Domain Wall Fermions at Ten Years

RIKEN BNL Research Center Workshop & Symposium...

March 15-17, 2007 at Brookhaven National Laboratory

The Old Fart Talk:

- Tedious Reminiscences
- Narrow Survey of the Current Situation
- Pretentious Prognostications about the Future
- Ideas of the Past Week

I. Tedious Reminiscences: Anomalies and DWF

- Fujikawa & the sociology of anomalies in the early 1980's

- No "anomalies" on the lattice: only exact symmetries and explicitly broken symmetries -> fermion doubling, so that gauge interactions are vector-like.

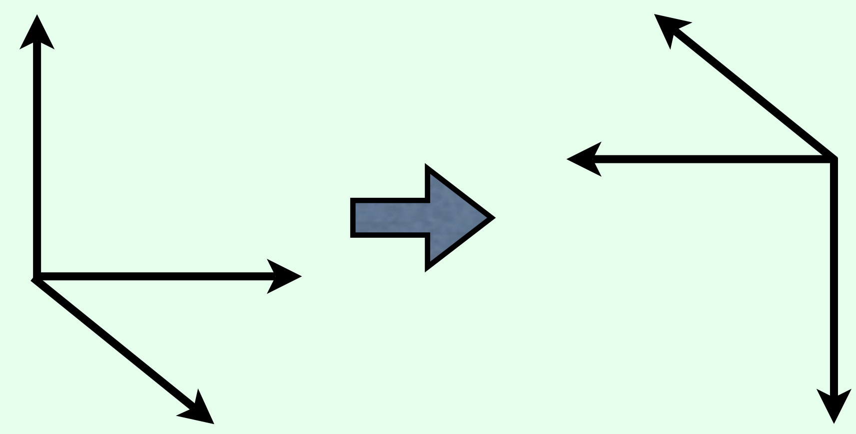
- Chiral gauge theories: how is a lattice supposed to know what to do?

- The game in the 80's for constructing chiral gauge theories: break all the anomalous symmetries explicitly, gauge the anomaly-free symmetry, look for a point on the phase diagram which has the target theory. Yikes!

The parity anomaly in even space dimensions

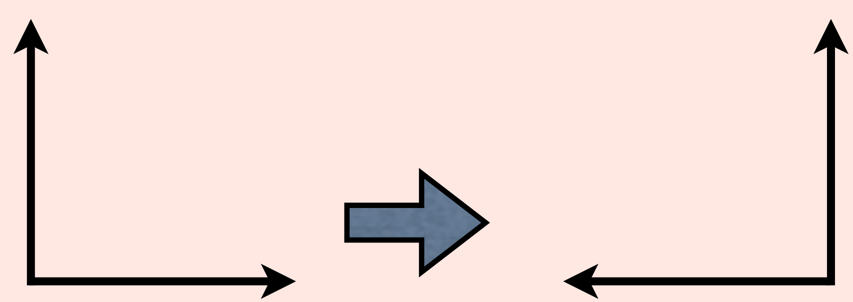
First: note that a Dirac mass violates parity

Parity in (3+1)d:



$$x \to -x,\; y \to -y,\; z \to -z$$

Parity in (2+1)d:



$$x \to -x,\; y \to +y$$

Parity in (3+1)d:

$$\Psi(\mathbf{x},t) \to \gamma_0 \Psi(\tilde{\mathbf{x}},t)\,, \quad \bar{\Psi}(\mathbf{x},t) \to \bar{\Psi}(\tilde{\mathbf{x}},t)\gamma_0\,, \quad m\bar{\Psi}\Psi \to m\bar{\Psi}\Psi$$

Parity in (2+1)d:

$$\Psi(\mathbf{x},t) \to \gamma_1 \Psi(\tilde{\mathbf{x}},t)\,, \quad \bar{\Psi}(\mathbf{x},t) \to \bar{\Psi}(\tilde{\mathbf{x}},t)\gamma_1\,, \quad m\bar{\Psi}\Psi \to -m\bar{\Psi}\Psi$$

