15.76075	15.37532	13.80421	11.67038	9.43986	7.38826	5.42456	0
0.2	17.52439	12.17074	7.16806	3.84552	1.94912	0.95169	0.45262	0.19370	0
0.3	12.27798	4.79840	1.60113	0.49238	0.14415	0.04083	0.01128	0.00267	0
0.4	6.07680	1.24867	0.22182	0.03653	0.00572	0.00087	0.00013	0.00002	0
0.5	2.24985	0.22220	0.01904	0.00150	0.00011	0.00001	0.00000	0.00000	0
0.6	0.61709	0.02594	0.00093	0.00003	0.00000	0.00000	0.00000	0.00000	0
0.7	0.12258	0.00206	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.02120	0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00638	0.00010	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00339	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00226	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00126	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.04765	-0.00056	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.33827	-0.01324	-0.00046	-0.00001	0.00000	0.00000	0.00000	0.00000	0
1.5	-1.42013	-0.13814	-0.01173	-0.00094	-0.00007	-0.00001	0.00000	0.00000	0
1.6	-4.15080	-0.86142	-0.15360	-0.02535	-0.00401	-0.00062	-0.00009	-0.00002	0
1.7	-8.79371	-3.54541	-1.20046	-0.37218	-0.10952	-0.03114	-0.00866	-0.00266	0
1.8	-12.81426	-9.38605	-5.68671	-3.10102	-1.58799	-0.78077	-0.37320	-0.19043	0
1.9	-10.41742	-12.25844	-12.48946	-11.53802	-9.94739	-8.15885	-6.45097	-5.18063	0

Improved gauge actions

Coefficient of $\bar{g}^2$ for Σ_0, in Feynman gauge for the Iwasaki action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	9.74581	10.60427	10.27779	9.18391	7.73742	6.24256	4.87641	3.82187	0
0.2	11.85223	8.16903	4.78908	2.56202	1.29622	0.63211	0.30036	0.14156	0
0.3	8.29469	3.22585	1.07380	0.32977	0.09646	0.02731	0.00753	0.00197	0
0.4	4.10879	0.84259	0.14958	0.02462	0.00385	0.00058	0.00009	0.00001	0
0.5	1.52461	0.15073	0.01293	0.00102	0.00008	0.00001	0.00000	0.00000	0
0.6	0.41974	0.01774	0.00064	0.00002	0.00000	0.00000	0.00000	0.00000	0
0.7	0.08394	0.00143	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.01460	0.00017	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00431	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00222	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00144	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00073	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.02859	-0.00034	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.20310	-0.00797	-0.00028	-0.00001	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.85426	-0.08344	-0.00710	-0.00057	-0.00004	0.00000	0.00000	0.00000	0
1.6	-2.50204	-0.52292	-0.09354	-0.01547	-0.00246	-0.00038	-0.00006	-0.00001	0
1.7	-5.30143	-2.16238	-0.73598	-0.22883	-0.06746	-0.01920	-0.00535	-0.00196	0
1.8	-7.68914	-5.73352	-3.50397	-1.91993	-0.98607	-0.48578	-0.23252	-0.13878	0
1.9	-6.17267	-7.43803	-7.67930	-7.15278	-6.20028	-5.10466	-4.04708	-3.60846	0

