# Domain Wall Fermions, Approximate Chiral Symmetry and Weak Matrix Elements

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# OUTLINE

- **Introduction**
- **Power divergent subtractions**
- **Contamination from residual chiral symmetry breaking?**
- **Transfer matrix analysis**
- **Conclusions.**

# Introduction

- **DWF should provide a refreshing "no-apologies" approach to standard lattice QCD.**
  - **Unitary.**
  - **As much chiral symmetry as needed(?)**
  - **The continuum limit becomes boring.**
  - **Bayesian statistics not required!**

- **Controlled chiral symmetry breaking allows important quantities to be accurately computed.**
  - $K^0 - \overline{K}^0$ **mixing:** $B_K$.
  - $\Delta S = 1$ **processes in** $K$ **decay:** $\Delta I = 1/2$ **rule,** $\epsilon'/\epsilon$.

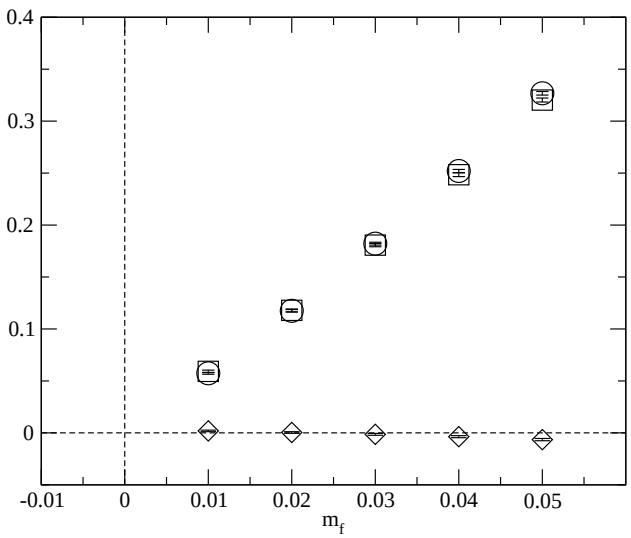
# But: DWF Chiral Symmetry is Not Perfect

- **The virtues of finite $L_s$**
  - Cost effective.
  - No diverging fermion forces during topology change.

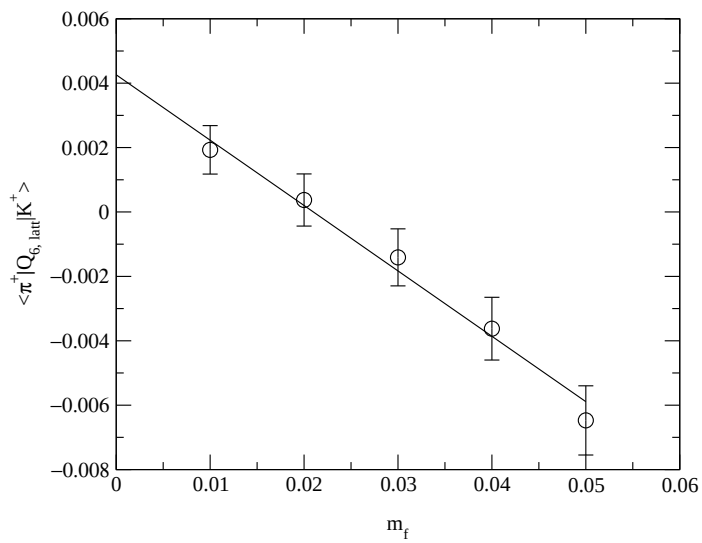
- **Problems of finite $L_s$**
  - Is wrong operator mixing sufficiently suppressed for $B_K$?
  - Does residual chiral symmetry breaking spoil $O_6$ subtraction?