Integrating out a massive fermion in (2+1)d induces a parity violating Chern-Simons term:

$$\propto \frac{m}{|m|}\epsilon_{abc}A_a\partial_b A_c \qquad \text{(Abelian)}$$

A remarkable expression:

$$\propto \frac{m}{|m|}\epsilon_{abc}A_a\partial_b A_c$$

IR: <u>Anomaly</u> - a massless fermion has to pick a value for m/|m| and break parity

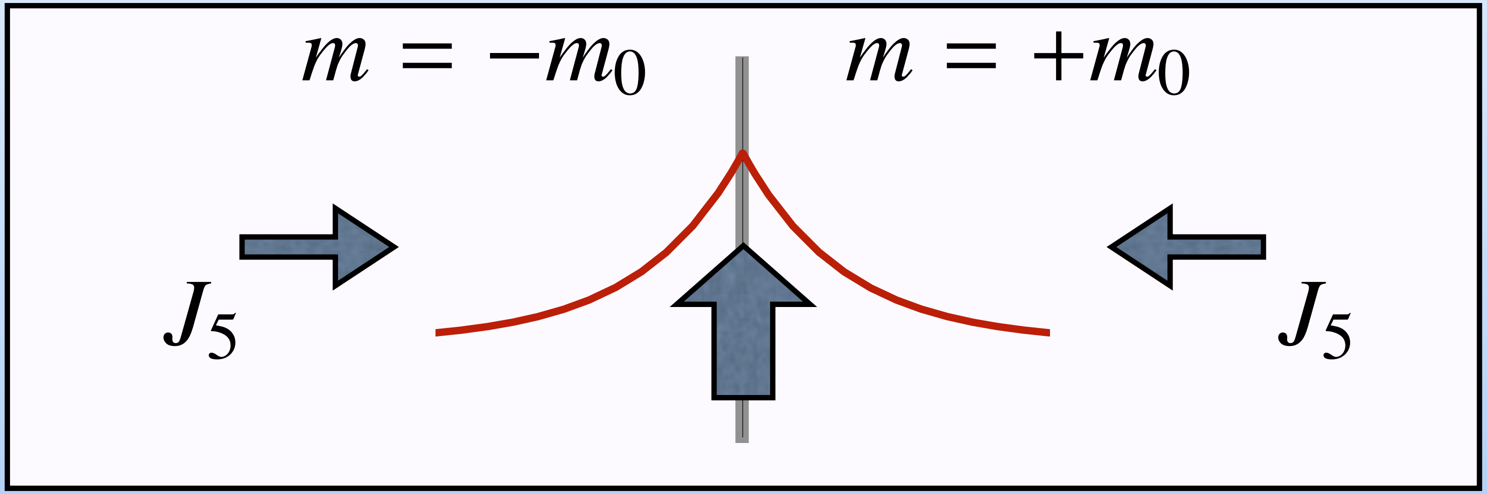
UV: Non-decoupling - same effect no matter how heavy the fermion is

Coefficient is quantized

Clearly related to anomalies in odd space dimensions

Formal connection between anomalies in different dimensions was made by Zumino & collaborators -- "descent equations";

Callan & Harvey came up with a physical model for the relation between anomalies in different dimensions, using topological defects.



Why was the Callan-Harvey paper so exciting?