Improved gauge actions

Coefficient of $\bar{g}^2$ for Σ_0, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	4.98528	5.30676	5.05573	4.45949	3.72080	2.98004	2.31491	2.23429	0
0.2	6.00048	4.05474	2.34771	1.24626	0.62735	0.30486	0.14449	0.09088	0
0.3	4.17765	1.60318	0.53006	0.16216	0.04732	0.01337	0.00368	0.00129	0
0.4	2.06536	0.42060	0.07446	0.01224	0.00190	0.00028	0.00004	0.00001	0
0.5	0.76488	0.07548	0.00647	0.00050	0.00004	0.00000	0.00000	0.00000	0
0.6	0.20938	0.00887	0.00032	0.00001	0.00000	0.00000	0.00000	0.00000	0
0.7	0.04121	0.00070	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.00686	0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00189	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00092	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00057	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00023	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.00978	-0.00012	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.06935	-0.00275	-0.00010	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.29330	-0.02912	-0.00250	-0.00021	-0.00002	0.00000	0.00000	0.00000	0
1.6	-0.86550	-0.18607	-0.03370	-0.00561	-0.00091	-0.00014	-0.00002	-0.00001	0
1.7	-1.83363	-0.78377	-0.27219	-0.08555	-0.02539	-0.00726	-0.00204	-0.00129	0
1.8	-2.60489	-2.08988	-1.32091	-0.73700	-0.38269	-0.18990	-0.09137	-0.08873	0
1.9	-1.97515	-2.63547	-2.86822	-2.75664	-2.43840	-2.03543	-1.62967	-2.06044	0

Improved gauge actions

Residual mass at $\beta = 6$, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.12388	-0.09847	-0.07790	-0.06075	-0.04657	-0.03511	-0.02607	-0.02315	0
0.2	-0.11106	-0.05897	-0.02996	-0.01467	-0.00700	-0.00327	-0.00151	-0.00088	0
0.3	-0.06467	-0.02060	-0.00617	-0.00178	-0.00050	-0.00014	-0.00004	-0.00001	0
0.4	-0.02819	-0.00494	-0.00081	-0.00013	-0.00002	0.00000	0.00000	0.00000	0
0.5	-0.00939	-0.00082	-0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00232	-0.00009	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00041	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00004	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.00045	0.00006	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.6	-0.00344	0.00018	0.00010	0.00002	0.00000	0.00000	0.00000	0.00000	0
1.7	-0.01392	-0.00044	0.00060	0.00032	0.00012	0.00004	0.00001	0.00001	0
1.8	-0.03840	-0.00709	0.00102	0.00207	0.00153	0.00091	0.00049	0.00063	0
1.9	-0.06511	-0.03141	-0.01099	0.00018	0.00543	0.00724	0.00724	0.01312	0

Improved gauge actions

Residual mass at $\beta = 6$, in Landau gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.11900	-0.09447	-0.07482	-0.05848	-0.04493	-0.03395	-0.02526	-0.01492	0
0.2	-0.10619	-0.05648	-0.02877	-0.01412	-0.00675	-0.00316	-0.00145	-0.00044	0
0.3	-0.06178	-0.01972	-0.00592	-0.00171	-0.00048	-0.00013	-0.00004	-0.00001	0
0.4	-0.02693	-0.00473	-0.00078	-0.00012	-0.00002	0.00000	0.00000	0.00000	0
0.5	-0.00898	-0.00079	-0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00223	-0.00009	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00040	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00013	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.00005	0.00010	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.6	-0.00219	0.00039	0.00014	0.00003	0.00001	0.00000	0.00000	0.00000	0
1.7	-0.01101	0.00043	0.00085	0.00039	0.00014	0.00004	0.00001	0.00000	0
1.8	-0.03342	-0.00459	0.00221	0.00262	0.00178	0.00102	0.00054	0.00019	0
1.9	-0.05992	-0.02729	-0.00787	0.00247	0.00708	0.00840	0.00805	0.00526	0

Improved gauge actions

Residual mass at $\beta = 5.2$, in Feynman gauge for the DBW2 action.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	-0.13036	-0.10536	-0.08446	-0.06655	-0.05141	-0.03898	-0.02908	-0.02605	0
0.2	-0.11886	-0.06424	-0.03301	-0.01629	-0.00781	-0.00367	-0.00169	-0.00100	0
0.3	-0.07010	-0.02268	-0.00686	-0.00199	-0.00056	-0.00015	-0.00004	-0.00001	0
0.4	-0.03087	-0.00549	-0.00091	-0.00014	-0.00002	0.00000	0.00000	0.00000	0
0.5	-0.01038	-0.00092	-0.00007	-0.00001	0.00000	0.00000	0.00000	0.00000	0
0.6	-0.00259	-0.00010	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.7	-0.00046	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	-0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	-0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	0.00013	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.5	-0.00007	0.00010	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.6	-0.00232	0.00042	0.00015	0.00003	0.00001	0.00000	0.00000	0.00000	0
1.7	-0.01154	0.00058	0.00096	0.00043	0.00015	0.00005	0.00001	0.00001	0
1.8	-0.03502	-0.00438	0.00274	0.00303	0.00203	0.00115	0.00060	0.00075	0
1.9	-0.06255	-0.02799	-0.00726	0.00376	0.00860	0.00989	0.00935	0.01579	0