# Power Divergent Subtraction from $O_6$

- **Matrix element $\langle\pi|O_6|K\rangle$**
  - **Contains divergent, unphysical $\langle\pi|\bar{s}d|K\rangle$ contribution.**
  - **Removed by subtraction specified by chiral perturbation theory.**
  - **How large is the systematic error on this 5% difference?**




(a) Two terms to be subtracted




(b) Result after subtraction

# Recall Leading Order Argument

- $O_6$, $(8,1)$ operator contains $(\overline{3},3)$ and $(3,\overline{3})$ counter terms.

$$
\begin{aligned}
(O_6)_{\text{finite}} = \; & \overline{s}\gamma^\mu(1-\gamma^5)d \sum_{q=u,d,s} \overline{q}\gamma^\mu(1+\gamma^5)q \\
& +\eta_0\overline{s}(1+\gamma^5)q_b\,(M_f)_{2,b} + \eta_0\overline{q}_a(1-\gamma^5)d\,(M_f^*)_{3,a} \\
& +\eta_1\overline{s}(1+\gamma^5)q_b\,(\Omega)_{2,b} + \eta_1\overline{q}_a(1-\gamma^5)d\,(\Omega^*)_{3,a}
\end{aligned}
$$

  - $\eta_0$ and $\eta_1$ are $\mathcal{O}(1/a^3)$.
  - $M_f$ is the usual $(3,\overline{3})$ fermion mass matrix.
  - $\Omega$ is a $(3,\overline{3})$ spinor matrix in the DWF $s$-derivative connecting the $s = L_s/2$ and $s = L_s/2-1$ slices.

$$
\begin{aligned}
& \overline{\Psi}_{x,L_s/2}(1+\gamma^5)\Psi_{x,L_s/2-1} + \overline{\Psi}_{x,L_s/2-1}(1-\gamma^5)\Psi_{x,L_s/2} \\
& \quad \to \overline{\Psi}_{x,L_s/2}(1+\gamma^5)\Omega\Psi_{x,L_s/2-1} + \overline{\Psi}_{x,L_s/2-1}(1-\gamma^5)\Omega^\dagger\Psi_{x,L_s/2}
\end{aligned}
$$

  - We expect $\eta_1 \propto e^{-\alpha L_s}$.

# Recall Leading Order Argument (con't)

- **With standard conventions:**

$$M_f \rightarrow \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$\Omega \rightarrow 1$$

- **We recover the usual result:**

$$(O_6)_{\text{finite}} = \bar{s}\gamma^\mu(1-\gamma^5)d \sum_{q=u,d,s} \bar{q}\gamma^\mu(1+\gamma^5)q$$
$$+\eta_0(m_d+m_s)\bar{s}d + \eta_0(m_d-m_s)\bar{s}\gamma^5 d + \eta_1\bar{s}d$$

- **Residual chiral symmetry breaking gives the irrelevant constant $\eta_1$.**

# Mixing of Order $M_f \times \Omega$?

- **A new counter term appears:**

$$\delta O_6 = \eta_2 \overline{s}(1+\gamma^5) q_{a'} \, (M_f^*)_{b,b'} (\Omega^*)_{c,c'} \epsilon^{2,b,c} \epsilon^{a',b',c'}$$
$$+ \eta_2 \overline{q}_{a'} (1-\gamma^5) d \, (M_f)_{b,b'} (\Omega)_{c,c'} \epsilon^{3,b,c} \epsilon^{a',b',c'}.$$

- **For conventional values for $M_f$ and $\Omega$:**

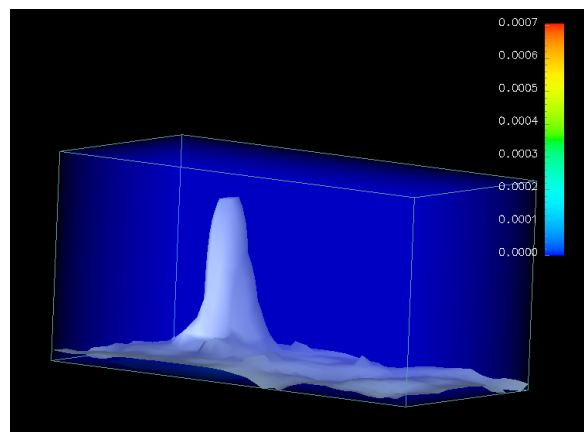
$$\delta O_6 = \eta_2 (m_s - m_d) \overline{s} \gamma^5 d + \eta_2 (m_d + m_s + 2m_u) \overline{s} d.$$