Because it was an example of a theory with accidental low energy chiral symmetry:

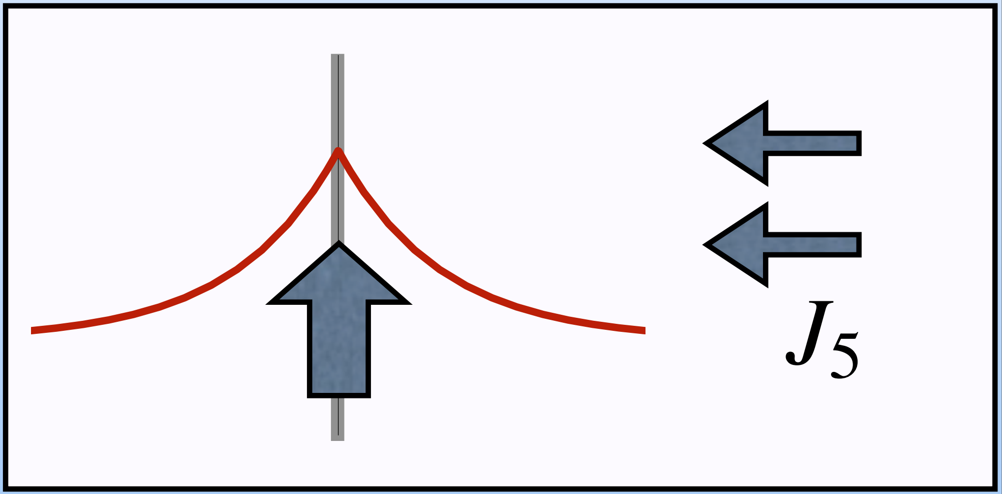
- No relevant operator could spoil the chiral symmetry, since LH and RH modes were physically separated by P violating bulk mass term
- The UV theory possessed no chiral symmetry at all
- Anomalies were explicitly accounted for.

Applicable to lattice QCD? No interest in 1988.

Revived DWF as a theory for chiral gauge theories (1992). Found that CH scenario works perfectly on the lattice, with chiral symmetry violating Wilson operators in the bulk.

Problem: on finite lattice need bizarre gauge dynamics or explicit gauge symmetry violation so that gauge fields only see chiral mode instead of vector.

Lattice exposes CH error: Anomalous current flow from one side (with Golterman, Jansen (1992))