Improved actions: Δ vs. c_{mix}

We would now like to understand why Δ and c_{mix} have a different behavior when improved gauge actions are used

We can begin by noticing that, unlike the residual mass, there is no tree-level contribution to Δ and c_{mix}

Moreover, no tadpole enters in the one-loop calculations of these quantities, and certainly there cannot be any tadpole dominance here

Furthermore, still at variance with the residual mass, the half-circle diagram is also absent

In fact, in the case of Δ it cancels in the difference of V and A, while for c_{mix} it is not present from the start

Only the vertex diagram contributes to the calculation of Δ

In the case of c_{mix} the sail diagrams are present as well, because the operator contains a gauge field coming from the covariant derivative

This could be the explanation for the different behavior of Δ and c_{mix} when improved gauge actions are turned on

Improved actions: Δ vs. c_{mix}

Looking at the past perturbative results for improved gauge actions with Wilson and overlap fermions, as well as with domain-wall fermions at infinite N_s, one can notice that the numbers for the vertex diagrams do not change very much in going from the plaquette to improved gauge actions

For example: for Wilson fermions the lattice finite constant of the vertex diagram of V_μ is 8.765394 for the plaquette and 7.050662 for the DBW2 action

For A_μ these values are 3.943879 and 5.555827 respectively

It is in general the tadpole and half-circle diagrams that end up generating the largest changes in these cases

These past results make plausible the fact that Δ is essentially not changed by the use of improved gauge actions also for domain-wall fermions at finite N_s

They however refer to multiplicative renormalization factors of bilinear operators, and things could go differently for the power-divergent mixing coefficient c_{mix} ...

Improved actions: Δ vs. c_{mix}

In looking for an explanation for the behavior of c_{mix} we have then computed this quantity using improved gauge actions also for standard Wilson fermions

For domain-wall at infinite N_s, and overlap, this coefficient is instead zero

Improved actions: Δ vs. c_{mix}

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For domain-wall at infinite N_s, and overlap, this coefficient is instead zero

Coefficient of $\bar{g}^2$ of c_{mix} for Wilson fermions (including fermionic improvement):

action	$c_{sw} = 0$	$c_{sw} = 1$	$c_{sw} = 2$
plaquette	16.243762	8.798732	0.174259
Lüscher-Weisz	13.517293	6.887859	-0.749841
Iwasaki	9.461302	4.216070	-1.756792
DBW2	4.688662	1.443948	-2.158332

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DBW2	4.688662	1.443948	-2.158332

It can be seen that the combined use of fermionic improvement (with $c_{sw} = 1$) and improved gauge actions (especially the DBW2 action) suppresses this mixing considerably (a factor 11.25 in this case)

This effect also recalls to mind what happens when UV-filtering and fermionic improvement are used together (Capitani, Dürr and Hoelbling, 2006)

Improved actions: Δ vs. c_{mix}

All these numbers seem to make plausible the behavior of c_{mix} with improved gauge actions that we have observed for domain-wall fermions at finite N_s, and in some sense the domain-wall formulation can be thought as having many Wilson fermions with the appropriate damping factors

On closer inspection, it turns out that in both cases (Wilson and domain-wall) the bulk of this decrease is given by the sail diagrams (present because of the covariant derivative in the operator)

They appear then to be responsible for the different behavior of Δ and c_{mix}

We can also take out as a lesson from the above considerations that the effect of improved gauge actions in the case of Δ and c_{mix} is substantially the same in the case of Wilson and domain-wall fermions