- **Trouble?**
  - $\eta_2 \approx 0.003$ **enhanced** $20\times$ **gives 6% correction to physical slope!**
  - **Could discover and remove using** $m_u$ **dependence?**

- **Ruled out by perturbative** $U_L(3) \times U_R(3)$ **symmetry:**
  **Even powers of** $\Omega$ **required.**

# Mixing of Order $M_f \times \Omega \times \Omega$?

- **If $\Omega^2$ term were of order $m_{\rm res}^2$ we could drop it: $(0.003)^2$ enhanced $20\times$ gives 0.02% effect.**
- **Residual mass reflects two effects:**
  - **Eigenstates of $H_T$ above the mobility edge: $m_{\rm res} \propto \frac{1}{L_s} e^{-\lambda_c L_s}$.**
    $\Omega^2 \propto \left(\frac{1}{L_s} e^{-\lambda_c L_s}\right)^2$
  - **Eigenstates of $H_T$ below the mobility edge: $m_{\rm res} \propto \frac{\rho(0)}{L_s}$.**
    $\Omega^2 \propto \frac{\rho(0)}{L_s}$




[ *Peter Boyle*]

- **Thus, $\Omega^2$ term might $\mathcal{O}(m_{\rm res})$!**

# Recall Transfer Matrix Description of DWF

- **A DWF Green's function of polynomials of left- and right-handed fields can be written (for $m_f = 0$):**

$$\langle \mathcal{O}_L[P_L q, \overline{q} P_R]\, \mathcal{O}_R[P_R q, \overline{q} P_L] \rangle_{L_s} = Z(L_s)\, \langle 0_S | \mathcal{O}_L[P_L \hat{a}, -\hat{a}^\dagger P_R]\, T^{L_s}\, \mathcal{O}_R[P_R \hat{a}, \hat{a}^\dagger P_L] | 0_S \rangle.$$

- **The Fock-space transfer matrix $T$ is given by:**

$$T = e^{-\hat{a}^\dagger H_T \hat{a}}$$

**where** $\hat{a} = \begin{pmatrix} B^{\frac{1}{2}}\, (\hat{q})_R \\ B^{\frac{1}{2}}\, (\hat{q})_L \end{pmatrix}, \quad \hat{a}^\dagger = \begin{pmatrix} -(\hat{\overline{q}})_R B^{\frac{1}{2}}, & (\hat{\overline{q}})_L B^{\frac{1}{2}} \end{pmatrix}$ **and**

$$e^{-H_T} = \begin{pmatrix} B + C B^{-1} C^\dagger & C B^{-1} \\ B^{-1} C^\dagger & B^{-1} \end{pmatrix}$$

# Expand $T^{L_s}$ for Large $L_s$

- **Introduce positive and negative energy eigenstates of $H_T$:**

$$H_T \phi^+_{k^+} = E^+_{k^+} \phi^+_{k^+} \quad 1 \le k^+ \le N^+$$

$$H_T \phi^-_{k^-} = -E^-_{k^-} \phi^-_{k^-} \quad 1 \le k^- \le N^-$$

- **We can then expand $\hat{a}$ and $\hat{a}^\dagger$ in terms of the $\phi^\pm$ basis and write:**

$$\begin{aligned}
\hat{T} &= \frac{\exp\left\{-\hat{a}^\dagger H \hat{a}\right\}}{\exp\left\{+\sum_{k^-=1}^{N^-} E^-_{k^-}\right\}} \\
&= \exp\left\{-\sum_{k^+=1}^{N^+} E^+_{k^+} \hat{o}^\dagger_{k^+} \hat{o}_{k^+} - \sum_{k^-=1}^{N^-} E^-_{k^-} \hat{p}^\dagger_{k^-} \hat{p}_{k^-}\right\}.
\end{aligned}$$

$$\hat{T}^{L_s} \approx |0_H\rangle\langle 0_H| + \sum_{k^+=1}^{N^+} e^{-E^+_{k^+} L_s} \hat{o}^\dagger_{k^+} |0_H\rangle\langle 0_H| \hat{o}_{k^+} + \sum_{k^-=1}^{N^-} e^{-E^-_{k^-} L_s} \hat{p}^\dagger_{k^-} |0_H\rangle\langle 0_H| \hat{p}_{k^-} + \ldots.$$

