Schrödinger Functional Boundary Conditions for Domain Wall Quarks

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The Schrödinger functional, a short reminder

The Schrödinger functional is the functional integral on a hyper cylinder,

$$\mathcal{Z} = \int_{\text{fields}} \mathrm{e}^{-S}$$

with periodic boundary conditions in spatial directions and Dirichlet conditions in time. With $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$,

$$\begin{aligned} P_+\psi(x)|_{x_0=0} &= \rho & \qquad P_-\psi(x)|_{x_0=T} &= \rho' \\ \bar{\psi}(x)P_-|_{x_0=0} &= \bar{\rho} & \qquad \bar{\psi}(x)P_+|_{x_0=T} &= \bar{\rho}', \\ A_k(x)|_{x_0=0} &= C_k & \qquad A_k(x)|_{x_0=T} &= C_k' \end{aligned}$$

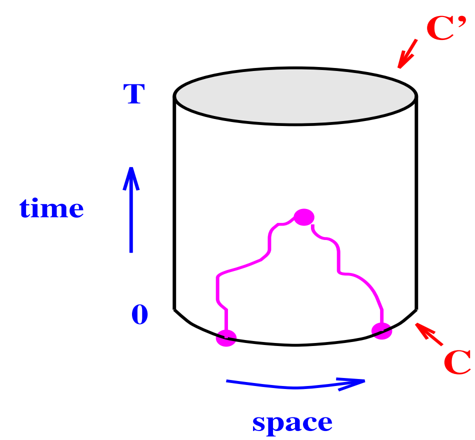
Correlation functions are then defined as usual

$$\langle O \rangle = \left\{ Z^{-1} \int_{\text{fields}} O\, \mathrm{e}^{-S} \right\}_{\rho=\rho'=0;\ \bar{\rho}=\bar{\rho}'=0}$$

O may contain quark boundary fields

$$\zeta(\mathbf{x}) \equiv P_{-}\zeta(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}(\mathbf{x})}$$

$$\bar{\zeta}(\mathbf{x}) \equiv \bar{\zeta}(\mathbf{x})P_{+} = -\frac{\delta}{\delta\rho(\mathbf{x})}$$



- N.B.: The fermionic boundary values $\rho, \bar{\rho}$ and $\rho', \bar{\rho}'$ act as external sources, and are always set to zero in correlation functions

$\Rightarrow$ alternatively, one can directly identify

$$\begin{aligned} \zeta &= P_{-}\psi(x)|_{x_0=0}, \qquad \bar{\zeta} = \bar{\psi}(x)P_{+}|_{x_0=0} \\ \zeta' &= P_{+}\psi(x)|_{x_0=T}, \qquad \bar{\zeta}' = \bar{\psi}(x)P_{-}|_{x_0=T} \end{aligned}$$

SF boundary conditions for Ginsparg-Wilson type quarks

Main goal: apply SF renormalization schemes to Ginsparg-Wilson type quarks (overlap, DWF,...);

- use cheaper regularization to obtain universal running (of coupling, quark masses, composite operators) in the continuum limit

- use GW regularisation of SF to match SF scheme at a low energy scale

$\Rightarrow$ mainly need SF with massless quarks!

Proposed solutions:

- direct orbifold construction (Taniguchi '04)

- "symmetries plus universality" (Lüscher '06)

- here: orbifold construction of chirally rotated SF (see also Taniguchi '06)

SF boundary conditions and chiral rotations

Consider isospin doublets χ' and $\bar{\chi}'$ satisfying homogeneous SF boundary conditions ($P_\pm = \frac{1}{2}(1 \pm \gamma_0)$),

$$P_+ \chi'(x)|_{x_0=0} = 0, \qquad P_- \chi'(x)|_{x_0=T} = 0,$$
$$\bar{\chi}'(x) P_-|_{x_0=0} = 0, \qquad \bar{\chi}'(x) P_+|_{x_0=T} = 0.$$

perform a chiral field rotation,

$$\chi' = \exp(i\alpha\gamma_5\tau^3/2)\chi, \qquad \bar{\chi}' = \bar{\chi}\exp(i\alpha\gamma_5\tau^3/2),$$