However, we should not overlook that, while it is true that improved gauge actions and fermionic improvement strongly suppress c_{mix} also in the case of the Wilson action, the numbers for domain-wall fermions are much smaller than the ones for Wilson fermions (apart from extreme choices of N_s and M)

Toward the borders

It is also interesting to see what happens when M is chosen dangerously close to 0 or 2, that is at the edges of its (at lowest order) allowed values

We have found as a general phenomenon that the exponential decay in N_s of the chiral violations slows down when approaching these borders

In some cases it can even happen that for a fixed M chosen very near to one of the edges the chiral violations increase with N_s, at least up to a certain point

One can observe this behavior already by looking at the tadpole diagrams, which can be calculated exactly

For $M \to 0$ they assume in fact the asymptotic expressions (in Feynman gauge)

$$\frac{4}{3}\, M\, T_d \left(N_s + 3 + \frac{2}{N_s} \right)$$

in the case of Σ_0, and

$$\frac{1}{3}\, M\, T_d \left(2N_s + 3 + \frac{1}{N_s} \right)$$

in the case of Σ_1

Toward the borders

For $M \to 2$ one just takes into account the fact that these tadpoles are respectively odd and even upon reflection around the point $M = 1$

We can then immediately see that for a given N_s the tadpoles tend to zero when M approaches 0 or 2

From the general numerical tables we can also see that the decay in N_s (at a constant M) of the tadpole of Σ_0 tends to become slower and slower as M nears the borders

The rate of decay probably vanishes at some point, and the tadpole then reaches the asymptotic regime $\frac{4}{3}\, M\, T_d \left(N_s + 3 + \frac{2}{N_s}\right)$, where actually the chiral violations grow with N_s (at fixed M)

Near $M = 2$, the residual mass also initially grows with N_s (at fixed M), before the exponential decay finally sets in

This can be seen already at $M = 1.9$

We can observe a similar behavior also for the other quantities considered in this article, which we have run for values of M close to $|1 - M| \sim 0.99$ and $|1 - M| \sim 0.999$

Toward the borders

Residual mass near $M = 2$, at $\beta = 5.2$ in Feynman gauge:

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$
1.91	0.043	0.094	0.120	0.127	0.120	0.107
1.92	0.034	0.085	0.115	0.128	0.128	0.120
1.93	0.026	0.074	0.106	0.125	0.132	0.130
1.94	0.018	0.061	0.094	0.117	0.130	0.134
1.95	0.011	0.048	0.080	0.104	0.121	0.132

Residual mass near $M = 2$, at $\beta = 5.2$ in Landau gauge:

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$
1.91	0.049	0.099	0.124	0.130	0.122	0.108
1.92	0.040	0.089	0.119	0.131	0.130	0.122
1.93	0.031	0.078	0.110	0.128	0.134	0.132
1.94	0.023	0.065	0.098	0.120	0.132	0.137
1.95	0.015	0.052	0.083	0.107	0.124	0.134

Toward the borders

Coefficient of $\bar{g}^2$ for Δ near $M = 2$, in Feynman gauge:

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$
1.94	3.699	3.151	2.488	1.915	1.467	1.125
1.95	4.246	3.858	3.220	2.594	2.063	1.637
1.96	4.885	4.753	4.220	3.592	2.995	2.475
1.97	5.630	5.883	5.586	5.063	4.474	3.897
1.98	6.494	7.301	7.442	7.225	6.823	6.332
1.99	7.495	9.073	9.945	10.377	10.524	10.481

Coefficient of $\bar{g}^2$ for Δ near $M = 2$, in Landau gauge:

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$
1.94	2.112	1.897	1.558	1.237	0.973	0.763
1.95	2.416	2.310	1.997	1.654	1.346	1.090
1.96	2.774	2.836	2.602	2.268	1.929	1.621
1.97	3.193	3.503	3.432	3.179	2.856	2.522
1.98	3.682	4.346	4.564	4.522	4.335	4.071
1.99	4.251	5.402	6.098	6.488	6.673	6.718