# Introduce 3 Flavors and Midpoint Phase Rotation

- **Single-flavor $T^{L_s}$ becomes product of three:**

$$\hat{T}^{L_s} \approx$$

$$\left( |0_H\rangle\langle 0_H| + \sum_{k^+=1}^{N^+} e^{-E^+_{k^+} L_s} (\hat{o}^u_{k^+})^\dagger |0_H\rangle\langle 0_H| \hat{o}^u_{k^+} + \sum_{k^-=1}^{N^-} e^{-E^-_{k^-} L_s} (\hat{p}^u_{k^-})^\dagger |0_H\rangle\langle 0_H| \hat{p}^u_{k^-} \right)$$

$$\cdot \left( |0_H\rangle\langle 0_H| + \sum_{k^+=1}^{N^+} e^{-E^+_{k^+} L_s} (\hat{o}^d_{k^+})^\dagger |0_H\rangle\langle 0_H| \hat{o}^d_{k^+} + \sum_{k^-=1}^{N^-} e^{-E^-_{k^-} L_s} (\hat{p}^d_{k^-})^\dagger |0_H\rangle\langle 0_H| \hat{p}^d_{k^-} \right)$$

$$\cdot \left( |0_H\rangle\langle 0_H| + \sum_{k^+=1}^{N^+} e^{-E^+_{k^+} L_s} (\hat{o}^s_{k^+})^\dagger |0_H\rangle\langle 0_H| \hat{o}^s_{k^+} + \sum_{k^-=1}^{N^-} e^{-E^-_{k^-} L_s} (\hat{p}^s_{k^-})^\dagger |0_H\rangle\langle 0_H| \hat{p}^s_{k^-} \right)$$

- **The midpoint rotation $\Omega$ introduces:**

$$T^{L_s} \to T^{L_s/2}\, e^{i \sum_q \phi_q \left\{ \sum_{k^+=1}^{N^+} (\hat{o}^q_{k^+})^\dagger \hat{o}^q_{k^+} - \sum_{k^-=1}^{N^-} (\hat{p}^q_{k^-})^\dagger \hat{p}^q_{k^-} \right\}}\, T^{L_s/2}$$

**for the case** $\quad \Omega = \begin{pmatrix} \Omega_u & 0 & 0 \\ 0 & \Omega_d & 0 \\ 0 & 0 & \Omega_s \end{pmatrix} = \begin{pmatrix} e^{i\phi_u} & 0 & 0 \\ 0 & e^{i\phi_d} & 0 \\ 0 & 0 & e^{i\phi_s} \end{pmatrix}$

# Examine $M_f \times \Omega \times \Omega$ contribution to $O_6$

- **Possible counter terms:**

$$\delta O_6 = \overline{s}(1+\gamma^5)q_a\,(\Omega)_{2,a}\eta_3 \mathrm{tr}(M_f^\dagger \Omega) + \overline{q}_a(1-\gamma^5)d\,(\Omega^*)_{2,a}\eta_3 \mathrm{tr}(M_f \Omega^\dagger)$$

- **Extract the allowed 2nd order polynomial in $\Omega_{ij}$:**

$$\Omega_{uu}\Omega_{dd} + \Omega_{uu}\Omega_{ss} + \Omega_{dd}\Omega_{ss}$$

- **Substitute conventional values for $M_f$ and $\Omega$:**

$$\delta O_6 = \eta_3(m_s - m_d)\overline{s}\gamma^5 d + \eta_3(m_d + m_s + 2m_u)\overline{s}d.$$

- **Should be subtracted!**
  - **Reversed sign implies standard subtraction fails.**
  - **Must fit to $m_u$ dependence and remove.**
  - **Size of term in question:** $0.5 \cdot 0.003 \times 20 \approx 3\%$.

**$O_6$ subtraction can be performed**

**with systematic errors $<< 1\%$.**