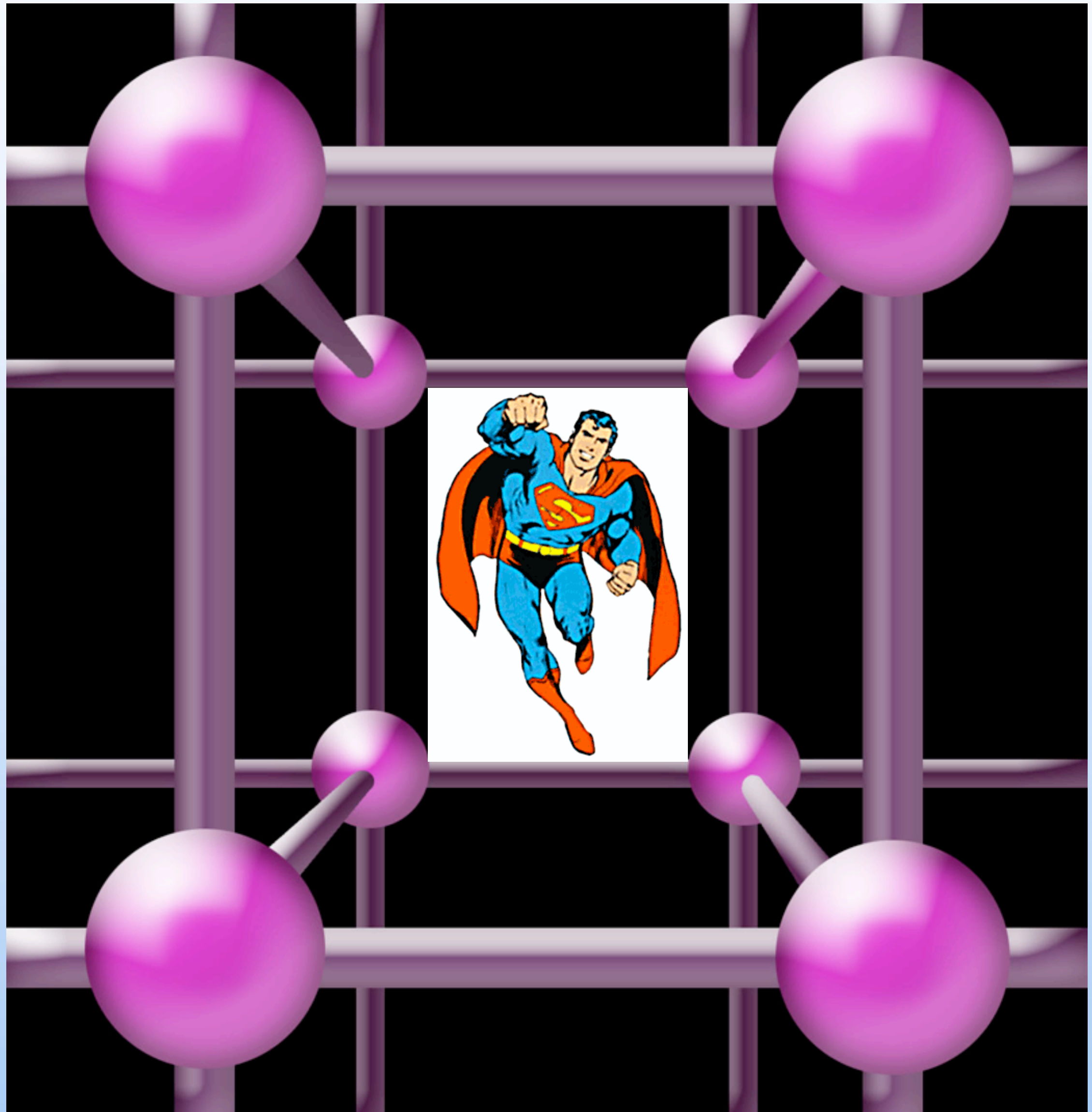


Subsequently:

- Shamir develops DWF as a tool for QCD
- Narayanan & Neuberger develop the effective 4d theory, leading to the overlap operator.
- GW equation? Pretty, but no new insights.

II. Narrow Survey of the Current Situation

(Chirality and lattice SUSY)



One of the challenges for lattice SUSY is the fact that SUSY theories typically possess chiral symmetries.

- N=1 SYM: Gauge + gaugino fields, classical U(1) (broken by anomalies)

- N=4 SYM: Gauge + 4 gauginos + 6 scalrs, SU(4) R-symmetry

Need accidental SUSY + accidental chiral symmetry...compatible??

Simplest case: N=1 SYM

$$\mathcal{L} = \bar{\lambda}\, i\bar{\sigma}^m D_m\, \lambda - \frac{1}{4} v_{mn} v^{mn}$$

The only relevant operator that can be added to this Lagrangian is a gaugino mass term:

$$\delta\mathcal{L} = m\lambda\lambda + h.c.$$

The gaugino mass breaks:

- Supersymmetry
- Z_{2N} chiral symmetry (the R-symmetry)

...so imposing a Z_{2N} chiral symmetry on the theory forbids the gaugino mass, and the IR theory is *accidentally* supersymmetric! (D.K., 1984) ☜ *(my first paper!)*

- SU(N) gauge theory with one Weyl fermion λ in adjoint representation
- Only relevant operator is a fermion mass:

$$\cancel{Z_{2N}} \quad m\,\lambda\lambda \quad \cancel{\text{SUSY}}$$

Violates both SUSY & discrete Z_{2N} chiral symmetry

- Realize Z_{2N} symmetry on the lattice, and SUSY follows "accidentally"

Lattice SUSY Yang Mills:

Need to construct a lattice theory with:

- SU(N) gauge symmetry
- 2-component adjoint fermion
- Z_{2N} discrete chiral symmetry