Toward the borders

For Δ and c_{mix} the rate of the exponential decay in N_s keeps decreasing when one approaches the borders, until a likely final disappearance, even though the absolute values of these quantities instead increase in M (for a fixed N_s) when $M \to 0$ or $M \to 2$

In fact, c_{mix} tends to diverge very fast when $M \to 0$

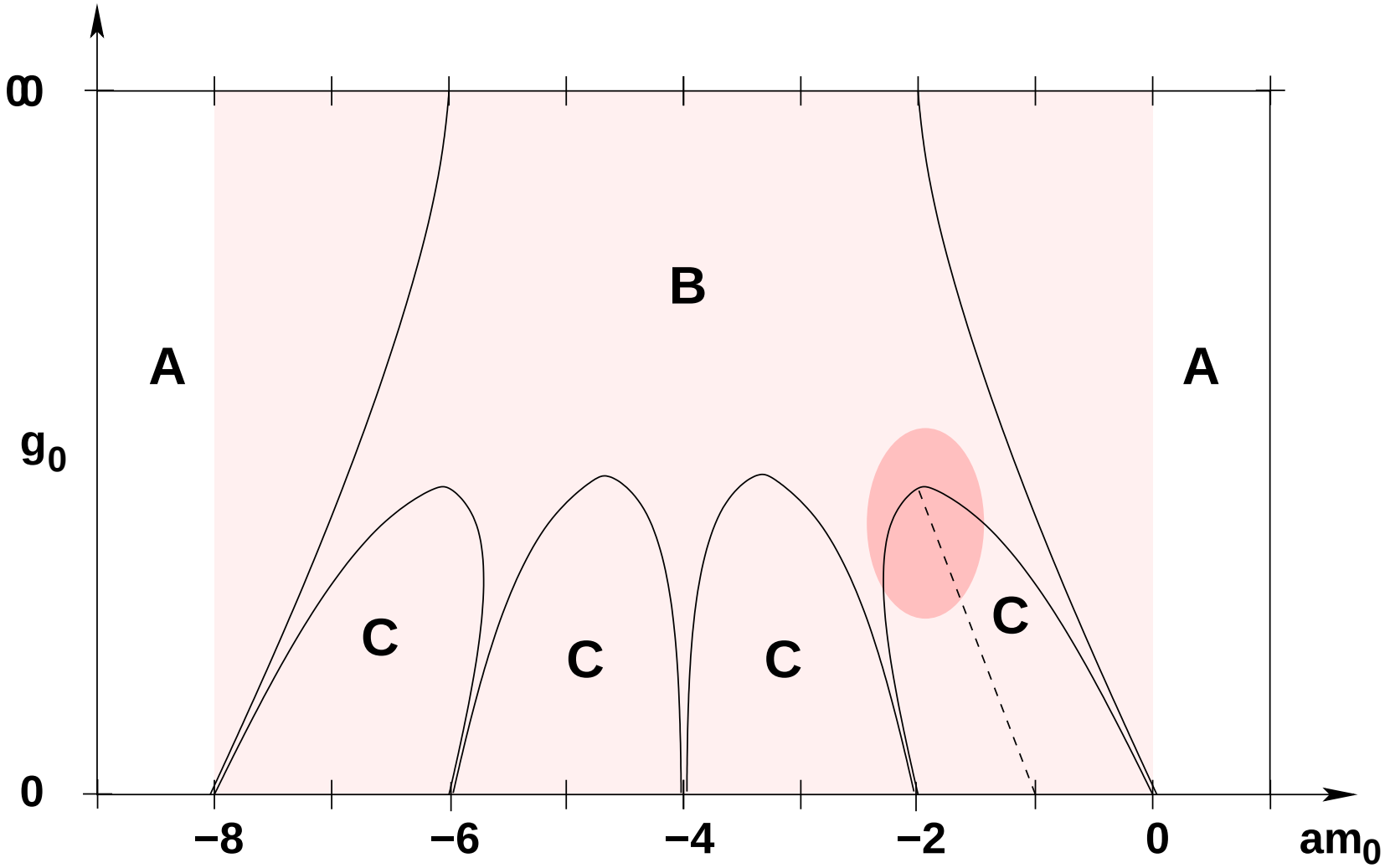
The slowing of the exponential decays of m_{res}, Δ and c_{mix} when one moves M such that $|1 - M| \to 1$ can be related to the decrease of the mobility edge λ_c towards zero in these extreme regions of M (Golterman & Shamir)