the rotated fields satisfy chirally rotated boundary conditions

$$P_+(\alpha)\chi(x)|_{x_0=0} = 0, \qquad P_-(\alpha)\chi(x)|_{x_0=T} = 0,$$
$$\bar{\chi}(x)\gamma_0 P_-(\alpha)|_{x_0=0} = 0, \qquad \bar{\chi}(x)\gamma_0 P_+(\alpha)|_{x_0=T} = 0,$$

with the projectors

$$P_\pm(\alpha) = \tfrac{1}{2}\left[1 \pm \gamma_0 \exp(i\alpha\gamma_5\tau^3)\right].$$

Special cases of $\alpha = 0, \pi/2$:

$$P_\pm(0) = P_\pm, \qquad P_\pm(\pi/2) \equiv Q_\pm = \tfrac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3),$$

The chiral rotation introduces a mapping between renormalised correlation functions

$$\langle O[\chi,\bar\chi]\rangle_{P_\pm} = \langle \tilde O[\chi,\bar\chi]\rangle_{P_\pm(\alpha)}$$

with
$$\tilde O[\chi,\bar\chi] \quad = \quad O\left[\exp(i\alpha\gamma_5\tau^3/2)\chi, \bar\chi \exp(i\alpha\gamma_5\tau^3/2\right],$$

Boundary quark fields are included by replacing

$$\bar\zeta(\mathbf{x}) \leftrightarrow \bar\chi(0,\mathbf{x})P_+ \qquad \zeta(\mathbf{x}) \leftrightarrow P_-\chi(0,\mathbf{x})$$

<u>Note</u>: The chirally rotated framework is only chosen for technical convenience. Using the above dictionary any standard SF correlator can be easily translated to this rotated framework (for an even number of fermions)

Orbifold technique

Orbifold techniques have been used to implemement the standard SF conditions for Ginsparg-Wilson quarks (Taniguchi '04). Here:

- start with standard lattice action for a single massless quark flavour

$$S_f[\psi,\bar{\psi},U] = a^4 \sum_x \bar{\psi}(x) D_N \psi(x), \quad D_N = 1 - A(A^\dagger A)^{-1/2}, \quad A = 1 - aD_W$$

  where

$$\psi(x_0 + 2T, \mathbf{x}) = -\psi(x), \qquad \bar{\psi}(x_0 + 2T, \mathbf{x}) = -\bar{\psi}(x)$$

- introduce a reflection ($R^2 = id$)

$$R : \psi(x) \rightarrow i\gamma_0\gamma_5\psi(-x_0, \mathbf{x}), \qquad \bar{\psi}(x) \rightarrow \bar{\psi}(-x_0, \mathbf{x}) i\gamma_0\gamma_5$$

- the gauge field is extended to $[-T, T]$ and then periodically continued:

$$U_k(-x_0, \mathbf{x}) = U_k(x_0, \mathbf{x}), \qquad U_0(-x_0 - a, \mathbf{x})^\dagger = U_0(x)$$

- Decompose fields into even and odd with respect to R,

$$R\psi_\pm = \pm\psi_\pm, \qquad R\bar{\psi}_\pm = \pm\bar{\psi}_\pm$$

- even/odd fields satisfy the boundary conditions at $x_0 = 0$

$$(1 \mp i\gamma_0\gamma_5)\psi_\pm(0,\mathbf{x}) = 0 \qquad \bar{\psi}_\pm(0,\mathbf{x})(1 \mp i\gamma_0\gamma_5) = 0$$

- and with complementary projectors at $x_0 = T$, due to antiperiodicity:

$$(1 \pm i\gamma_0\gamma_5)\psi_\pm(T,\mathbf{x}) = 0 \qquad \bar{\psi}_\pm(T,\mathbf{x})(1 \pm i\gamma_0\gamma_5) = 0$$

- $[D_N, R] = 0$

$$\Rightarrow S_f[\psi,\bar{\psi},U] = S_f[\psi_+ + \psi_-, \bar{\psi}_+ + \bar{\psi}_-, U] = S_f[\psi_+,\bar{\psi}_+,U] + S_f[\psi_-,\bar{\psi}_-,U]$$

$\Rightarrow$ the functional integral factorises!