...but *not* lattice supersymmetry

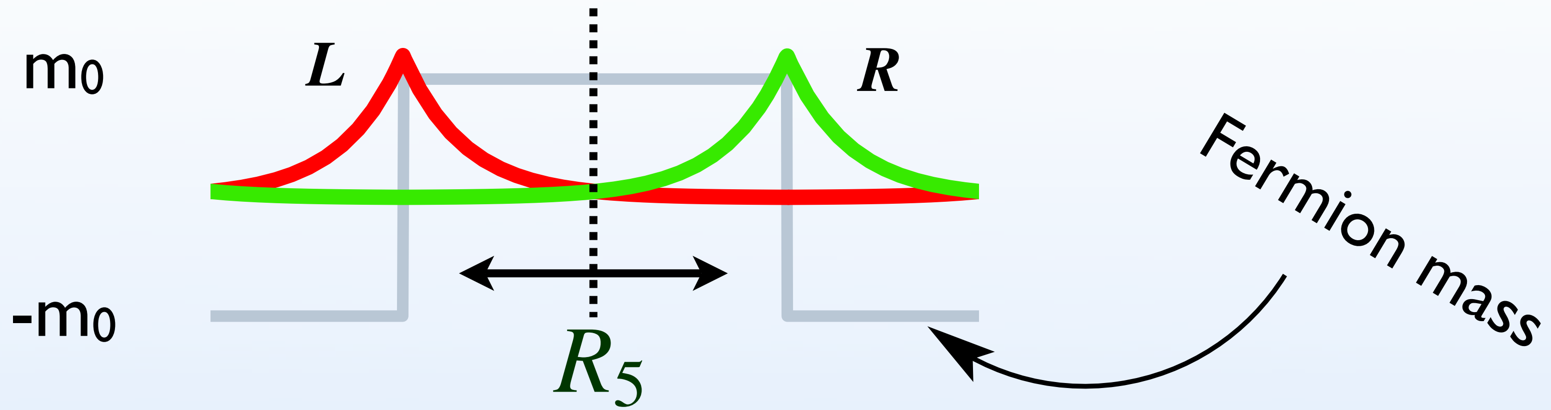
Chiral lattice fermion

Majorana condition

Neuberger developed overlap theory (1998)

D.K., M. Schmaltz (2000): used domain wall fermions to construct a lattice theory for supersymmetric YM theory

Domain wall fermion



$$\Psi = \begin{pmatrix} \alpha \\ \bar{\beta} \end{pmatrix} \qquad \bar{\Psi} = \begin{pmatrix} \bar{\alpha}^T & \beta^T \end{pmatrix}$$

= massless 4d Dirac fermion

5d fermion

Zero-mode components

"Majorana" constraint: $\Psi = R_5 C \bar{\Psi}^T \longrightarrow \alpha = \beta$

$$\Psi = R_5 C \bar{\Psi}^T$$

only possible because fermion is in a real representation of the gauge group (adjoint)

$$\bar{\Psi}\, D\!\!\!\!/\, \Psi \longrightarrow \Psi^T (R_5^T C^T D\!\!\!\!/\,)\Psi$$

$$\det(D\!\!\!\!/\,) \longrightarrow \mathrm{Pf}(R_5^T C^T D\!\!\!\!/\,)$$

Pfaffian

The Pfaffian is an analytic square root of the Dirac operator.
No fermion sign problem in the continuum.
No fermion sign problem on the lattice (Neuberger, Kikukawa)

But what about SUSY theories with bigger chiral symmetries and scalars?