Choosing a value of M too close to 0 or 2 can then become dangerous, because the mobility edge has to remain well above zero in order to perform reliable Monte Carlo simulations, otherwise the restoration of chiral symmetry can become problematic

The fall of the mobility edge to zero signals the onset of the Aoki phase

It corresponds to moving, for g_0 not too large, from the rightmost C phase towards the B phase, which one eventually enters through one of the thin "fingers"

Toward the borders



from: Golterman and Shamir, PRD 68, 074501 (2003)

Toward the borders

All this can be inferred just from our 1-loop perturbative calculations of m_{res}, Δ and c_{mix}, and it would be put on a more solid basis if it were supported by investigations of the order parameter $\langle \overline{q}\gamma_5 q \rangle$

We can thus briefly sum up the behavior of the chiral violations when M changes as follows:

- the coefficient of their exponential decay is zero or rather small when M is in the vicinity of 1
- it then grows when M moves towards either 0 or 2
- it decreases again at last, and it gets rather small for $M \sim 0$ and $M \sim 2$

A remarkable thing for values of M very near the edges is that m_{res} and Δ happen to be smaller for small N_s than for larger values of N_s

This means that the chiral violations initially grow in N_s instead of decaying

This observed initial growth seems to be just a temporary one, before eventually m_{res} or Δ starts to decay as expected

The onset of this behavior takes place around $M = 1.9$

Toward the borders

The exponential decay which follows these regions of initial growth in N_s sets in at ever higher values of N_s when one gets nearer and nearer to 2 (for $M > 1.95$ the maximum of m_{res} is reached for $N_s > 28$!)

But it could well be that the value of N_s for which the chiral violations reach their maximum (at fixed M) is further and further shifted toward higher values of N_s until one cannot observe any exponential decay at all even for very large N_s, and possibly for all N_s, as in the exact asymptotic results of the tadpoles

In the latter case eventually a behavior like the one of the tadpoles sets in, that is the exponential decay disappears altogether and one can only see a steady increase with N_s

We can speak of a sort of suppression of the chiral violations for small N_s, which becomes stronger as M approaches the borders

It could be that in this case the density of eigenvalues or the radius of the modes in $m_{res} \sim R_e^4\, \rho_e(\lambda_c)\, \frac{\exp(-\lambda_c N_s)}{N_s} + R_l^4\, \rho_l(0)\, \frac{1}{N_s}$ change in such a way to produce this kind of effect, or that this equation breaks down in this region

Toward the borders

For the the difference of the vector and axial-vector currents we can observe a behavior similar to the one of the residual mass

For c_{mix} instead this does not seem to happen, that is we always observe a clear exponential-like decay, without any initial growth in N_s, at least for $N_s \geq 8$ and for the values of M that we have investigated, up to $2 - M = 10^{-10}$

Since c_{mix} apparently misses some of these effects, it could be that this quantity is described by a somewhat different formula

Similar phenomena are also seen when improved gauge actions are used

It would be interesting to carry out further studies about these phenomena which happen near the borders of M, and more investigations in the future could clarify these issues

We have however seen that already at one loop we can observe interesting features, in some cases corresponding to what is expected from general considerations which can be derived from other methods

Conclusions

- A disturbing finding of our calculations is the pathological behavior of renormalization factors, which are no longer gauge invariant , although the deviations from gauge invariance are not numerically large, even for values of M far away from 1
- Furthermore, the coefficients of the logarithmic divergences turn out to depend on N_s and M, and only for $N_s = \infty$ they are equal to the ones calculated in the continuum
- In particular, the anomalous dimension of the vector (as well as the axial-vector) current is not zero , and not constant in N_s and M

- Although it could be that perturbation theory is not the best tool to study chiral violations, we have still obtained some interesting results