- interpret even and odd fields as quark flavours

$$\chi = \sqrt{2}\begin{pmatrix}\psi_- \\ \psi_+\end{pmatrix}, \qquad \bar{\chi} = \sqrt{2}\left(\bar{\psi}_- \quad \bar{\psi}_+\right)$$

- functional integral:

$$\int \prod_{-T\leq x_0<T} \mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x)\mathrm{e}^{-S_f[\psi,\bar{\psi},U]} \propto \int \prod_{0\leq x_0\leq T} \mathrm{d}\chi(x)\mathrm{d}\bar{\chi}(x)\mathrm{e}^{-\frac{1}{2}S_f[\chi,\bar{\chi},U]}$$

- equivalent to theory in the interval $[0,T]$ with boundary conditions

$$Q_+\chi(x)|_{x_0=0} = 0, \qquad Q_-\chi(x)|_{x_0=T} = 0,$$
$$\bar{\chi}(x)Q_+|_{x_0=0} = 0, \qquad \bar{\chi}(x)Q_-|_{x_0=T} = 0$$

The dynamical field variables are

$$Q_-\chi(0,\mathbf{x}), \qquad \chi(x)|_{0<x_0<T}, \qquad Q_+\chi(T,\mathbf{x})$$

and

$$\bar{\chi}(0,\mathbf{x})Q_-, \qquad \bar{\chi}(x)|_{0<x_0<T}, \qquad \bar{\chi}(T,\mathbf{x})Q_+$$

Using the orbifold symmetry the Dirac operator can be reduced implicitly to the interval $[0,T]$:

$$S_f[\chi,\bar{\chi},U] = a^4 \sum_{-T<x_0\leq T} \bar{\chi}(x) D_N \chi(x) = 2a^4 \sum_{0\leq x_0\leq T} \bar{\chi}(x)\mathcal{D}_N\chi(x),$$

- but the explicit form of $\mathcal{D}_N$ is unnecessarily complicated!

$\Rightarrow$ use an alternative set-up:

Alternative set-up:

- Start with $2(T+a)$ anti-periodic fields ψ, $\bar{\psi}$

$$\psi(x_0+2(T+a),\mathbf{x}) = -\psi(x), \qquad \bar{\psi}(x_0+2(T+a),\mathbf{x}) = -\bar{\psi}(x),$$

- introduce a reflection $(R^2 = id)$

$$R: \psi(x) \rightarrow i\gamma_0\gamma_5\psi(-a-x_0,\mathbf{x}), \qquad \bar{\psi}(x) \rightarrow \bar{\psi}(-a-x_0,\mathbf{x})i\gamma_0\gamma_5$$

- the gauge field is extended to $[-T-a, T+a]$ and then periodically continued

$$U_k(-a-x_0,\mathbf{x}) = U_k(x_0,\mathbf{x}), \qquad U_0(-2a-x_0,\mathbf{x})^\dagger = U_0(x)$$

  this implies that the boundary layer is doubled!

- decompose in even/odd fields and define doublets $\chi, \bar{\chi}$ as before the dynamical field variables are now $\chi(x)$ and $\bar{\chi}(x)$ for all $0 \le x_0 \le T$

- achievement: the construction is completely analogous to the Wilson-Dirac case (S.'05); a block structure (in time) of the Wilson-Dirac operator is obtained and inherited by the overlap operator

$\Rightarrow$ the Neuberger operator in the inverval is simply obtained by using the corresponding orbifolded Wilson-Dirac kernel:

$$\mathcal{D}_N = 1 - A(A^\dagger A)^{-1/2}, \qquad A = 1 - a\mathcal{D}_W$$

with the kernel $\mathcal{D}_W$,

$$a\mathcal{D}_W\chi(x) = -U(x,0)P_-\chi(x+a\hat{\mathbf{0}}) + (K\psi)(x) - U(x-a\hat{\mathbf{0}})^\dagger P_+\chi(x-a\hat{\mathbf{0}}),$$

where we have set $\chi(x) = 0$ for $x_0 < 0$ and $x_0 > T$, and

$$K = 1 + \tfrac{1}{2}\sum_{k=1}^{3}\left\{a(\nabla_k + \nabla_k^*)\gamma_k - a^2\nabla_k^*\nabla_k\right\} + \delta_{x_0,0}i\gamma_5\tau^3 P_- + \delta_{x_0,T}i\gamma_5\tau^3 P_+$$

Symmetries

In a massless theory in finite volume the identification of flavour and chiral symmetries is a mere convention!