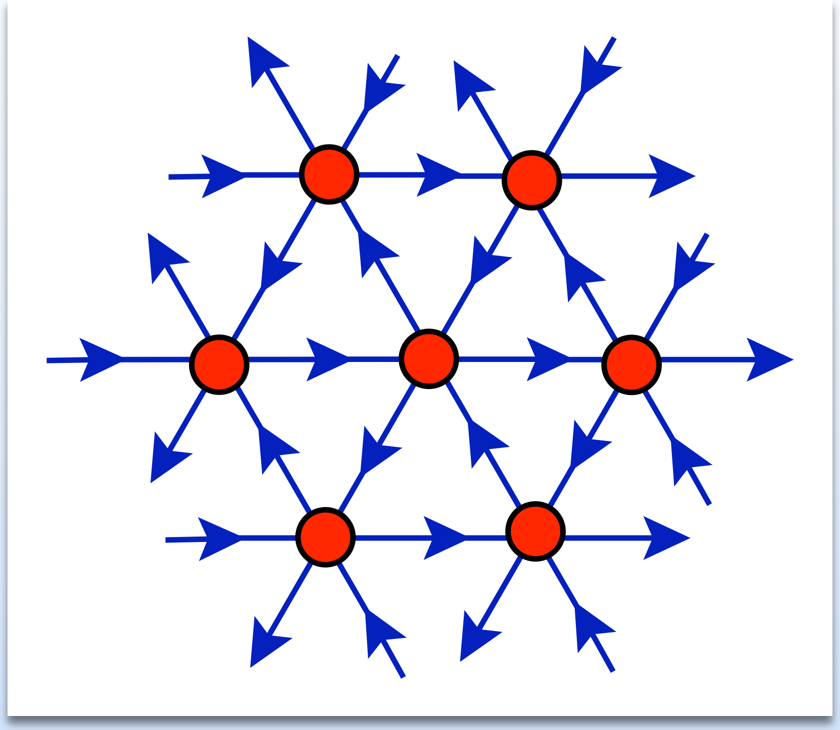
With scalars, need some exact SUSY on the lattice to protect against radiative corrections

Need DW scalars? Don't know how.

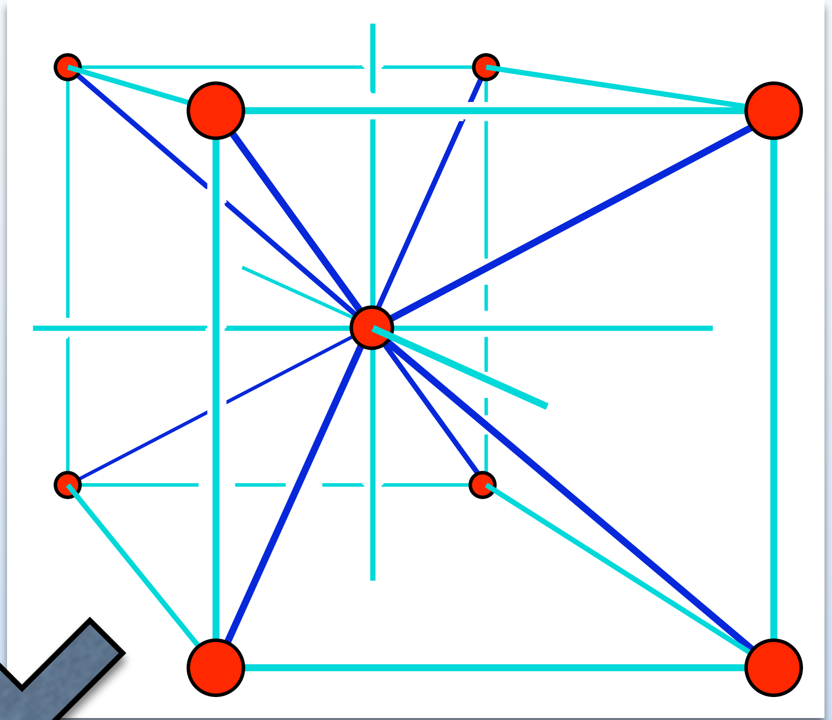
Still, SUSY theories with large, accidental chiral symmetries can be constructed...

Example: target = SYM with 16 supercharges

d=2



d=3



Site: 2 fermions, 2 real bosons

Light blue link: 2 fermions

Dark blue link: 2 real bosons, 2 fermions

Kaplan and Unsal, hep-lat/0503039

The secret weapon??

Staggered

Fermions!