- take the standard Schrödinger functional with projectors $P_{\pm}$ as SU(2) flavour symmetric reference basis
- in the rotated SF, the SU(2) flavour symmetry is realised à la Ginsparg-Wilson:

$$\begin{aligned} \gamma_5\tau^{1,2}\mathcal{D}_N + \mathcal{D}_N\gamma_5\tau^{1,2} &= \mathcal{D}_N\gamma_5\tau^{1,2}\mathcal{D}_N \\ \tau^3\mathcal{D}_N - \mathcal{D}_N\tau^3 &= 0 \end{aligned}$$

Note that the flavour algebra closes $[\hat{\gamma}_5 = \gamma_5(1 - a\mathcal{D}_N)]$:

$$\hat{T}^1 = \hat{\gamma}_5\tau^2/2, \quad \hat{T}^2 = -\hat{\gamma}_5\tau^1/2, \quad \hat{T}^3 = \tau^3/2, \quad [\hat{T}^a, \hat{T}^b] = i\epsilon^{abc}\hat{T}^c$$

- Chiral symmetry is broken by the SF boundary conditions: expect: the standard GW relation is violated by terms which decrease exponentially with the distance from the boundaries

- form of non-singlet chiral symmetries:

$$[\tau^{1,2}, \mathcal{D}_N] \neq 0, \qquad \{\gamma_5\tau^3, \mathcal{D}_N\} \neq a\mathcal{D}_N\gamma_5\tau^3\mathcal{D}_N$$

  expect: both flavour components of $\mathcal{D}_N$ become equal and the GW relation holds up to corrections which decrease exponentially with the distance from the boundaries (checked at tree level).

- GW versions of parity and time reversal, e.g. :

$$P : \chi(x) \to i\gamma_0\gamma_5\tau^3\chi(\tilde{x}), \qquad \tilde{x} = (x_0, -\mathbf{x}), \qquad \mathcal{D}_N P + P\mathcal{D}_N = \mathcal{D}_N P\mathcal{D}_N$$

- in contrast to the case of Wilson quarks, parity and flavour are realised exactly, expect no extra counterterms!

Real & Positive Determinant?

- The determinant is real, due to the hermiticity property

$$\gamma_5 \tau^1 \mathcal{D}_N \gamma_5 \tau^1 = \mathcal{D}_N^\dagger.$$

Furthermore, this equation implies that the determinants of the single flavour operators are complex conjugate to each other:

$$\mathcal{D}_N = \mathrm{diag}\left(\mathcal{D}_N^{(1)}, \mathcal{D}_N^{(2)}\right), \qquad \det \mathcal{D}_N^{(1)} = \left(\det \mathcal{D}_N^{(2)}\right)^* .$$

The determinant is therefore non-negative

$$\det \mathcal{D}_N = \det\left(\mathcal{D}_N^{(1)}\right) \det\left(\mathcal{D}_N^{(2)}\right) \geq 0$$

One furthermore expects that the SF boundary conditions introduce a gap $\propto 1/T$ in the spectrum, so that the determinant should be positive provided the volume is not too large.

Conclusions and Outlook

- Successful implementation of chirally rotated SF boundary conditions for even number of GW quarks

- In the continuum limit the chirally rotated SF with an even number of massless GW quarks is equivalent to the standard SF;
  - parity and flavour symmetries are exact on the lattice!
  - solution is technically simple: just requires the insertion of the corresponding Wilson kernel into the Neuberger relation

- Construction applies directly to Domain Wall Quarks, technically simple, no obstruction for mass term of Pauli-Villars fields.

- Work in progress: extension to odd numbers of flavours