Staggered fermions as Dirac-Kahler/geometric fermions:

Recall basics of p-forms

$$F = f + f_\mu \, dx_\mu + \frac{1}{2!} f_{[\mu\nu]} \, dx_\mu \wedge dx_\nu + \dots$$

0-form 1-form 2-form +...

All f's are functions of x. Two types of differential operators:

$$d\,F = \partial_\mu f \, dx_\mu + \frac{1}{2!} \partial_\mu f_\nu \, dx_\mu \wedge dx_\nu + \dots$$

curl: p -> p+1

$$\delta\,F = \partial_\mu f_\mu + \partial_\mu f_{[\mu\nu]} dx_\nu + \dots$$

div: p -> p-1

The Dirac equation can be formulated in terms of p-forms, for the right number of flavors (Kahler)

Example: d=2, 2 flavors of Dirac fermion.
Write as a 2x2 matrix, and then expand in the gamma matrix basis:

$$\Psi_{\alpha i} = \left[\psi + \psi_\mu \gamma_\mu + \tfrac{1}{2}\psi_{[12]}(\gamma_1\gamma_2 - \gamma_2\gamma_1)\right]_{\alpha i}$$

- Under $SO(2)_L \times U(2)_f$ symmetry, the fermion transforms as $\Psi \to \Lambda \Psi U^\dagger$

- The components $\psi,\ \psi_\mu,\ \psi_{[\mu\nu]}$ transform as **tensors** under the diagonal subgroup

$$SO(2) \subset SO(2)_L \times U(2)_f$$

Now that the fermions are classified as tensors, instead of spinors, they have a natural geometric interpretation when latticizing them:

ψ	0-forms	$\Rightarrow$	sites
ψ_μ	1-forms	$\Rightarrow$	links
$\psi_{[\mu\nu]}$	2-forms	$\Rightarrow$	plaquettes

Furthermore, the d and δ operations have natural interpretations as lattice difference operators.

- Latticized Dirac-Kahler fermions are equivalent to staggered fermions.

- Staggered fermions have a well defined geometric significance

- Point group of the lattice lies in a nontrivial subgroup of (Lorentz x Flavor)

- Key to SUSY lattices: staggered scalars, stagger gauge bosons. Nontrivial lattice symmetry repps can become quite different continuum reps

Lorentz x Flavor

Diagonal subgroup

Discrete lattice symmetry

Lorentz x Flavor

Diagonal subgroup

Gauge bosons

Discrete lattice symmetry

Lorentz x Flavor

Diagonal subgroup

Fermions

Discrete lattice symmetry

Lorentz x Flavor

Diagonal subgroup

Scalars

Discrete lattice symmetry

Staggered/Dirac-Kahler fermions are a natural for lattice SUSY...but too restrictive: only certain SUSY theories have the compatible numbers of fermions.

Would like to do N=1 SUSY QCD with N_f flavors! But currently can't.

One could conceivably engineer the desired chiral symmetry with DWF...but how to preserve the lattice SUSY? An open question.

III. Pretentious Prognostications and My Thoughts of Last Week

(Lattice fermion challenges continued)

For me - most exciting challenge for the future: properties of dense matter

(fermion sign problem)

Here and now: An interesting direction?

2 nucleons in box (NPLQCD)

A fundamental obstacle for more nucleons?

$$O(x,y) = u(x_1)\cdots u(x_m)d(y_1)\cdots d(y_n)$$

Correlator $O(z_i)O^\dagger(z_f)$ requires (m! x n!) propagator contractions?!

Triton: 5! x 4! = 2,880

^{4}He: 6! x 6! = 518,400

^{238}U~ 10^{1516}

No! Calculating <det[$D^{-1}(x_i,x_j)$]> for N fermions...difficulty scales as N^3, not N!

Can this observation help us with other aspects of dense fermions?

Lattice fermions are wonderful aggravating, puzzling, maddening, delightful objects...years of fun and games with them still lie before us.